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Effective viscosity of a two-dimensional suspension of interacting active particles

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Suspensions of hydrodynamical active particles exhibit interesting rheological properties. For a dilute suspension of microswimmers, it has been shown that the effective viscosity of the suspension depends on the volume fraction of swimmers, and it behaves differently for pushers and pullers. Here we develop a theoretical framework to study the rheological properties of an interacting suspension. Taking into account the hydrodynamic interaction between swimmers and considering the small Péclet number condition, we calculate the effective viscosity of a two-dimensional suspension. For a dilute suspension, a perturbative result is obtained up to the second order of the surface fraction of swimmers. Our results show that the effective viscosity for the suspension can be very different for pushers and pullers.

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I. INTRODUCTION

Motivated by its application and relevance in biophysics and engineering, the rheology of active suspensions has attracted much attention recently [1,2]. Active suspensions composed of self-propelled active agents, either living microorganisms or synthetic active colloidal particles, have a nontrivial response to an external shear flow [3,4]. Thermal fluctuations of individual active colloids and their mutual interactions are the major sources that make their response nontrivial. Reduction or increase in the effective apparent viscosity [5,6] and non-Newtonian behavior [7] are among the main important features of such rheological responses. In addition to the rheological properties, a concentrated population of suspended active agents as a nonlinear dynamical system exhibit fascinating spatial and temporal patterns [8–12].

Despite the fact that thermodynamic pressure exhibits anomalous properties in nonequilibrium active suspensions [13], viscosity as a kinetic coefficient could be defined and measured in such systems. The effective viscosity of an active suspension is at the heart of recent theoretical studies and experiments. In an active suspension, the force distribution and subsequently the flow field produced by an individual particle are essential in studying their responses to external forces. As a result of zero net force in self-propelled particles, the force distribution can be either dipolar or quadrupolar at the leading order of multipole expansion. Pushers, pullers, and neutral swimmers are the examples of real organisms with different force distributions. New recent experiments reveal that the effective viscosity of an active suspension highly depends on such force distributions. An increase in the effective viscosity in motile algae *Chlamydomonas* (puller) [5], reduction of effective viscosity in *Bacillus subtilis* (pusher), and suspension of *Escherichia coli* [6,7] have been reported. It has been shown very recently that in a suspension of *E. coli,* the activity of bacteria can turn the suspension into a superfluid state [14,15].

The question we want to address here is how the two-body hydrodynamic interaction between swimmers influences the effective viscosity of the suspension. The main emerging approaches to the modeling of a suspension of microswimmers typically abstract away the details of the actual propulsion mechanism and use simple tractable and rigid geometries for swimmers [16–18]. Saintillan [19] by extending previous classical theories for passive suspensions [20] has used a simple kinetic model for studying the effective rheology of active suspensions in extensional flows. Haines *et al.*, by modeling bacteria as self-propelled disks [21] and as a rigid prolate spheroid with a point force [22], have obtained analytical expressions for the effective viscosity up to the first order in volume fraction. In both cases hydrodynamic interactions were neglected. Gyrya *et al.* has studied the hydrodynamic interaction between two microscopic swimmers, modeled as self-propelled dumbbells, and has performed a simulation to identify the connection between interactions and rotational noise as key (interchangeable) ingredients for reduction of viscosity [23].

In this work, we use a microscopic hydrodynamical model for a self-propelled swimmer and extend our previous work [24] to calculate the rheological properties of an interacting suspension. This work can be considered as an extension of Batchelor's classical $O(c^2)$ correction in a colloidal suspension to a system composed of active colloids [25,26].

This paper organized as follows: in Sec. II we introduce our model for a single swimmer and give analytic results for the hydrodynamic interaction between two swimmers in an external shear flow. Then in Sec. III we develop the statistical frame work and mean field approximation for a collection of particles. Rheological properties of a suspension of active particles are given in Sec. IV, and finally the results will be discussed in Sec. V.

II. MODEL AND TWO-PARTICLE INTERACTION

To calculate the rheological properties of a suspension of active particles, we start with a detailed microscopic model for an individual active particle. For this task we use a minimal model of a low Reynolds autonomous swimmer that is able to describe both pushers and pullers [27]. In low Reynolds regime where the effects of viscous forces dominate over inertial effects, it is a well-known fact that a minimal model for a swimmer needs at least two internal degrees of freedom [28]. A model system composed of three spheres connected linearly by two rods with variable lengths can capture the hydrodynamic features of such swimmers. Figure 1 shows

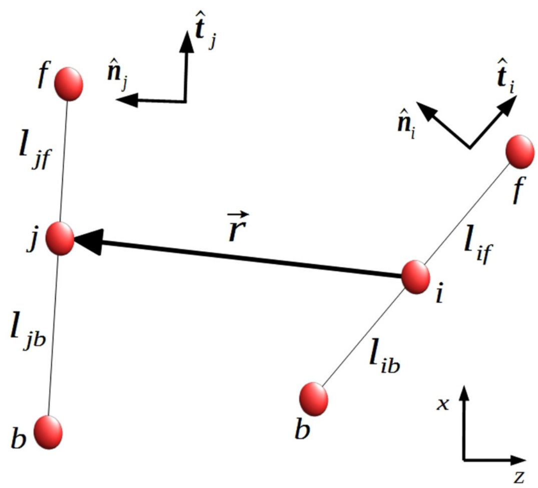

FIG. 1. Geometry of two three-linked spheres' swimmers.

a schematic view of two interacting swimmers (denoted by swimmer i and j) and their internal structures. We denote by a the radius of spheres and assume that the rods are very thin with no hydrodynamic coupling to the fluid. Denoting the position vector of central spheres of each swimmer by $\mathbf{r}_i$ and $\mathbf{r}_j$, the other spheres are labeled by ib, if, jb, and jf and their position vectors are given as

$$\mathbf{r}_{if} = \mathbf{r}_i + \ell_{if}\hat{\mathbf{t}}_i, \quad \mathbf{r}_{ib} = \mathbf{r}_i - \ell_{ib}\hat{\mathbf{t}}_i,$$
$$\mathbf{r}_{jf} = \mathbf{r}_j + \ell_{jf}\hat{\mathbf{t}}_j, \quad \mathbf{r}_{jb} = \mathbf{r}_j - \ell_{jb}\hat{\mathbf{t}}_j, \tag{1}$$

where $\hat{\mathbf{t}}_{i,j}$ denote the orientation vector of the swimmers. In our quasi-two-dimensional study, we assume that all swimmers are moving in the (x-z) plane. In this case we have $\hat{\mathbf{t}}_{i,j} = (\cos\theta_{i,j}, \sin\theta_{i,j})$. We assume that for each swimmer the arm lengths change sinusoidally around a mean value L as

$$\ell_{if} = L + u\sin(\omega t + \varphi_i),$$
$$\ell_{ib} = L(1+\xi) + u\sin(\omega t + \varphi_i + \delta\varphi),$$
$$\ell_{jf} = L + u\sin(\omega t + \varphi_j),$$
$$\ell_{jb} = L(1+\xi) + u\sin(\omega t + \varphi_j + \delta\varphi),$$

where ξ is an asymmetry parameter for swimmers. In the case where the arm lengths of each swimmer are equal, i.e., $\xi = 0$, we have a symmetric swimmer. ω denotes the internal frequency of swimmers, and $\delta\varphi$ is the internal phase difference for each swimmer where for simplicity we have assumed that all swimmers have equal internal phase differences.

Phases of the swimmers are given by φ_i and φ_j. For an autonomous swimmer, in addition to its translational and rotational velocities, phase is another variable that is necessary to describe its dynamical state. To have physical intuition about the phase variable, one can consider a beating flagellum that provides the driving force for a microorganism. It is shown that assigning an effective limit-cycle oscillator to the flagellum, it is possible to extract its phase [29]. In this article, we will consider the simple case where all swimmers are in phase ($\varphi_i = \varphi_j = 0$). As we discuss at the end, for uniform distribution of phase difference, the general behavior does not change.

The asymmetry parameter ξ plays essential role in the details of flow field produced by an individual swimmer. We will show that the force distribution due to a single swimmer shows a quadrupolar distribution for $\xi = 0$, and it shows a dipolar field when $\xi \neq 0$. For $\xi = 0$ the swimmer is neutral, for $\xi > 0$ the swimmer is a pusher, and for $\xi < 0$ the swimmer is a puller [24].

As we aim to investigate the response of a suspension of swimmers to external forcing, let us assume that the above swimmers are immersed in an external simple shear flow. We denote the external shear flow by

$$\mathbf{u}^{sh.} = \Gamma\cdot\mathbf{x}, \quad \Gamma = \dot{\gamma}\,\hat{\mathbf{z}}\hat{\mathbf{x}}, \tag{2}$$

where $\dot{\gamma}$ is the shear rate applied to the fluid. We can show that the above swimmers are autonomous systems that can propel themselves by internal motions defined by Eq. (1). To obtain the swimming velocity, we should solve the dynamical equations of the above model swimmers that are coupled to the dynamics of the ambient fluid. Due to linearity of the hydrodynamic equations at micrometer scale (negligible Reynolds number), the hydrodynamic forces acting by the spheres on the fluid are related to the velocity of each sphere linearly. Consider a collection of colloidal particles moving in an inertialess fluid; their interaction can lead to the equations of motion between their velocities $\mathbf{v}_\alpha = \dot{\mathbf{r}}_\alpha$ and hydrodynamic forces they apply to the fluid, $\mathbf{f}_\alpha$, as

$$\mathbf{v}_\alpha = \sum_\beta \mathbf{M}_{\alpha\beta}(\mathbf{r}_\alpha,\mathbf{r}_\beta)\cdot\mathbf{f}_\beta + \Gamma\cdot\mathbf{r}_\alpha, \tag{3}$$

where indices α and β run over all spheres (i,j,ib,if,jb,jf). The hydrodynamic kernel $\mathbf{M}_{\alpha\beta}$ has the following tensorial structure at the Oseen approximation [30]:

$$\mathbf{M}_{\alpha\beta}(\mathbf{r}_\alpha,\mathbf{r}_\beta) = \begin{cases} \frac{1}{8\pi\eta_\circ|\mathbf{r}_{\alpha\beta}|}\left[\mathbf{I} + \frac{\mathbf{r}_{\alpha\beta}\mathbf{r}_{\alpha\beta}}{|\mathbf{r}_{\alpha\beta}|^2}\right] & \alpha\neq\beta \\ \frac{1}{6\pi\eta_\circ a}\mathbf{I} & \alpha=\beta. \end{cases}$$

Here $\mathbf{r}_{\alpha\beta} = \mathbf{r}_\alpha - \mathbf{r}_\beta$ and $\eta_\circ$ is the viscosity of the fluid. In writing the above expression for the hydrodynamic kernel, we have assumed that the lengths of rods are much larger than the sphere size $L \gg a$. In addition to the above equations, we should include the facts that the swimmers are force and torque free:

$$\mathbf{f}_i + \mathbf{f}_{ib} + \mathbf{f}_{if} = \mathbf{0}, \quad \ell_{if}\hat{\mathbf{t}}_i\times\mathbf{f}_{if} = \ell_{ib}\hat{\mathbf{t}}_i\times\mathbf{f}_{ib},$$
$$\mathbf{f}_j + \mathbf{f}_{jb} + \mathbf{f}_{jf} = \mathbf{0}, \quad \ell_{jf}\hat{\mathbf{t}}_j\times\mathbf{f}_{jf} = \ell_{jb}\hat{\mathbf{t}}_j\times\mathbf{f}_{jb}. \tag{4}$$

The sets of equations given in (1), (3), and (4) form a closed set of equations that we can solve and find all dynamical characteristics of the swimmers, namely, their velocities and force distributions.

To simplify the results we analyze the dynamics of the swimmers in the case where $a \ll L$, $u \ll L$, and $|\xi| \ll 1$. This helps us to expand the final results in terms of small quantities that can be constructed by different relevant length scales. If we assume that the swimmers are far enough away from each other such that the distance between them is larger than the length of swimmers $|\mathbf{r}| = |\mathbf{r}_i - \mathbf{r}_j| \gg L$, we can expand the forces and velocities in powers of the small quantity L/r. Averaging over time, the linear and angular velocity of the first swimmer up to the leading order can be obtained. As a result of the linearity of dynamical equations, the velocities and forces can be decomposed into their contributions coming from the shear flow and the parts due to the interactions. We can decompose the translational and angular velocities as

$$\begin{aligned}\mathbf{v}_i &= \mathbf{v}_i^0 + \mathbf{v}_i^{\text{bg}} + \mathbf{v}^{\text{int}}(\mathbf{r}_i - \mathbf{r}_j,\theta_i,\theta_j),\\ \Omega_i &= \Omega_i^0 + \Omega_i^{\text{bg}} + \Omega^{\text{int}}(\mathbf{r}_i - \mathbf{r}_j,\theta_i,\theta_j),\end{aligned} \tag{5}$$

where $\mathbf{v}_i^0$ and Ω_i^0 are the intrinsic swimming velocities of individual swimmers in an ambient fluid and are given by

$$\mathbf{v}_i^0 = v_\circ(1-\xi)\,\hat{\mathbf{t}}_i, \quad \Omega_i^0 = \mathbf{0}, \tag{6}$$

where $v_\circ = \frac{7}{24}\frac{a}{L^2}u^2\omega\sin\delta\varphi$. The velocity of the swimmer due to the background shear flow is given by

$$\mathbf{v}_i^{\text{bg}} = \dot\gamma\, x\cos\theta\,\hat{\mathbf{t}}_i - \frac{1}{12}L\dot\gamma\,\xi\ \sin\theta\cos\theta\left(1+\frac{7}{4}\frac{a}{L}\right)\hat{\mathbf{t}}_i, \tag{7}$$

$$\Omega_i^{\text{bg}} = \dot\gamma\sin^2\theta, \tag{8}$$

and the interaction contributions are given by

$$\begin{aligned}\mathbf{v}^{\text{int}}(\mathbf{r}_i - \mathbf{r}_j,\theta_i,\theta_j) = &-\frac{11}{8}\xi v_\circ[1-3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2]\frac{aL}{r^2}\ \hat{\mathbf{r}} - \frac{3}{8}\frac{aL^2}{r^2}\dot\gamma\ \sin\theta_j\cos\theta_j\left(1+\xi+\frac{3}{4}\frac{a}{L}\right)[1-3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2]\hat{\mathbf{r}}\\ &+\frac{3}{7}\frac{L^3}{r^3}v_\circ q_1(\theta_i,\theta_j,\hat{\mathbf{r}})\,\hat{\mathbf{t}} - \frac{6}{7}v_\circ\frac{L^3}{r^3}\{[1-3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2]\hat{\mathbf{t}}_j + 3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)[5(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2-3]\hat{\mathbf{r}}\}\\ &+\frac{17}{64}\frac{a^2L^2}{r^3}\dot\gamma\sin\theta_j\cos\theta_j\ q_1(\theta_i,\theta_j,\hat{\mathbf{r}})\,\hat{\mathbf{t}}_j + \frac{1}{16}\frac{aL^3}{r^3}\xi\dot\gamma\ \sin\theta_j\cos\theta_j\left(7+\frac{13}{4}\frac{a}{L}\right)\\ &\times\{[1-3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2]\hat{\mathbf{t}}_j + 3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)[5(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2-3]\hat{\mathbf{r}}\}\end{aligned} \tag{9}$$

$$\begin{aligned}\Omega^{\text{int}}(\mathbf{r}_i - \mathbf{r}_j,\theta_i,\theta_j) = &\frac{45}{56}\,\xi\ v_\circ\frac{a^2}{r^3}\,q_2(\theta_i,\theta_j,\hat{\mathbf{r}})\ - \frac{9}{8}\frac{aL^2}{r^3}\dot\gamma\ \sin\theta_j\cos\theta_j\ q_2(\theta_i,\theta_j,\hat{\mathbf{r}})\ + \frac{27}{28}v_\circ\ \frac{L^3}{r^4}q_3(\theta_i,\theta_j,\hat{\mathbf{r}})\\ &+\frac{9}{16}\frac{aL^3}{r^4}\dot\gamma\ \sin\theta_j\cos\theta_j\ q_3(\theta_i,\theta_j,\hat{\mathbf{r}}),\end{aligned} \tag{10}$$

where the angular functions $q_i,(i=1,2,3)$ are defined in the Appendix. The forces on spheres of the swimmer i can be written as ($\alpha\in\{i,if,ib\}$)

$$\mathbf{f}_\alpha = \mathbf{f}_\alpha^0 + \mathbf{f}_\alpha^{bg} + \mathbf{f}_\alpha^{\text{int}}. \tag{11}$$

Here $\mathbf{f}_\alpha^0$ are forces due to intrinsic swimming without shear and given by

$$\mathbf{f}_{if}^0 = \frac{3}{7}\pi\eta_\circ a v_\circ(5+7\xi)\,\hat{\mathbf{t}}_i, \qquad \mathbf{f}_{ib}^0 = \frac{3}{7}\pi\eta_\circ a v_\circ(5-17\xi)\,\hat{\mathbf{t}}_i.$$

The role of the asymmetry parameter ξ can be understood from the forces that are exerted by a single swimmer on the fluid. For $\xi=0$ the above force distribution shows a quadrupolar distribution, and it shows a dipolar field when $\xi\neq 0$. For $\xi=0$ the swimmer is neutral, for $\xi>0$ the swimmer is a pusher, and for $\xi<0$ the swimmer is a puller. Pullers use their head to generate their motion, and pushers use their tail to produce motion.

The forces acting on the swimmer due to the background flow are given by

$$\begin{aligned}\mathbf{f}_{if}^{bg} &= -6\pi\eta_\circ aL\dot\gamma\sin\theta_i\cos\theta_i\left[1+\frac{3}{4}\left(\frac{a}{L}\right)+\frac{1}{12}\xi+\cdots\right]\hat{\mathbf{t}}_i,\\ \mathbf{f}_{ib}^{bg} &= 6\pi\eta_\circ aL\dot\gamma\sin\theta_i\cos\theta_i\left[1+\frac{3}{4}\left(\frac{a}{L}\right)+\frac{1}{12}\xi+\cdots\right]\hat{\mathbf{t}}_i,\end{aligned}$$

and the forces due to the interaction, $\mathbf{f}_\alpha^{\text{int}}$, are [31]

$$\begin{aligned}(\pi\eta_\circ a v_\circ)^{-1}\mathbf{f}_{if}^{\text{int}} = &\left\{-\frac{36}{7}\frac{L^3}{r^3}q_1(\theta_i,\theta_j,\hat{\mathbf{r}}) + \frac{3}{2}\frac{aL^3}{r^3}\dot\gamma/v_\circ\sin\theta_j\cos\theta_j q_1(\theta_i,\theta_j,\hat{\mathbf{r}}) - \frac{L^4}{r^4}\left[\frac{108}{7}q_5(\theta_i,\theta_j,\hat{\mathbf{r}}) + \frac{99}{7}q_4(\theta_i,\theta_j,\hat{\mathbf{r}})\right]\right.\\ &\left.+\frac{L^6}{ar^4}\dot\gamma/v_\circ\sin\theta_j\cos\theta_j\left[\frac{3}{2}\frac{a^2}{L^2}q_5(\theta_i,\theta_j,\hat{\mathbf{r}}) - \frac{7}{8}\frac{a}{L}\xi q_4(\theta_i,\theta_j,\hat{\mathbf{r}})+\cdots\right]+\cdots\right\}\hat{\mathbf{t}}_i\\ &+\left[\frac{9}{4}\frac{L^5}{ar^4}q_3(\theta_i,\theta_j,\hat{\mathbf{r}}) + \frac{9}{16}\frac{aL^3}{r^4}\dot\gamma/v_\circ\sin\theta_j\cos\theta_j q_3(\theta_i,\theta_j,\hat{\mathbf{r}})+\cdots\right]\hat{\mathbf{n}}_i,\end{aligned} \tag{12}$$

$$(\pi\eta_\circ a v_\circ)^{-1}\mathbf{f}_{ib}^{\text{int}} = \left\{ -\frac{18}{7}\frac{L^3}{r^3}q_1(\theta_i,\theta_j,\hat{\mathbf{r}}) - \frac{3}{8}\frac{aL^3}{r^3}\dot{\gamma}/v_\circ \sin\theta_j\cos\theta_j q_1(\theta_i,\theta_j,\hat{\mathbf{r}}) + \frac{L^4}{r^4}\left[\frac{54}{7}q_5(\theta_i,\theta_j,\hat{\mathbf{r}}) + \frac{99}{14}q_4(\theta_i,\theta_j,\hat{\mathbf{r}})\right] + \frac{L^6}{ar^4}\dot{\gamma}/v_\circ \sin\theta_j\cos\theta_j\left[-3\frac{a^2}{L^2}q_5(\theta_i,\theta_j,\hat{\mathbf{r}}) + \frac{7}{2}\frac{a}{L}\xi q_4(\theta_i,\theta_j,\hat{\mathbf{r}}) + \cdots\right] + \cdots\right\}\hat{\mathbf{t}}_i + \left[\frac{9}{4}\frac{L^5}{ar^4}q_3(\theta_i,\theta_j,\hat{\mathbf{r}}) + \frac{9}{16}\frac{aL^3}{r^4}\dot{\gamma}/v_\circ\sin\theta_j\cos\theta_j q_3(\theta_i,\theta_j,\hat{\mathbf{r}}) + \cdots\right]\hat{\mathbf{n}}_i, \tag{13}$$

where the angular functions $q_i,(i=4,5)$ are defined in the Appendix.

Next, we will use the above analytical results that are obtained for the velocity and force distribution to evaluate the rheological properties of a suspension.

III. A FLUCTUATING SUSPENSION

Consider a two-dimensional dilute suspension of N swimmers moving in a background shear flow. Denoting the surface density of swimmers by $n_\circ$ their surface fraction is given by $c = n_\circ s_\circ$ where $s_\circ = 2a \times 2L$ is the effective surface occupied by a single swimmer. As we have mentioned before, each swimmer has translational and orientational degrees of freedom given by $\mathbf{r}_i$ and θ_i. Taking into account the rotational fluctuations of the swimmers, the equations of motion read as

$$\begin{aligned}\dot{\mathbf{r}}_i &= \mathbf{v}_i^0 + \mathbf{v}_i^{\text{bg}} + \sum_{j\neq i}\mathbf{v}^{\text{int}}(\mathbf{r}_i-\mathbf{r}_j,\theta_i,\theta_j),\\ \dot{\theta}_i &= \Omega_i^0 + \Omega_i^{\text{bg}} + \sum_{j\neq i}\Omega^{\text{int}}(\mathbf{r}_i-\mathbf{r}_j,\theta_i,\theta_j) + \sqrt{2D_r}\zeta_i(t),\end{aligned} \tag{14}$$

where D_r represents the rotational diffusion constant of swimmers and $\zeta_i(t)$ are Gaussian noises with zero mean and temporal δ correlations, i.e., $\langle\zeta_i(t)\zeta_j(t')\rangle = \delta_{ij}\delta(t-t')$. It is shown previously that for such swimmers, the rotational diffusion has a value like $D_r = k_BT/(24\pi\eta_\circ aL^2)$ [24]. In writing the above equations of motion, we have assumed that $D_r \ll \omega$. As a result of this approximation, the internal motion of the swimmers are faster in comparing with the rotational diffusion. This allowed us to use interaction contributions that are averaged over internal motion of the swimmers. The validity of the above assumption can be verified in real systems. For a swimmer with the largest length $L \sim 1\,\mu$m and the smallest length $a \sim 0.1\,\mu$m, we see that $D_r \sim 0.5\,\text{s}^{-1}$. For a real microswimmer, for example, *Chlamydomonas,* the undulation frequency of the flagellum is about 100 Hz, which justifies our assumption [32,33].

Instead of solving the above Langevin description, we use the Fokker-Planck method to analyze the statistical properties of the suspension. The statistical properties of the suspension are entirely determined by the N-particle probability distribution function $P_N(\mathbf{r}_1,\theta_1;\mathbf{r}_2,\theta_2;\cdots;\mathbf{r}_N,\theta_N;t)$, which is the probability of finding particle i at position $\mathbf{x}_i$ moving along direction given by θ_i at time t. The normalization condition for this probability distribution function is given by

$$\prod_{j=1}^N \int d\mathbf{r}_j \int_0^{2\pi} d\theta_j P_N(\mathbf{r}_1,\theta_1;\mathbf{r}_2,\theta_2;\cdots;\mathbf{r}_N,\theta_N;t) = 1.$$

The probability distribution function obeys the following dynamical Fokker-Planck equation:

$$\frac{\partial P_N}{\partial t} = -\sum_{i=1}^N \nabla_{\mathbf{r}_i}\cdot(\dot{\mathbf{r}}_i P_N) - \sum_{i=1}^N \frac{\partial}{\partial\theta_i}(\dot{\theta}_i P_N).$$

As a first approximation in dealing with a dilute suspension, we assume that the N-particle distribution function can be factorized as

$$P_N(\mathbf{r}_1,\theta_1;\mathbf{r}_2,\theta_2;\cdots;\mathbf{r}_N,\theta_N;t) = \prod_{i=1}^N P(\mathbf{r}_i,\theta_i,t).$$

Defining the one-particle probability density as $p(\mathbf{r}_i,\theta_i,t) = NP(\mathbf{r}_i,\theta_i,t)$, we can see that the governing equation for this probability density can be written as

$$\begin{aligned}\frac{\partial\, p(\mathbf{r},\theta,t)}{\partial t} &= -\nabla_{\mathbf{r}}\cdot[\dot{\mathbf{r}}\, p(\mathbf{r},\theta,t)] + D_r\frac{\partial^2 p(\mathbf{r},\theta,t)}{\partial\theta^2}\\ &\quad - \frac{\partial}{\partial\theta}[\bar{\Omega}^{\text{int}}(\mathbf{r},\theta,t)\ p(\mathbf{r},\theta,t)]\\ &\quad - \frac{\partial}{\partial\theta}[\Omega^{\text{bg}}\ p(\mathbf{r},\theta,t)],\end{aligned} \tag{15}$$

where $\bar{\Omega}^{\text{int}}$ is defined as

$$\bar{\Omega}^{\text{int}}(\mathbf{r},\theta,t) = \int d\mathbf{R}d\theta'\ \Omega^{\text{int}}(\mathbf{R},\theta,\theta')P(\mathbf{r}',\theta',t),$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. We further assume that the suspension is spatially homogeneous, that is, $p(\mathbf{r},\theta,t) = p(\theta,t)$. In this case, by integrating over spatial degrees of freedom and using a divergence theorem, (15) reduces to

$$\begin{aligned}\frac{\partial\, p(\theta,t)}{\partial t} &= D_r\frac{\partial^2 p(\theta,t)}{\partial\theta^2} - \frac{\partial}{\partial\theta}[\bar{\Omega}^{\text{int}}(\mathbf{r},\theta,t)\ p(\theta,t)]\\ &\quad - \frac{\partial}{\partial\theta}[\Omega^{\text{bg}}\ p(\theta,t)].\end{aligned} \tag{16}$$

After replacing the value of angular velocity from the previous section, and performing the summation of contributions from other swimmers, we will obtain

$$\begin{aligned}\bar{\Omega}^{\text{int}}(\mathbf{r},\theta,t) &= \frac{15\pi}{512}\frac{a^3 n_\circ}{L^2\ell_c}\xi\ \Phi_\circ\int_\theta^{\theta+2\pi} d\theta'\sin[2(\theta-\theta')]\ p(\theta',t)\\ &\quad + \frac{9\pi}{128}\frac{aL^2}{\ell_c}n_\circ\dot{\gamma}\int_\theta^{\theta+2\pi} d\theta'\\ &\quad \times \sin[2(\theta-\theta')]\sin(2\theta')\, p(\theta',t).\end{aligned}$$

As the hydrodynamic interaction is divergent at short distances, we should define a short length cutoff to regularize the integrals. ℓ_c is a microscopic cutoff length beyond which the long-range hydrodynamic interaction affects. Here we assume that $\ell_c \gtrsim 6L$, As the analytic far field velocity expression is accurate for distances further than around three swimmer lengths [34].

Therefore, the steady state dimensionless Fokker-Planck equation simplifies to

$$\begin{aligned}\frac{\partial^2 p(\theta,t)}{\partial\theta^2} &= \mathrm{Pe}_r\frac{\partial}{\partial\theta}[\sin^2\theta\;\, p(\theta,t)]\\ &\quad+\kappa\frac{\partial}{\partial\theta}\left\{\int_\theta^{\theta+2\pi}\mathrm{d}\theta'\sin[2(\theta-\theta')]\,p(\theta',t)\;\,p(\theta,t)\right\}\\ &\quad+z_\circ c\,\mathrm{Pe}_r\frac{\partial}{\partial\theta}\left\{\int_\theta^{\theta+2\pi}\mathrm{d}\theta'\sin[2(\theta-\theta')]\sin(2\theta')\right.\\ &\quad\left.\times\;p(\theta',t)\;\,p(\theta,t)\right\},\end{aligned}\tag{17}$$

where $z_\circ=\frac{9\pi}{15}\frac{L}{\ell_c}$, and the two dimensionless numbers are given by $\mathrm{Pe}_r=\frac{\dot\gamma}{D_r}$, which is the rotational Péclet number, and $\kappa=\frac{45\pi}{1792}\frac{av_\circ}{L\ell_c D_r}\xi c$. The above equation is the central equation that determines the single-particle distribution function in an interacting system. It contains different terms with different underlying physics. The term proportional to Pe_r shows the contribution from external shear flow, and it will result in angular asymmetry in the distribution function due to the external force. The term proportional to κ reflects the fact that the activity of the swimmers has nontrivial effects on the distribution function. The last term that is proportional to $c\mathrm{Pe}_r$ shows the contributions due to the hydrodynamic interactions between the swimmers. Contributions due to activity and interaction make the above equation hard to solve.

For our system we can compare the relative importance of two terms κ and $c\mathrm{Pe}_r$. We can see that

$$\frac{c\mathrm{Pe}_r}{\kappa}\sim\left(\frac{L}{a}\right)^2\times\left(\frac{L}{u}\right)^2\times\frac{\dot\gamma}{\omega}\gg 1.$$

This means that we can neglect the effects due to the term proportional to κ and make the equation simpler. So the equation for the orientational probability distribution takes the following form:

$$\begin{aligned}\frac{\partial^2 p(\theta,t)}{\partial\theta^2} &= \mathrm{Pe}_r\frac{\partial}{\partial\theta}[\sin^2\theta\;\, p(\theta,t)]\\ &\quad+z_\circ c\,\mathrm{Pe}_r\frac{\partial}{\partial\theta}\left\{\int_\theta^{\theta+2\pi}\mathrm{d}\theta'\sin[2(\theta-\theta')]\right.\\ &\quad\left.\times\sin(2\theta')\,p(\theta',t)\;\,p(\theta,t)\right\}.\end{aligned}\tag{18}$$

To solve the above equation, we proceed by searching for perturbative solutions expanded in terms of two small numbers Pe_r and $c\mathrm{Pe}_r$. The final result for the distribution function can be arranged as

$$\begin{aligned}p(\theta)=\frac{1}{2\pi}\bigg[&1-\frac{1}{4}\mathrm{Pe}_r(1+z_\circ c)\sin(2\theta)\\ &-\frac{1}{8}\mathrm{Pe}_r^2(1+z_\circ c)\sin^4\theta+\cdots\bigg].\end{aligned}\tag{19}$$

As one can see, the above distribution function is sensitive to the surface fraction of swimmers c. It should be noted that this distribution function is normalized to particle density and has the same angular dependence as passive dumbbells. This function has peaks at $\theta=3\pi/4$ and $\theta=-\pi/4$, which arise as the competition between the simple shear flow and rotational Brownian motion of the swimmers. Once $p(\theta)$ is known from the Fokker-Planck equation, and it can be used to evaluate the average stress imposed on the fluid by swimmers.

IV. LINEAR RHEOLOGY

For a system composed of N swimmers, we assume that all swimmers interact through the two-body interactions obtained above. In this case the stress tensor due to all swimmers can be written as

$$\mathcal{S}=\sum_{i=1}^{N}(\mathbf{r}_i\mathbf{f}_i+\mathbf{r}_{if}\mathbf{f}_{if}+\mathbf{r}_{ib}\mathbf{f}_{ib}),\tag{20}$$

where summation should be performed over all swimmers. As we have shown, the hydrodynamic force acting by each swimmer can be divided into a part that depends on the state of that swimmer and a second part that emerges from the interaction with other swimmer. The interaction contribution depends on the relative state of interacting swimmers. We separate the interaction part of the stress tensor as

$$\mathcal{S}=\sum_{i=1}^{N}\mathcal{S}^0(\mathbf{r}_i,\theta_i)+\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{S}^{\mathrm{int}}(\mathbf{r}_i,\theta_i,\mathbf{r}_j,\theta_j).\tag{21}$$

For swimmers with fast internal motion and regarding our discussion above, we can first average over internal motion of swimmers and then average over fluctuations to obtain the averaged stress as

$$\begin{aligned}\langle\mathcal{S}\rangle &= N\int d\mathbf{r}d\cos\theta\,P(\mathbf{r},\theta)\langle\mathcal{S}^0\rangle_t\\ &\quad+\frac{N(N-1)}{2}\int d\mathbf{r}d\cos\theta\int d\mathbf{r}'d\cos\theta'\,P(\mathbf{r},\theta)\\ &\quad\times P(\mathbf{r}',\theta')\langle\mathcal{S}^{\mathrm{int}}\rangle_t,\end{aligned}$$

where the subscript t denotes the averaging over internal motion. Using the results that have been obtained above for the forces, and averaging over the angular distribution of swimmers we could obtain the average stress tensor. Up to the leading order, the time average of the stress is

$$\begin{aligned}\langle\mathcal{S}\rangle_t=\bigg\{&-\frac{15}{7}\pi\eta_\circ\,\xi\,La\,v_\circ+3\pi\eta_\circ\,aL^2\,\dot\gamma\,\sin\theta_i\cos\theta_i\left[1+\frac{3}{4}\left(\frac{a}{L}\right)+3\xi\right]-\frac{1}{2}\pi\eta_\circ\frac{a^2L^4}{r^3}q_1\\ &\times\left[\frac{v_\circ}{7a}(144+132\xi)+\frac{1}{8}\dot\gamma\,\sin\theta_j\cos\theta_j\big(15+16\xi\big)\right]+\frac{3}{2}\pi\eta_\circ\frac{L^5}{r^4}q_5\left(\frac{12}{7}\frac{L^2}{a}v_\circ+9\frac{a^3}{L}\dot\gamma\,\sin\theta_j\cos\theta_j\right)\bigg\}\,\hat{\mathbf{t}}_i\,\hat{\mathbf{t}}_i.\end{aligned}\tag{22}$$

At this point we have not yet performed the radial and angular average; this is just the time average. After performing the angular average with respect to the orientational distribution function, the effective viscosity of the suspension can be obtained as $\eta = \eta_\circ(1 + \frac{n_\circ \mathcal{S}_{zx}}{2a\dot{\gamma}\,\eta_\circ})$. Up to the leading orders we will have

$$\frac{\eta}{\eta_\circ} = 1 + c\left[z_1\frac{L}{a} - z_2\xi w\right] + c^2\left[z_3\frac{L^2}{a\ell_c} + w\left(z_4\frac{L^5}{a^3\ell_c^2} - z_5\frac{L^2}{a\ell_c}\right) - \xi w\left(z_6\frac{L^2}{a\ell_c} + z_7\frac{L}{\ell_c}\right)\right] + O(c^3), \quad (23)$$

where $w = (\eta_\circ v_\circ L^2)/(k_B T)$, $z_1 = 3\pi/64$, $z_2 = 180\pi/28$, $z_3 = 15\pi^2/8192$, $z_4 = 504\pi^3/112$, $z_5 = 621\pi^3/224$, $z_6 = 207\pi^3/112$, $z_7 = 405\pi^3/57344$.

Our result for the effective viscosity of an active suspension can be considered as an extension of the classical formula for the effective viscosity of a passive suspension. It has been shown that for a suspension composed of passive colloidal particles, the effective viscosity has a contribution proportional to c (volume fraction in three-dimensional systems). In the passive case, both the thermal fluctuations and the interactions contribute to the effective viscosity as a second order effect proportional to c^2.

The dimensionless number w, which appeared in our results, is a measure of the activity of swimmers. For a suspension of active particles, the terms proportional to w represent the activity contribution to the effective viscosity. As one can distinguish from the above result, the activity has a contribution in both c and c^2 terms. The hydrodynamic short length cutoff ℓ_c, appears only in terms proportional to c^2. This shows that similar to passive suspensions, in active suspensions the hydrodynamic interaction also can contribute in second order corrections. As we can see, for $\ell_c \to \infty$ the interaction contribution to the effective viscosity vanishes, and only the passive and intrinsic activity of swimmers contributes in effective viscosity [24]. The dependence to the asymmetry parameters ξ in our results is interesting. This shows that the effective viscosity of pullers and pushers is different. The effective viscosity is reduced for pusher swimmers ($\xi > 0$), and it is increased for puller swimmers ($\xi < 0$). These are the facts that are seen in experimental observations [5–7].

In addition to effective viscosity, the normal stress difference can also be calculated:

$$N = \mathcal{S}_{zz} - \mathcal{S}_{xx} = \mathrm{Pe}_r^2(1 + z_\circ c)\,\pi\eta_\circ v_\circ\left[\frac{15}{224}La\,\xi - \frac{17\pi}{14\,336}\frac{L^3}{\ell_c} \times c(144 + 132\xi) + \frac{9\pi}{112}\frac{L^7}{a^2{\ell_c}^3}\,c\right]. \quad (24)$$

As could be seen, the magnitude of this normal stress is proportional to swimming speed, and the type of swimmer could affect the decrease or increase of normal stress difference through the sign of ξ. For pushers we have $N > 0$, while for pullers $N < 0$ in agreement with previous work [35]. This normal stress difference is a measure for non-Newtonian behavior in an active suspension that is observed experimentally [5].

V. CONCLUDING REMARKS

We have presented a microscopic model for studying the rheological properties of a quasi-two-dimensional suspension of self-propelled particles. To study the effects of swimming activity in the rheological properties of a suspension, it is very important to note that the thermal fluctuations of the swimmers should be considered. Neglecting the thermal fluctuations of the swimmer will result in rheological properties that are independent from swimming activity. In other words the fluctuations provide a mechanism that result in activity-dependent response (rheological properties) of the system. This is due to the linearity of the Stokes equations for the fluid. For hydrodynamical swimmers, their swimming activity enters through the boundary condition on their surface. The linearity of fluid equations shows that the swimming problem in the presence of an external flow can be separated into two problems, a single swimmer immersed in a quiescent fluid and a passive body in the presence of an external force. To obtain the rheological properties or the response of the system to external force one will need to solve the second problem that is independent of the swimming activity.

Taking into account the orientational fluctuations and hydrodynamic interaction between swimmers, we show that the hydrodynamic interaction contributes to the effective viscosity as a second order correction proportional to c^2. Our results for the effective viscosity of the suspension extend the classical theories for passive dumbbells [36,37]. We show that in addition to the surface fraction c, the geometry of the swimmers L/a, their asymmetry factor ξ, and their swimming strength $w = \eta_\circ v_\circ L^2/k_B T$ have considerable effects in their rheological properties. For a typical micrometer-scale self-propelled particle swimming with speed $v_\circ \sim 1\,\mu$m, the activity parameter reads as $w \sim 1$. This shows that the effects due to activity in the viscosity are crucial. We found that swimming activity results in an increase in viscosity in suspensions of pullers, but a decrease in suspensions of pushers, in agreement with experimental observations [5–7].

It should be mentioned that we have neglected translational fluctuations and have considered only the rotational fluctuations. To check the relevance of this approximation, one should note that the rotational diffusion of an active particle will give rise to an effective translation diffusion given by $D_t^{\text{eff}} \sim \frac{v_\circ^2}{D_r}$ [38]. For a typical microswimmer, rotational and translational diffusion coefficients are given by $D_r \sim 10^{-1}\,\mathrm{s}^{-1}$ and $D_t \sim 10^{-12}\,\mathrm{m}^2\mathrm{s}^{-1}$, respectively. Regarding the fact that the swimming speed is about $10^{-6}\,\mathrm{ms}^{-1}$, we can easily see that $D_t^{\text{eff}} \sim 10 D_t$. This will ensure that orientational diffusion has a stronger effect than the translational one.

In this article, we have presented the results for a simple case where all swimmers are in phase. The model swimmer discussed in this article allows us to consider a more general case of a system with out-of-phase swimmers. For out-of-phase swimmers, the hydrodynamic interactions will depend on the phase difference $\varphi_i - \varphi_j$. Without presenting the details of the calculations here, we have examined the case that the phases are rigid and distributed randomly. Averaging over a uniform distribution of the phases, the terms proportional to z_5 and

z_6 in Eq. (23) will vanish. The terms promotional to z_6 and z_7 have similar signs, and at the limit of $a \ll L$, the term proportional to z_5 is negligible as compared with the term proportional to z_4. This shows that the rheological properties of two systems, one with in-phase swimmers and the other with uniformly distributed phases, are similar. However, in a more realistic system with diffusing phases, coupling between phase and orientational degree of freedom can arise because of nontrivial dynamics of the phase variables. Such coupling can introduce interesting effects in the system.

In this article we have considered the active Brownian particles mechanism for swimmers. Run and tumble is another mechanism that most microorganisms use to swim. A natural extension of this work is the generalization of our results for the latter case.

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APPENDIX: THE ANGULAR FUNCTIONS

The angular functions $q_i(\theta_i,\theta_j,\hat{\mathbf{r}}),(i = 1,2,3,4,5,6)$ are

$$q_1(\theta_i,\theta_j,\hat{\mathbf{r}}) = 1 - 3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)^2 - 3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2 - 6(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j) + 15(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2,$$

$$q_2(\theta_i,\theta_j,\hat{\mathbf{r}}) = (\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}_i)[(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i) + 2(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j) - 5(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2],$$

$$q_3(\theta_i,\theta_j,\hat{\mathbf{r}}) = (\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}_i)[1 + 2(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j)^2 - 5(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)^2 - 5(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2 - 20(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j) + 35(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2],$$

$$q_4(\theta_i,\theta_j,\hat{\mathbf{r}}) = (\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)[-3 - 4(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i) - 2(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j)^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j) + 15(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)^2 + 20(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j) + 5(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2 - 35(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2],$$

$$q_5(\theta_i,\theta_j,\hat{\mathbf{r}}) = 3(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i) + 4(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j) + 2(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j)^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i) - 15(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)^2 - 20(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)(\hat{\mathbf{t}}_i\cdot\hat{\mathbf{t}}_j)$$
$$- 5(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)^2 + 35(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_j)(\hat{\mathbf{r}}\cdot\hat{\mathbf{t}}_i)^3.$$

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