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Effects of correlations and fees in random multiplicative environments: Implications for portfolio management

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Geometric Brownian motion (GBM) is frequently used to model price dynamics of financial assets, and a weighted average of multiple GBMs is commonly used to model a financial portfolio. Diversified portfolios can lead to an increased exponential growth compared to a single asset by effectively reducing the effective noise. The sum of GBM processes is no longer a log-normal process and has a complex statistical properties. The nonergodicity of the weighted average process results in constant degradation of the exponential growth from the *ensemble average* toward the *time average*. One way to stay closer to the ensemble average is to maintain a balanced portfolio: keep the relative weights of the different assets constant over time. To keep these proportions constant, whenever assets values change, it is necessary to rebalance their relative weights, exposing this strategy to fees (transaction costs). Two strategies that were suggested in the past for cases that involve fees are rebalance the portfolio periodically and rebalance it in a partial way. In this paper, we study these two strategies in the presence of correlations and fees. We show that using periodic and partial rebalance strategies, it is possible to maintain a steady exponential growth while minimizing the losses due to fees. We also demonstrate how these redistribution strategies perform in a phenomenal way on real-world market data, despite the fact that not all assumptions of the model hold in these real-world systems. Our results have important implications for stochastic dynamics in general and to portfolio management in particular, as we show that there is a superior alternative to the common buy-and-hold strategy, even in the presence of correlations and fees.

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I. INTRODUCTION

Multiplicative stochastic processes are commonly used to model assets prices as these are strongly affected by *relative* fluctuations. The most popular framework used in finance to model stock prices is GBM. It is used in the classic Black-Scholes model [1] as well as in many other models [2,3]. It was noted that in random multiplicative environments the median represents the typical path much better than the mean. Hence, it is not recommended to simply maximize the expectation of the wealth [4] but rather to maximize the expectation of the log of the wealth [5]. In fact, utilizing the log space of wealth can be traced back to Daniel Bernoulli [6]. Inspired by the St. Petersburg paradox, he invented the log utility function in 1708 [7], which reflects a declining marginal utility of the wealth, and later also led to the concept of risk aversion [8]. In gambling and repeated investments, Kelly was the first to suggest the usage of logarithmic utility [9,10]. Finite sum of multiplicative stochastic processes is not ergodic [11], which implies that the time average is not the same as the ensemble average. There is a growing interest in stochastic ergodicity breaking in a wide range of physical systems [12,13].

The nonergodicity of this process implies that systems that are naïvely expected to flourish (arithmetic mean larger than 1), in reality may be doomed to extinction (geometric mean lower than 1). A possible workaround for this unfortunate outcome can be achieved by diversification and cooperation [14,15]. The dominance of the geometric mean for the typical path in multiplicative processes was demonstrated by proving that the price trajectory behaves almost surely as [16]

$$\lim S(t)(t \to \infty) = \begin{cases} \infty, & \mathrm{E}\{\ln[S(1)]\} > 0 \\ \text{oscillation}, & \mathrm{E}\{\ln[S(1)]\} = 0 \\ 0, & \mathrm{E}\{\ln[S(1)]\} < 0. \end{cases} \quad (1)$$

Also, for assets with price dynamics that follow GBM, the price will be nowhere near its expected value for large times [17]. The noise fluctuations in GBM process have a negative effect on the growth, as can be seen in the Ito correction term [18]. This correction term is of σ^2 magnitude and reflects the difference between the arithmetic mean and the geometric mean. A useful way to reduce the noise is by diversification in which N GBM processes are summed together. The sum of GBM processes is not ergodic, thus there is a difference between the time average, $\lim t \to \infty$, and the ensemble average, $\lim N \to \infty$, where the ensemble average > time average. In reality, the interesting dynamics is when N and t are finite. One of the results derived from the nonergodicity of the process is that for small t the growth is close to the ensemble average but decreases over time toward the time average as described in Ref. [11]. By infinite diversification (i.e., $\lim N \to \infty$), the stochastic noise is removed and the ensemble average is achieved. One way to recover from the nonergodicity without infinite diversification and to gain a steady growth over time is by keeping constant weight for each GBM in the weighted sum as described in Ref. [19]. The constant rebalanced portfolios are also called balanced portfolios. Balanced portfolios should not be confused with buy-and-hold (passive) portfolios in which the number of shares held in each asset are kept fixed, hence when the wealth changes, wealth fractions also change. The wealth of a balanced portfolio composed of GBM assets is a log-normal process. It was shown that balanced portfolios, in the absence of transaction costs, have steady expected growth [17]. In practice, the constant rebalance required for a balanced portfolio exposes it to transaction costs. The effects of transaction costs on a Kelly portfolio with one risky asset and one risk-free asset with zero growth were studied in Ref. [20]. It was shown in Ref. [20] that there is an optimal rebalance period that is proportional to $\alpha^{\frac{2}{3}}$ where α is the transaction costs parameter. In this model, transaction costs were calculated as a fraction of the volume traded: for each transfer of wealth W, $\alpha \cdot W$ is paid as fees ($0 < \alpha < 1$). A partial rebalance strategy in which only part of the wealth required for full rebalance is transferred, was studied in Ref. [20] as well, and it was shown that this strategy outperforms the periodic rebalance strategy. Different return distributions, including binary and log normal, were considered and exhibited similar results [20].

Another important strategy is the growth-optimal portfolio pioneered by J. Kelly in 1956 [9]. Although Kelly formulated his model in terms of gambling and repeated investments, the results are applicable to portfolio management as well [21]. Kelly portfolio is a portfolio in which the optimal asset allocation is determined by maximizing the expected value of the *log of the wealth* and under the assumption that the assets properties do not change over time, the wealth fraction of each asset should be kept fixed (i.e., balanced portfolio). Kelly portfolios have many important mathematical properties. For example, the Kelly optimal portfolio lies on the Markowitz efficient frontier for assets with log-normal return distribution, small means, and small variances [10]. The applicability of the Kelly portfolio in real financial markets was shown in Ref. [21]. In Ref. [22], the authors extended the concept of balanced portfolios and presented an algorithm that asymptotically performs as well as the best balanced portfolio in hindsight. Although the Kelly and the balanced portfolios were designed to maximize the expected growth, many investors are also interested in minimizing the risk. The trade-off between growth and risk in Kelly portfolios was studied and resulted in the fractional Kelly approach, which allows the investors to choose their desired growth-risk trade-off [23]. Another source of risk in the Kelly portfolio is that in real life the "game" probabilities or the return distribution are estimated and are subject to random noise. Thus, choosing the right fractions can be complicated and one can choose fractions that can lead to losses and even ruin in some cases. In a setup that includes cooperation of M players who share their wealth and redistribute it at each time step in a random multiplicative setup, it was shown that a cooperative strategy can reduce the risk significantly [14]. More generally, the concept of sharing resources to reduce risk and increase growth has been shown also in a broader economic context [24].

The analysis of the complex noise dynamics in stochastic processes in general and in financial markets in particular is subject to a lot of researches in statistical physics and econophysics [25,26]. In this work we investigate the noise dynamics of the sum of stochastic processes viewed as a financial portfolio. Specifically we are investigating the dynamics of such portfolios in the presence of correlation, rebalancing, and transaction costs. We show that even with minimal rebalancing it is possible to stay closer to the ensemble average and to avoid the degradation toward the time average. We also show that these partial rebalancing strategies have a practical use in the presence of transaction costs and in real market trading.

A. Modeling scheme

The basic model we consider is a portfolio, composed of $N \geqslant 1$ asset(s). The price of asset i (s_i where $i \in \{1,\dots,N\}$) follows GBM and can be defined by a stochastic differential equation (SDE):

$$ds_i = s_i[u_i dt + \sigma_i dB_i(t)], \tag{2}$$

where $B_i(t)$ is a standard Brownian motion, u_i is the drift, and σ_i is the standard deviation. The solution of this SDE yields the price for each asset at any time $t > 0$:

$$s_i(t) = s_i(0)e^{[(u_i-\frac{\sigma_i^2}{2})t+\sigma_i B_i(t)]}. \tag{3}$$

Without loss of generality we can assume that $\forall i, s_i(0) = 1$. For GBM the arithmetic mean is $E[s_i(t)] = e^{u_i t}$, while the variance is $\text{Var}[s_i(t)] = e^{2u_i t}(e^{\sigma_i^2 t} - 1)$, and the geometric mean is $GM[s_i(t)] = e^{t(u_i-\frac{\sigma_i^2}{2})}$. The wealth held in each asset is defined by

$$W_i(t) = s_i(t)q_i(t), \tag{4}$$

where $q_i(t)$ is the number of shares held by asset i at time t. The wealth of the portfolio is the sum of the wealth of all the assets and is defined by

$$W(t) = \sum_{i=1}^{N} q_i(t)s_i(t). \tag{5}$$

As noted in the introduction, the expected value of a GBM random variable does not represent well its long-term behavior. Thus, we define $g_p(t)$, the exponential growth rate of the portfolio:

$$g_p(t) = \frac{\ln\left[\frac{W(t)}{W(0)}\right]}{t}. \tag{6}$$

Without loss of generality, we can assume that $W(0) = 1$. Thus, Eq. (6) reduces to

$$g_p(t) = \frac{\ln[W(t)]}{t}. \tag{7}$$

We are primarily interested in the expectation and the variance of the growth ($E[g_p(t)]$ and $\text{Var}[g_p(t)]$, respectively). Note that the expected growth is achieved by taking the natural logarithm, ln, of the geometric average. In addition to the growth measures, many investors are also interested in minimizing the risk. We use the following risk measure, proposed in Ref. [23]:

$$P_r(T,l) = P[W(T) \leqslant W(0)(1-l)]. \tag{8}$$

This measure represents the probability of losing at least a fraction l of the initial wealth at time T. The expected growth of each asset is:

$$\frac{E\{\ln[s_i(t)]\}}{t} = u_i - \frac{\sigma_i^2}{2}. \tag{9}$$

We mark the expected growth of each asset as $E[g_i(t)]$ and the expected growth of the portfolio as $E[g_p(t)]$. Figure 1 shows $\ln[W(t)]$ for typical paths of both single asset [Fig. 1(a)] and $N = 2$ assets [Fig. 1(b)]. It can be seen that although for each asset $E\{\ln[W(t)]\} = 0$, combining the two assets to a portfolio yields a positive $E\{\ln[W(t)]\}$. This demonstrates the advantages of diversification. Also, as expected, in the portfolio with $N = 2$ assets, $\ln[W(t)]$ has a lower variance.

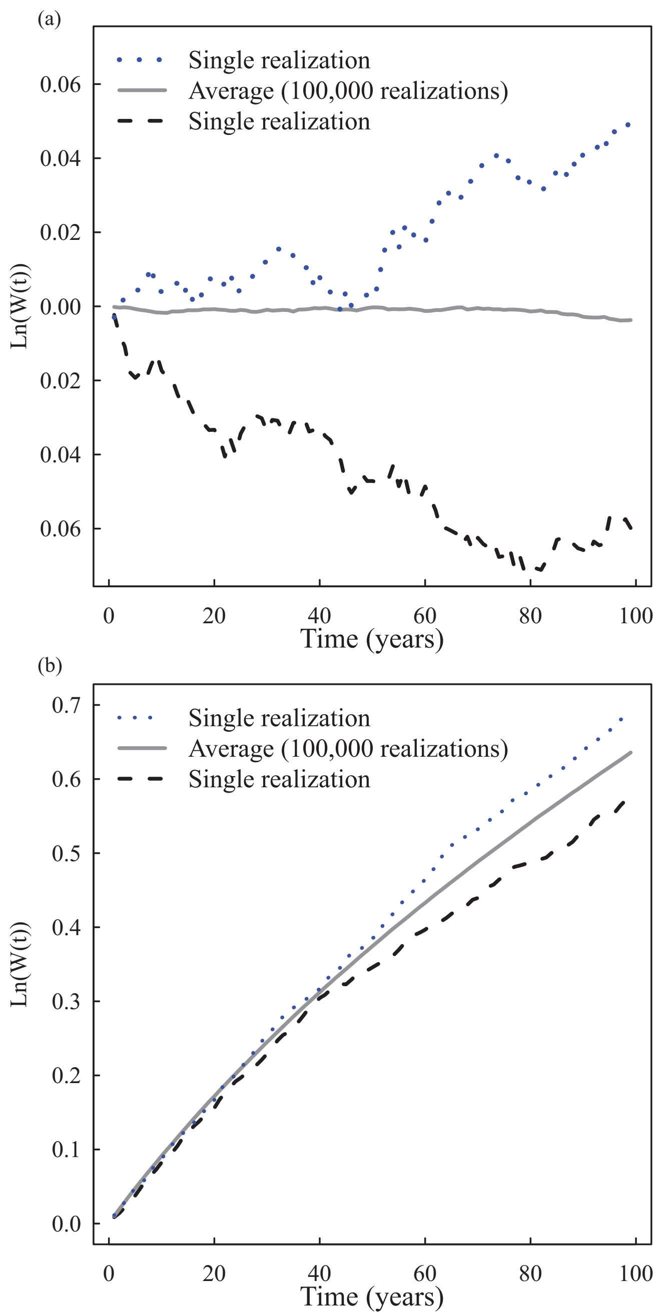

FIG. 1. Representative trajectories of the logarithm of the wealth for single and multiple assets. Time dynamics of the logarithm of the wealth for passive portfolios of $N = 1$ (a) and $N = 2$ (b) assets are shown. Solid gray lines represent the expectation, $E\{\ln[W(t)]\}$, over 100 000 random realizations, while blue and black dashed lines represent single random realizations. Wealth dynamics of each asset was simulated with the following parameters $u = 0.02$ and $\sigma = 0.2$ (see methods for more details about the simulations).

The rest of the paper is organized as follows: in the first two sections of the results chapter we study and compare two strategies for portfolio management. The first is a balanced portfolio in which the fractions invested in each asset are kept fixed and the second is a passive portfolio in which the quantities invested in each asset are kept fixed. In Sec. II A, we study the dynamics of balanced portfolios and analyze its properties without transaction costs, but for correlated and uncorrelated assets. In Sec. II C we study the dynamics of passive portfolios for short and long times. In the Sec. II D 1 we study the periodic rebalance strategy, where the portfolio is rebalanced every τ days instead of continuously. In Sec. II D 2 we study the partial rebalance strategy, in which the fixed fraction approach is relaxed and only part of the required wealth for full rebalance is being transferred from one asset to another. In Sec. II E we apply the above redistribution strategies to real world data, and in the last chapter we conclude.

II. RESULTS

A. Theoretical analysis of a balanced portfolio

In this section, we analyze the mathematical properties of a balanced portfolio. We derive analytic expressions for the expected growth, the variance of the growth, and the risk measure defined above [Eq. (8)]. We start by analyzing the behavior of balanced portfolios for N correlated GBM assets. We define u_i,σ_i to be the drift and the standard deviation of asset i, σ_{ij} to be the covariance between assets i and j, and f_i to be the wealth fraction invested in asset i.

Proposition 1: The wealth of a balanced portfolio is a log-normal process and defined by the following SDE:

$$ds_p = s_p[u_p dt + \sigma_p dB(t)], \tag{10}$$

where

$$u_p = \sum_{i=1}^{N} u_i f_i \tag{11}$$

and

$$\sigma_p = \sqrt{\sum_{i=1}^{N}\sum_{j}^{N} f_i f_j \sigma_{ij}}. \tag{12}$$

The expected growth of balanced portfolio is given by

$$E[g_p(t)] = \sum_{i=1}^{N} u_i f_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} f_i f_j \sigma_{ij}. \tag{13}$$

The variance of the growth of balanced portfolio is given by

$$\text{Var}[g_p(t)] = \frac{1}{t}\sum_{i=1}^{N}\sum_{j}^{N} f_i f_j \sigma_{ij}. \tag{14}$$

The risk measure defined by Eq. (8) is given by

$$P[W(t) \leqslant W(0)(1-l)]$$
$$= \frac{1}{2}\left(1 + \text{erf}\left\{\frac{\ln[W(0)(1-l)] - t\left(u_p - \frac{\sigma_p^2}{2}\right)}{\sigma_p\sqrt{2t}}\right\}\right). \tag{15}$$

Equation (13) shows that balanced portfolio has a steady exponential growth over time. The proof can be found in Appendix A.

B. The effect of correlations on balanced portfolios

Portfolio diversification is thought to lower the risk and increase the growth. In reality, however, assets are correlated, thus the effect of diversification is lessened. Correlation between assets can result from various reasons. For instance, stocks of companies from the same industry are often highly correlated because they are influenced by the same economic factors. Also stocks of companies from different industries and markets show significant correlations [27].

In this section, we study the effect of diversification on a given balanced portfolio of N correlated assets. To focus on the effects of correlation, we simplify the model so that the price dynamics of all assets share the same properties (i.e., $\forall i \; u_i = u$ and $\sigma_i = \sigma$), and we do not consider yet transaction costs. For a given balanced portfolio of N identically distributed correlated assets, we find the size (N_{eff}) of an equivalent portfolio with the same expected growth, composed of N_{eff} *independent identically distributed* (i.i.d.) assets with the same u and σ. In other words: a balanced portfolio with N correlated assets is equivalent, in terms of the expected growth and its variance, to a balanced portfolio with N_{eff} uncorrelated assets.

Proposition 2:

$$N_{\text{eff}} = \frac{N^2}{N + \sum_{i=1}^{N} \sum_{j \neq i}^{N} \rho_{ij}}, \tag{16}$$

where ρ_{ij} is the Pearson correlation coefficient between assets i,j.

In the simple case of uniform correlation coefficient, where $\rho_{ij} = \rho \; \forall i \neq j$, we get

$$N_{\text{eff}} = \frac{N}{1 + (N-1)\rho}. \tag{17}$$

If $\rho < 0$, then N_{eff} is given by

$$N_{\text{eff}} = \begin{cases} \text{undefined}, & \text{N} > 1 - \frac{1}{\rho}, \\ \infty, & \text{N} = 1 - \frac{1}{\rho}, \\ \frac{N}{1+(N-1)\rho}, & \text{N} < 1 - \frac{1}{\rho}. \end{cases} \tag{18}$$

The proof and an illustration of the dramatic effect of the correlation can be found in the Appendix B

C. Passive portfolio dynamics

As can be seen from Eq. (9), the expected growth of a GBM asset is reduced due to the noise (standard deviation) term. Diversification is often used to lower the investment risk by reducing the standard deviation. The simplest approach of a diversified portfolio is a passive portfolio in which the investor buys stocks and holds them, keeping the number of shares held in each asset constant. In this case the portfolio's wealth is a stochastic process, which is a finite weighted sum of log-normal processes. The wealth dynamics is defined by

$$W(t) = \sum_{i=1}^{N} q_i s_i(t), \tag{19}$$

where q_i are constants. A finite sum of log-normal processes is not a log-normal process and does not have a known return distribution. Unlike a balanced portfolio, the expected growth of a passive portfolio declines over time. For $t \to 0$ it has the same expected growth as a balanced portfolio, but it declines over time toward the expected growth of a single asset and the effect of diversification diminishes. The dynamics of a weighted average of N i.i.d. GBM processes, which can be viewed as a passive portfolio, was also studied in Ref. [11]. The authors proved that the long-time average growth is

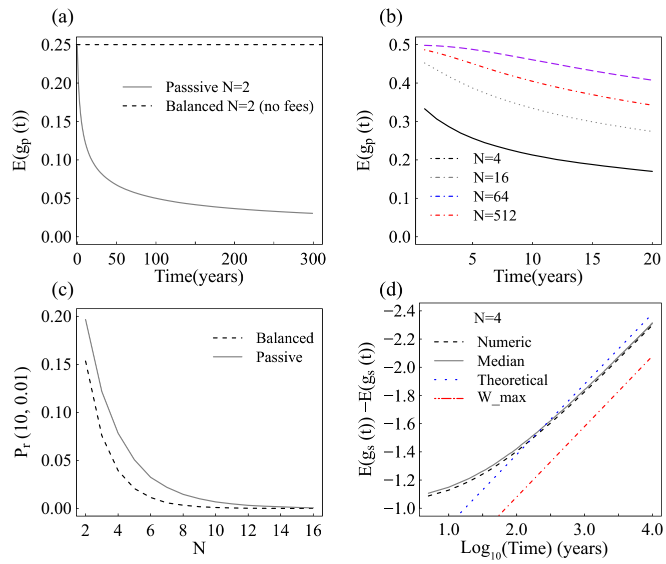

FIG. 2. Expected growth of a passive portfolio over time. The parameters used in the simulations shown in (a) and (b) are $u = 0.5$ and $\sigma = 1$ for all assets. (a) shows the results of a portfolio with $N = 2$ i.i.d. simulated assets. The gray solid line represent the passive portfolio's growth, while the black dashed line represents the balanced portfolio's growth. (b) The effect of N on the expected growth. (c) The risk measure as function of N is presented. The parameters used in (c) are $T = 10, \sigma = 0.4$, and $u = 0.13$. (d) The expected growth of a passive portfolio over time in log scale is shown. The blue doted line represents the results that follow from Eq. (24). The parameters used in this simulation for all setups are $N = 4$ and $\sigma = 0.9$

$$\lim_{t \to \infty} \frac{E\{\ln[W(t)]\}}{t} = u - \frac{\sigma^2}{2}, \tag{20}$$

where u and σ are the drift and the standard deviation of the assets, respectively. This proves that in the long run, a passive portfolio of N assets has the same expected growth as that of a single asset. They also proved that the ensemble average growth is

$$\lim_{N \to \infty} \frac{E\{\ln[W(t)]\}}{t} = u. \tag{21}$$

The maximal expected growth of the passive portfolio is the expected growth of a balanced portfolio, which for a finite N is smaller than the arithmetic mean. Increasing the number of assets, increases the time elapsed until a passive portfolio and a balanced portfolio have similar expected growth.

Figure 2(a) shows the expected growth over time for a simulation of a passive portfolio of $N = 2$ i.i.d. assets. Each asset has an expected growth of $u - \frac{\sigma^2}{2} = 0$, but as can be seen, due to diversification, the portfolio has a positive expected growth that declines over time. Increasing the number of assets, increases the expected growth of the portfolio for short times and also leads to slower decline of the growth towards the single asset value [Fig. 2(b)].

In addition to a higher expected growth, a balanced portfolio also presents a lower risk relative to a passive portfolio. The risk measure [Eq. (8)] as a function of N is shown in Fig. 2(c). It shows that a balanced portfolio has lower probability to lose a fraction of its initial wealth. Increasing the number of assets, decreases the risk in both setups.

The problem of finding an analytic expression to the expected growth of a passive portfolio is not trivial. One approach was taken in Ref. [28], where it was proven that for $t >> \frac{\ln(N)}{\sigma^2}$ a passive portfolio of i.i.d. assets with expected growth of 0 (i.e., $u = \frac{\sigma^2}{2}$), will be dominated by the asset with the maximal value. Then, the median, which approximates the exponent of the expected growth, can be found using extremal statistic theory [29], leading to the results

$$\text{Median}[W(t)] \approx e^{\sqrt{2\sigma^2 t \ln(N)}}, \tag{22}$$

and in terms of expected growth,

$$E[g_p(t)] = \sqrt{\frac{2\sigma^2 \ln(N)}{t}}. \tag{23}$$

Combining theoretical analysis and numerical approximation, we suggest a different approach to this problem, by showing the relation between a passive and a balanced portfolio. The wealth fractions of a passive portfolio change whenever the assets values change. We define $f_i(t)$ to be the wealth fraction invested in asset i at time t. The main idea behind this approach is to express the expected growth of a passive portfolio as an integral over time of the expected growths of consecutive balanced portfolios, leading to Proposition 3 (the proof can be found in Appendix C) the long-time expected growth of a passive portfolio of N i.i.d. assets is

$$\begin{aligned} E[g_p(t)] &= \frac{1}{t}\int_0^t \left[u - \frac{\sigma^2}{2}\sum_{i=1}^{N} f_i(\tau)^2\right] d\tau \\ &= u - \frac{\sigma^2}{2} + \sigma^2\sqrt{\frac{\ln(N)}{t\sigma}}. \end{aligned} \tag{24}$$

The last term represents the excess growth of a passive portfolio. In Fig. 2(d), the expected growth of a passive portfolio derived by different methods [numerical simulations, the results achieved by Ref. [28], and those which follow from Eq. (24)] are compared.

D. Balanced portfolio in the presence of transaction costs

To keep the proportion fixed, it is needed to continually rebalance the portfolio whenever the assets values change. The presence of transaction costs can make frequent rebalancing expensive and inefficient as it involves buying and selling assets. There is a tradeoff between the benefits from the rebalancing and the costs of the fees. In this section, we study two rebalance strategies that aim at maximizing the expected growth of a balanced portfolio in the presence of transaction costs. We use a volume model for the transaction cost, in which for each transferred wealth W the costs are $\alpha \cdot W$ $(0 < \alpha < 1)$. We will also use the notation Fees in basis points (BPS). BPS is a common unit of measure for a percentages in finance. One basis point is equal to 10^{-4} and $\alpha = (\text{Fees} \cdot 10^{-4})$.

1. Periodic rebalance strategy

Instead of continually rebalancing the portfolio, this strategy applies a full rebalance periodically each τ time units. In between the rebalances, the portfolio behaves like a passive one. If $\tau \to \infty$ the portfolio is passive and if $\tau \to 0$ the portfolio is balanced. Short τ implies frequent rebalances, thus many buy and sell operations occur, which might lead to decline in the expected growth due to the transaction costs. Long τ implies longer periods with passive behavior, which can lead to an expected growth degradation as demonstrated in passive portfolios (see Sec. II C). For intermediate periods however, the passive portfolio growth degradation is relatively small. In other words, rebalance can be made less frequently without degrading the growth dramatically, yet benefiting from the reduction in the transaction costs. The optimal rebalance period is sensitive to the transaction costs parameter, Fees. To study this sensitivity we conducted simulations and measured the expected growth achieved using the optimal rebalance period, each time with different Fees. Increasing Fees increases the optimal rebalance period. Another interesting effect is that increasing Fees makes the growth less sensitive to the rebalance period (i.e., $\frac{\partial E[g_p(t)]}{\partial \tau}$ is smaller if the transaction cost is higher). The effect of Fees on the expected growth and the optimal rebalance period is shown in Fig. 3. Both the maximal growth and the minimal optimal rebalance period were obtained for the case with the smallest Fees as can be seen in Figs. 3(a) and 3(b). In the case with the highest transaction costs, the expected growth is less sensitive to the rebalance period and there is a wide range of rebalance periods that yield a comparable growth to the optimal expected growth. In Ref. [20], the authors found that for a Kelly portfolio with two assets (one risky asset and cash) the optimal rebalance period is proportional to $\sim\alpha^{\frac{2}{3}}$, where $\alpha = \text{Fees} \cdot 10^{-4}$. In the current setup, for a portfolio with $N = 2$ GBM assets, we obtained similar results by running a simulation and finding the optimal τ for each Fees and then fitting a linear regression line to the log scaled results. The 95% confidence interval of the fitted exponent lied in the range 0.66–0.77. The optimal rebalance period is also sensitive to the number of assets. Increasing the number of assets increases the time in which a passive portfolio behaves like a balanced portfolio. This can suggest that it is possible to increase the rebalance period without degrading the expected growth significantly and benefit from the reduction in the transaction costs. Indeed, in the presence of transaction costs, the optimal rebalance period is an increasing function of the number of assets. Increasing the number of assets also makes the expected growth less sensitive to the rebalance period (i.e., $\frac{\partial E[g_p(t)]}{\partial \tau}$ is smaller when the number of assets is higher). Comparing Figs. 3(a) and 3(b) shows the effect of N on the growth and on the optimal rebalance period: the larger the N, the larger the optimal τ is, and the expected growth is less sensitive to τ. The reason behind this is that large N is less sensitive to the growth degradation of passive portfolio and thus increasing the passive periods can be beneficial in the presence of fees.

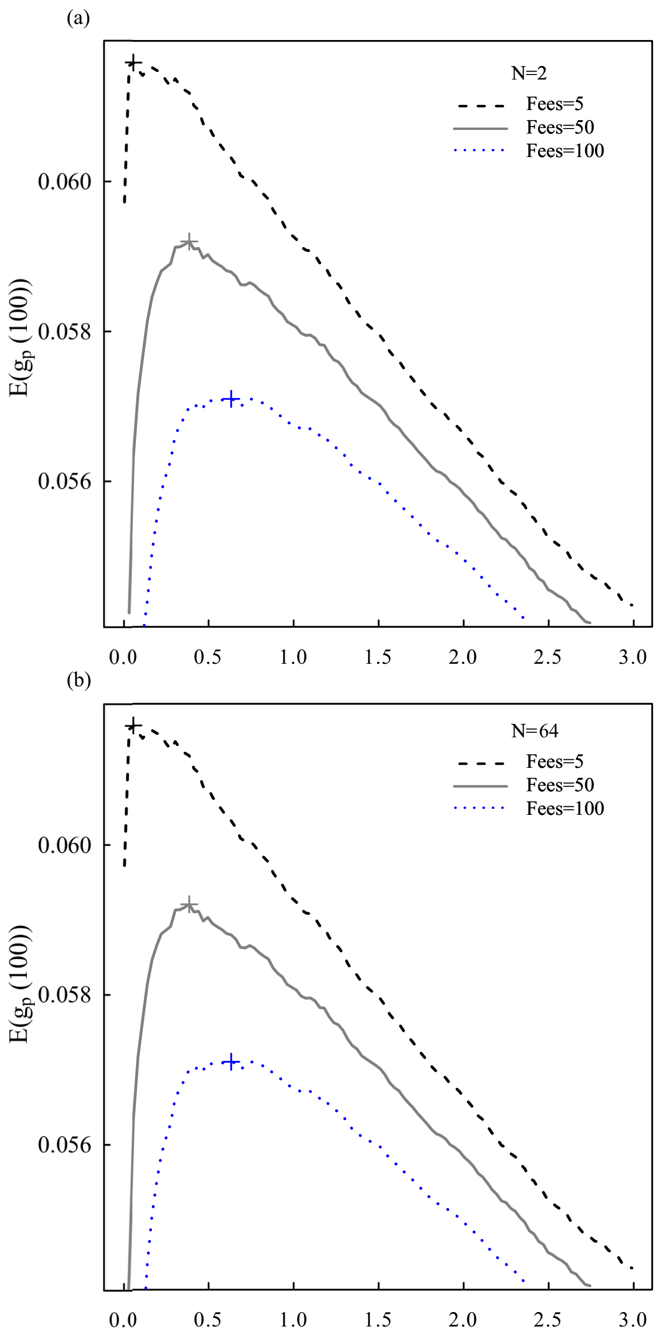

FIG. 3. Expected growth as a function of τ for different Fees. The parameters used for all simulations are $u = 0.125$, $\sigma = 0.5$, $T = 100$, and $N = 2$ (a) or $N = 64$ (b).

2. *Partial rebalance strategy*

Partial rebalance strategy refers to cases in which only a fraction D $(0 < D < 1)$ of the required capital for full rebalancing is transferred between the assets. This implies that during the rebalance step, the wealth invested in asset i at time t, $W_i(t)$, is updated to be

$$W_i^{\text{rebalanced}}(t) = W_i(t) - D[W_i(t) - W(t) f_i]. \qquad (25)$$

The transaction cost for this rebalance operation is thus $\alpha \cdot D \cdot |W_i(t) - W(t) f_i|$. $D = 1$ implies full rebalance, and $D = 0$ implies passive portfolio. If $\tau > 0$ and $D = 1$, the partial strategy is equivalent to the periodic strategy. The idea behind this strategy is that to reduce transaction costs, rebalancing should occur more frequently but smaller amounts should be transferred at each rebalance compared to the periodic rebalance strategy. In partial rebalancing, the portfolio is no longer balanced. The dynamics of the fractions can be formulated as follows:

$$W_i(t + dt) = W_i^{\text{rebalanced}}(t) R_i(t, t + dt), \qquad (26)$$

where $R_i(t, t + dt)$ is the return of asset i between the times t and $t + dt$, and f_i is the desired optimal fraction of asset i. If the fractions are time-varying, the fraction of asset i at time $t + dt$ is

$$f_i(t + dt) = \frac{W_i^{\text{rebalanced}}(t + dt)}{W(t + dt)}. \qquad (27)$$

Thus, after rebalance, using Eq. (25) for $t + dt$, we get

$$f_i(t + dt) = \frac{f_i(t) R_i(t, t + dt)}{\sum_{i=1}^{N} f_i(t) R_i(t, t + dt)} (1 - D) + D f_i. \qquad (28)$$

Although we have not found an analytical solution to this equation, several observations can be made. $f_i(t + dt)$ is a weighted average of the original optimal fraction f_i and $f_i(t)$. If $D = 1$ (full rebalance) the fractions remain constant. If $D = 0$ (passive portfolio), the fractions dynamics is subject to the noise of the returns. Increasing D reduces the variance of the fraction as it adds more weight to the constant desired fraction. If the fractions are time-varying, the dynamics of the total wealth is

$$W(t + dt) = W(t) \sum_{i=1}^{N} f_i(t) R_i(t), \qquad (29)$$

and the variance of the fractions has a negative effect on the expected growth, as it increases the variance of the wealth. Despite the fact that for partial rebalance the portfolio has no longer fixed fractions, we can still ask how far the fractions are from their optimal fixed values. It turns out that even for a very small D the variance of the fraction distribution is surprisingly small. Figure 4(a) shows that while the standard deviation of the fractions of the passive portfolio increases over time, the standard deviation of the fractions using partial rebalancing is smaller and approaches a constant value. This phenomenon is crucial for the understanding of the advantages of partial rebalancing over no rebalance at all. In Refs. [30] and [31], it was shown that for very large N and for any $D > 0$, the wealth fractions follow a power law distribution. Figure 4(c) shows that the partial strategy overcomes the period rebalancing for every chosen rebalance period. D_{opt} changes as a function of the rebalance period, τ, but for every τ there is a partial strategy with $D_{\text{opt}} < 1$ that overcomes the periodic rebalance. Figure 4(b) shows that the optimal full rebalance using periodic rebalance strategy (i.e., $D = 1$) is obtained by $\tau = 0.63$ years, represented by the blue dotted line, but the optimal result are achieved using partial rebalancing with $\tau = 0.1$, $D_{\text{opt}} = 0.2$ represented by the black dashed line. Increasing the fees reduces D_{opt} as can be seen in Fig. 4(d). Comparing Fig. 4(b)

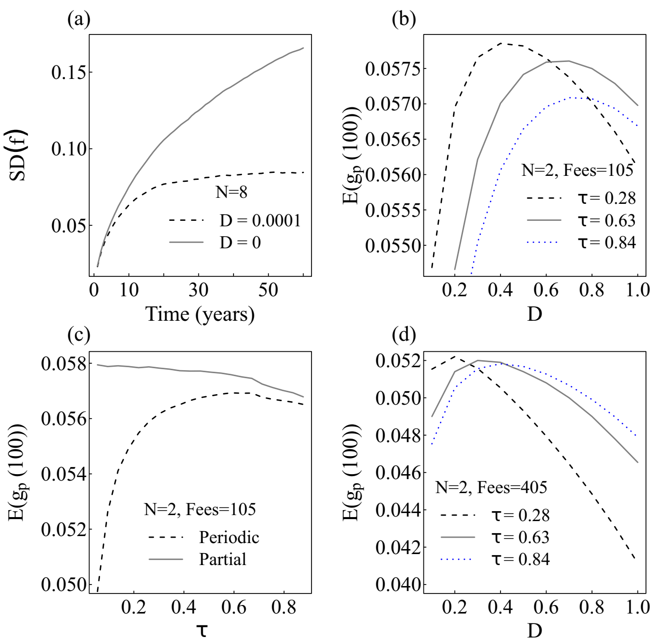

FIG. 4. Periodic and partial rebalancing strategies. Standard deviation of the wealth fractions for partially rebalanced and passive portfolios is shown in (a). Expected growth over time as a function of D for different rebalance periods is shown in (b) (Fees = 105) and (d) (Fees = 405). Comparison of the partial and periodic strategies is shown in (c). The parameters used in (a) are $u = 0.03$ and $\sigma = 0.2$, and the parameters used in (b–d) are $u = 0.125$, $\sigma = 0.5$, and $T = 100$.

to 4(d) one realizes that, as expected, the optimal full rebalance period is longer if the fees are larger as can be seen in the blue line of Fig. 4(b) but for all rebalance periods D_{opt} is smaller. Increasing the number of assets also decreases D_{opt}.

The optimal partial rebalance parameter (D_{opt}) is an increasing function of the rebalance period. Naïvely, one could expect that as long as the ratio $\frac{D}{\tau}$ remains the same, taking different values for D and τ would yield similar growth. This is because in each case similar amounts are eventually transferred, however, as demonstrated in Fig. 4(a), the optimal results are obtained for more frequent (small τ) and weaker (small D_{opt}) rebalance events.

Figure 5 shows the relationships between D_{opt} and τ_{opt}, where τ_{opt} is the optimal rebalance found using periodic strategy and D_{opt} is the optimal parameter found using partial strategy. In Fig. 5(a), D_{opt} and τ are found for different fees. It can be seen from the inset in Fig. 5(a) that $\tau_{\text{opt}} \sim \frac{1}{D_{\text{opt}}}$. Figure 5(b) shows that the partial strategy outperforms the periodic strategy. It also shows that the gap between the two strategies increases as the fees increase.

Figure 6 summarizes the results for all methods in a GBM market. The red line represents the steady growth of a balanced portfolio without transaction costs, the black line represents the declining growth of passive portfolio, the purple line represents a continuously balanced portfolio with transaction costs, the gray line represents the results achieved by the best partial rebalance strategy, and the blue line represents the results achieved by the best periodic rebalance strategy. It

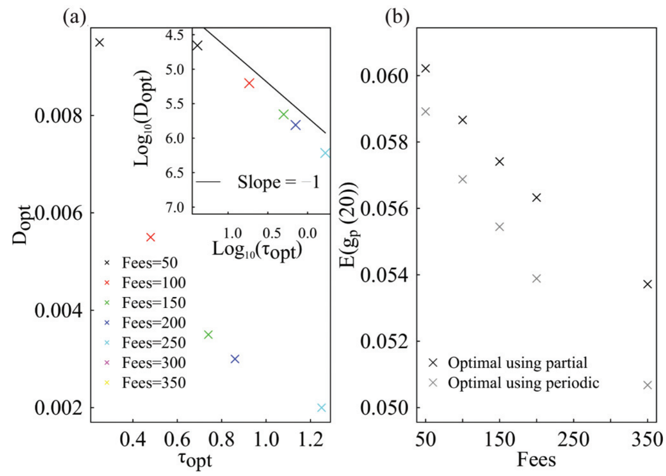

FIG. 5. Optimal values of the parameters and the expected growth for periodic and partial rebalance strategies. Optimal D and optimal τ for different fees is shown in (a). Optimal expected growth as function of fees for periodic and partial rebalance strategies is shown in (b). The inset of (a) shows the results on a log scale. The parameters used for this simulation are $u = 0.125$, $\sigma = 0.5$, $N = 2$, $T = 20$.

is evident that using a partial rebalance strategy, transaction costs are reduced dramatically, and the growth approaches the performance of a balanced portfolio without transaction costs.

E. Real market experiments

Using mathematical models such as GBM raises the question of whether these models reflect well the real world and whether the theoretical results remain valid in real stock markets. To answer these questions we tested the above

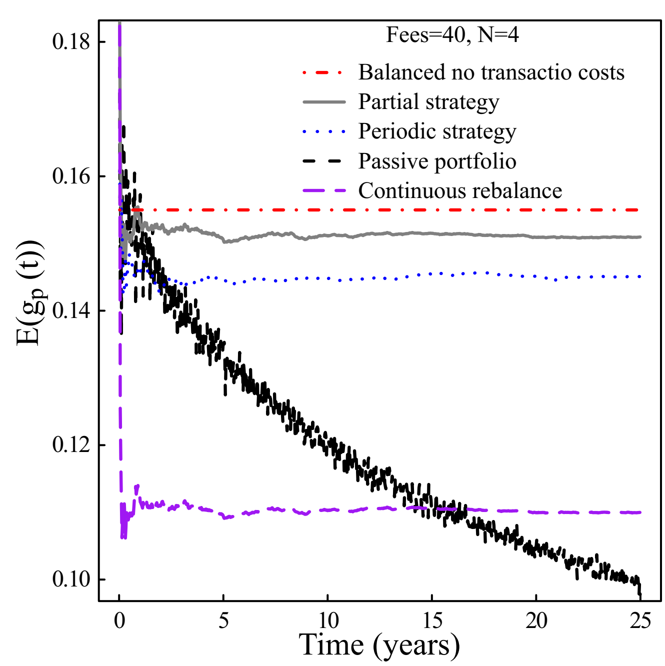

FIG. 6. Comparison of all methods for portfolio of uncorrelated assets. The parameters used for all simulations are $u = 0.2$, $\sigma = 0.6$, $N = 4$, fees = 40.

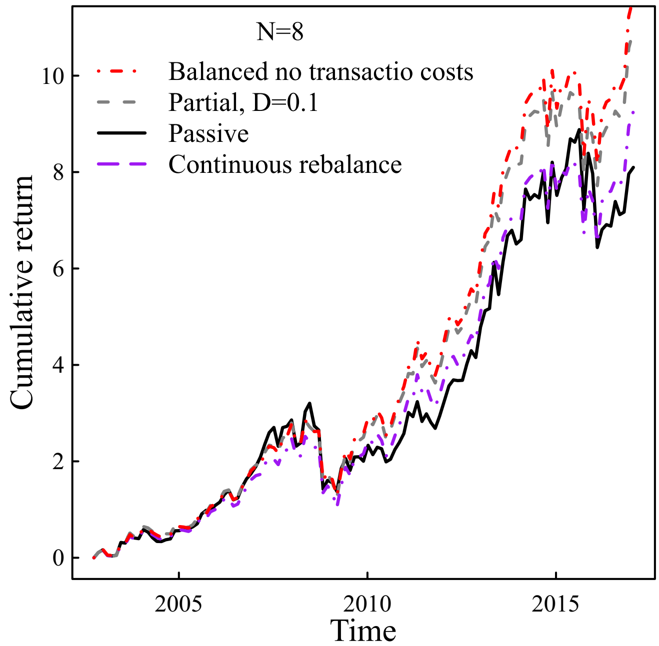

FIG. 7. Comparison between rebalancing strategies on a typical portfolio, for the period of 2002–2017. Stocks symbols that are part of this portfolio are: ADP, DE, EMN, JNJ, NBL, PWR, REGN, SCHW. Graph shows the cumulative return for different strategies. The best performance is achieved by continuous rebalance when no fees were considered, represented by the red line. When fees were considered the best strategy was the partial strategy with $D = 0.1$ represented by the gray line.

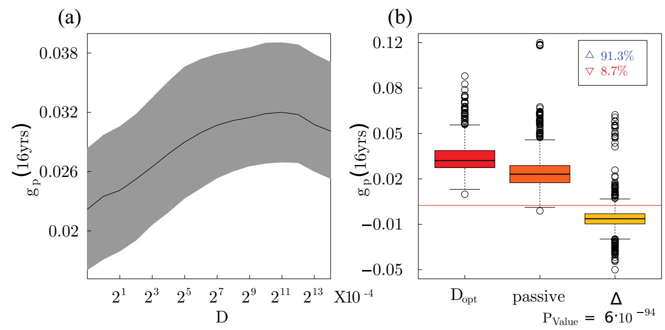

FIG. 8. Comparison between rebalancing strategies on many portfolios, for the period of 2002–2017. (a) Growth distribution of portfolios composed by eight randomly picked stocks. In each realization eight stocks were sampled and their mean growth was calculated for varying values of the partial redistribution parameter (D). Gray shaded area corresponds to 25%–75% of the growth distribution of the 1000 random choices of stocks, while the black line corresponds to the median of each distribution. The X axis is in a logarithmic scale. (b) Boxplots describing the mean growth distribution for $D = 0.1$ (left), $D = 0$ (middle), and the difference between the two (right) are shown. P value indicated at the bottom of the graph corresponds to a paired Mann-Whitney u test. Blue arrow indicates the fraction of realizations (in percentage) that a strategy with a diffusion factor $D = 0.1$ yielded a better result compared with a passive strategy. Red arrow indicates the complementary percentages in which the passive strategy is better.

rebalancing strategies against real market data using realistic transaction costs models. We show that our theoretical and simulated results also hold in the real world and that the rebalance strategies perform better, in a statistically significant way, than buy-and-hold portfolios when transaction costs are taken into account. To illustrate the power of these strategies we tested them in various setups, including randomly chosen assets from different stock markets, varying number of assets, and two transaction costs schemes. In all setups the rebalance strategies gained substantial excess growth on average and beat the passive portfolio in over 90% of the experiments, each experiment with different randomly chosen stocks. The fact that rebalanced strategies outperform buy-and-hold on randomly chosen stocks implies that it is very likely to improve any portfolio regardless of the stocks composing it. For simplicity we assigned a weight of $\frac{1}{N}$ to each stock in the portfolio. Nonequal fraction allocation could potentially lead to even more efficient rebalance results. In addition to the fees per trade model that was used in the theoretical part of the article, in this section we also validated our results, using another very popular transaction costs model, the fees per share model. In this scheme the fees are the maximum between a minimal fee and the number of traded shares times the cost per share (Max(minFees, costPerShare × numberOfTradedShares)). We used minFees = 1.5 dollar and costPerShare = 7 cents. An example of a typical experiment is presented in Fig. 7. In this experiment we ran a portfolio composed of $N = 8$ randomly chosen stocks from S&P 500, NYSE, and NASDAQ for the period of 2002–2017. It can be seen that the partial rebalancing strategy performed better than the buy-and-hold and the continuous rebalance with fees (the purple line) for most of the tested period.

As opposed to the simulation environment, in real markets there are rare extreme cases in which even the continuous rebalance without fees performs worse than the buy-and-hold but in the vast majority of the cases rebalance is beneficial. Figure 8(a) shows the growth rate distribution, as a function of D, of 1000 random choices of 8 stocks from S&P 500 for the period of 2001–2016, and a transaction fee of 0.4%. $D = 0$ represent the buy-and-hold strategy, and it can be seen that this strategy performs worse than other strategies with $D > 0$. Figure 8(b) compares the growth distributions between buy-and-hold ($D = 0$) and the optimal D, which was set here to $D = 0.1$, the right most column shows the difference in growth for each randomly picked set of stocks between the two strategies. It can be noted that partial rebalancing in this setup drives the growth of the portfolio by ∼50%. That is, instead of 0.02, the growth of the partially rebalanced portfolio is 0.03, and it outperforms the passive portfolio by more than 90% of the portfolios. A Mann-Whitney nonparametric u test was used to check if this difference is statistically significant yielding a very strong signal (P value $< 10^{-90}$).

In real life, rational investors also wish to minimize risk, thus every trading strategy should also be evaluated by risk measures in addition to the return and growth. To illustrate that the rebalance strategies also outperform the buy-and-hold strategy in terms of risk, we ran 50 different experiments. On each experiment we chose three assets randomly from NYSE and NASDAQ for the period of 2002–2017 and measured the

Sharpe ratio of each strategy. It turned out that the Sharpe ratio of the partial strategy was 17% larger than the one of the buy-and-hold strategy, making the partial rebalance strategy in real markets also very appealing in terms of risk. It should be noted that due to the incompleteness of the market the prices that were observed in the market will not necessarily be available for trading, due to competition with other traders or other real-life friction phenomenon. Though every real-data experiment is exposed to this problem, to confront this issue we also took into account intentionally very high commission rates that should include the uncertainty one has in real market scenarios. We repeated our live experiment this time, using minFees = 1.5 usd and costPerShare = 12 cents, which is much higher than the common commission rates, and still on average the Sharpe ratio of the partial rebalance strategy was 11% larger than the one of buy-and-hold.

III. METHODS

In this paper we studied management strategies applied to portfolios composed of GBM and real-world assets. Specifically, we studied the portfolio's wealth dynamics and the influence of updating strategies for varying values of the parameters. We combined both theoretical analysis and numerical simulations. For the theoretical analysis we used stochastic process theory and Itô calculus. The theoretical results were verified numerically using the following methodology.

A. Numerical simulation

To simulate portfolio's dynamics, Monte Carlo simulations were written in c-sharp [32] and the results were analyzed in R [33]. For each simulation $M = 100\,000$ realizations were generated. The size of M was determined to yield negligible standard errors compared to the measured quantities, where the standard error is $\frac{\sigma_s}{\sqrt{M}}$ (σ_s is the standard deviation of the sample).

The wealth of a portfolio composed of N assets is defined by

$$W(t) = \sum_{i=1}^{N} q_i(t) s_i(t), \tag{30}$$

where $q_i(t)$ is the number of shares invested in asset i at time t and $s_i(t)$ is the price of asset i at time t. The price of each asset followed a GBM stochastic process, which was simulated by the exact solution to the GBM equation:

$$s_i(t+dt) = s_i(t) e^{(u_i - \frac{\sigma_i^2}{2})dt + \sigma_i \sqrt{dt} z_i}, \tag{31}$$

where u_i and σ_i are constant.

The noise terms, $z_i \sim \mathcal{N}(0,1)$, were simulated using Mersenne Twister pseudo-random number generator (PRNG) [34]. To generate correlated dynamics, given a correlation matrix, N correlated random noise terms were simulated using Cholesky decomposition method [35]. To study the effect of a specific parameter, simulations were repeated using the same random seed with varying values of the parameter. The parameters that were changed during the simulations are

(1) u_i—the drift of asset i.

(2) σ_i—the standard deviation of asset i. The values of u_i and σ_i^2 were set to reflect real stocks and ranged between 0 and 1 (in 1/year units). In several experiments u and σ were chosen to reflect a negative growth of the asset (i.e., $u - \frac{\sigma^2}{2} \leqslant 0$) to illustrate the power of the diversification and to emphasize the difference between the arithmetic and the geometric mean.

(3) N—the number of assets. Finite portfolio sizes were considered with values ranging from 1 to 1024

(4) Fees—transaction costs in BPS. A wide range was tested from 0 to 405 BPS.

(5) α—transaction costs coefficient ($\alpha = \text{Fees} \cdot 10^{-4}$)

(6) T—the time horizon. Short and long time horizons were considered ranging from 0 to 1000 years.

(7) τ—the rebalance period. A wide range of rebalance periods was simulated staring from $\tau = \frac{1}{24}$ days, that was used to simulate continues rebalancing, to $\tau = 1$ year that was used to simulate infrequent rebalancing.

(8) D—the partial parameter. Values ranging from 0 to 1 were simulated where $D = 1$ used to simulate full rebalance and $D = 0$ used to simulate passive portfolio.

(9) l—the loss percentage for the risk Eq. (8)

(10) ρ_{ij}—the correlation coefficient between assets i and j.

(11) $W(0)$—the initial wealth. Without loss of generality, for convenience we set to $W(0) = 1$

(12) $s_i(0)$—the initial price of asset i. With out loss of generality, for convenience we set to $W(0) = 1$

Each realization built on N random trajectories corresponding to the prices of the N assets. Data from all realizations were then aggregated every dt and empirical expectations, standard deviations, and the medians for $\ln[W(t)$ and $G(t)]$ were calculated [see Eq. (7)]. The risk measure [Eq. (8)] was calculated by counting the fraction of realizations in which $W(T) \leqslant W(0)(1-l)$.

The simulation of a passive portfolio for one realization was done by calculating the price of all assets between time $t = 0$ and $t = T$, every predefined time step dt, using Eq. (31), and then calculating the portfolio's wealth $W(t)$ using Eq. (30). The simulation of a balanced portfolio for one realization was done by the following steps:

(1) Calculate the price of each asset, $s_i(t)$ using Eq. (31) every dt between time $t = 0$ and $t = T$.

(2) Set $W_i(t) = s_i(t) \cdot q_i(t)$.

(3) Rebalance the portfolio every τ time units. Before each rebalance prices were recalculated.

(a) The rebalance is done by updating the wealth of each asset i

$$W_i^{\text{rebalanced}}(t) = \begin{array}{l} W_i(t) + D[W(t) \cdot f_i(t) - W(t)] \\ -\alpha|D[W(t) \cdot f_i(t) - W(t)]|. \end{array} \tag{32}$$

(b) Update the quantity of each asset i

$$q_i(t) = \frac{W_i^{\text{rebalanced}}(t)}{s_i(t)}. \tag{33}$$

B. Empirical simulation

For the empirical analysis we used the statistical program R and the QUANTMOD package [36]. Daily values of 500 stocks traded in the S&P 500 were downloaded from Yahoo finance.

Stocks that were in trade since 2001 until 2017 were chosen leaving us with a list of 407 stocks. After deducting dividends and updating closing prices we used these data to simulate partial rebalance strategy in the presence of transaction fees. 1000 realizations were made for portfolios with 2, 4, 8, and 16 stocks, varied values of the partial parameter D, and the transaction costs.

IV. DISCUSSION

The assumption of market equilibrium is that all rational investors have the same information, thus it is impossible to "beat the market" and the recommendation is to passively hold a combination of the chosen portfolio and a risk-free asset. The mathematical properties of a balanced portfolio show that with no transaction costs it is better to hold a balanced portfolio, instead of the passive one. In the presence of transaction costs it is important to examine if it is still possible to use a balanced portfolio and to outperform the passive one. In this paper, we study this question using a realistic setup of correlated GBM assets with transaction costs and tested the results in real markets.

We proved that the wealth dynamics of a balanced portfolio is a log-normal process and derived analytical expressions for the drift and standard deviation of this process. We demonstrated that the diversification effect in passive portfolios degrades over time and that the expected growth declines toward the single asset's value. We also derived an analytical expression for the relation between passive and balanced portfolios. In addition to full rebalance, we have studied two rebalance strategies: periodic and partial. Using a periodic strategy, we demonstrated that it is possible to rebalance the portfolio less frequently and still gain a steady excess growth. We showed that using a partial rebalance strategy, i.e., transferring only part of the required wealth for a full rebalance, it is possible to dramatically reduce the impact of transaction costs. We showed that while the expected growth of a single asset is dominated by the geometric mean ($u - \frac{\sigma^2}{2}$), using diversification it is possible to approach the arithmetic mean, u, and for i.i.d. assets it is possible to achieve expected growth of $u - \frac{\sigma^2}{2N}$. The wealth fractions fluctuations (standard deviation of the fractions) is an additional source of noise that has a negative influence on the long term expected growth over time. While the standard deviation of the fractions in a passive portfolio are increasing over time, using partial strategy it is possible to reduce the standard deviation of these fractions to a steady level. This property makes the partial strategy superior, as it enables to keep the advantages of a balanced portfolio while reducing the transaction costs. Figure 6 summarizes the results for all methods in a GBM market.

While this paper focuses on the updating strategies part of portfolio management, we wish to refer also to the optimal asset allocation part. The problem of optimal asset allocation was addressed in the pioneering work of Markowitz [37], for which he was awarded the Nobel prize in 1990. The idea was to minimize the variance of the portfolio given a desired return. Knowing that $W(t)$ is very close to a multiplicative log-normal process it is sensible to maximize its expected growth as suggested in Ref. [9]. Maximizing the expected growth can be easily done with linear optimization by choosing the fractions that maximize equation (13). If the variance-return trade off approach of Markowitz is taken, it is still crucial to add additional constraint and to ensure that $u_p - \frac{\sigma_p^2}{2} > 0$; otherwise, it leads to a certain loss as shown in the introduction.

Indeed, it was shown that GBM does not necessary reflect real market's properties such as, fat-tail return distribution [38,39], volatility clustering, and long-range memory [40]. Despite this fact, we have tested the different rebalance strategies on real data, and to our surprise an optimal partial rebalance strategy has been shown to improve portfolio growth, even with naive fraction allocation, by a factor of ~50% in more than 90% of the cases compared to a buy-and-hold strategy. We have tested rebalancing strategies on a daily data. Before applying our results to real markets one has to carefully assess the parameters of the specific market, and other complications that may occur. For example, in high-frequency trading, D_{opt} is expected to be lower due to short time autocorrelation effects. Liquidity, book details, and discretization of stocks need also to be taken into account.

In the past, several mathematical models were purposed to model stocks to better reflect real financial markets properties. For example, the ARCH model [41], the GARCH model [42], and the Heston model [43]. The latter was shown to be consistent with return probability of the Dow Jones, S&P 500, and NASDAQ [44], as well as with high-frequency return data from DAX [45]. It will be interesting to test our results on these and other models.

More research questions that remain open include: (1) finding an analytical solution to the fractions equations introduced in Sec. II D 2, for small N values with and without transaction costs, (2) studying additional rebalance strategies, and (3) considering different transaction costs models. In real life the assets properties are subject to a substantial noise, thus another crucial topic that is yet to be explored is the consequences of choosing the wrong optimal fractions.

ACKNOWLEDGMENT

We thank Sorin Solomon for his helpful comments.

APPENDIX A

Proof of proposition 1:
The expected growth is given by

$$E[g_p(t)] = \frac{E[\ln(W(t)]}{t}, \tag{A1}$$

and the variance of the growth is

$$\text{Var}[g_p(t)] = \frac{E\{\ln[W(t)]^2\}}{t^2} - E[g_p(t)]^2. \tag{A2}$$

The wealth fraction of each asset is kept fixed and the wealth invested in the ith asset is $f_i \cdot W(t)$, where f_i is the fraction of the portfolio's total wealth invested in asset i and $\sum_{i=1}^N f_i = 1$. At time t we can write

$$W(t) = \sum_{i=1}^{N} W(t) f_i. \tag{A3}$$

After rebalance, the prices change and in time $t + dt$ we get

$$W(t + dt) = W(t)\sum_{i=1}^{N} f_i R_i(t,t + dt), \tag{A4}$$

where $R_i(t,t + dt)$ is the return of asset i between the times t and $t + dt$. Since the price dynamics of each asset follows GBM, we get, using the Mil'shtejn method [46]

$$S(t + dt) = S(t) + S(t)u_i dt + S(t)\sigma_i \sqrt{dt} Z_i + \tfrac{1}{2} S(t)\sigma_i^2((\Delta B_t)^2 - dt), \tag{A5}$$

$$S(t + dt) = S(t) + S(t)u_i dt + S(t)\sigma_i \sqrt{dt} Z_i + S(t)O\left(\tfrac{1}{2}\sigma_i^2 dt Z_i\right), \tag{A6}$$

where $\forall i\ Z_i \sim \mathcal{N}(0,1)$. Thus, if $dt \to 0$, then we can neglect the last term and get

$$R_i(t,t + dt) = \frac{S(t + dt)}{S(t)} = 1 + u_i dt + \sigma_i \sqrt{dt} Z_i, \tag{A7}$$

and therefore

$$W(t + dt) = W(t)\sum_{i=1}^{N} f_i(1 + u_i dt + \sigma_i \sqrt{dt} Z_i). \tag{A8}$$

We define a new random variable

$$N_p = \sum_{i=1}^{N} f_i \sigma_i Z_i, \tag{A9}$$

which is normally distributed as well, with variance

$$\mathrm{Var}(N_p) = \sum_{i=1}^{N}\sum_{j}^{N} f_i f_j \sigma_{ij}, \tag{A10}$$

where $\sigma_{ij} = Cov(\sigma_i Z_i,\sigma_j Z_j)$. Using N_p, we can rewrite Eq. (A8) and get

$$W(t + dt) = W(t)\left(1 + \sum_{i=1}^{N} u_i f_i dt + \sigma(N_p)\sqrt{dt} Z_p\right), \tag{A11}$$

where $Z_p \sim \mathcal{N}(0,1)$. Equation (A11) is valid for all t, thus we can conclude that under continuous rebalance the wealth of a balanced portfolio follows a GBM with $u_p = \sum_{i=1}^{N} u_i f_i$ and $\sigma_p = \sigma(N_p)$ because u_p and σ_p are constant. From GBM properties, we get

$$E[g_p(t)] = u_p - \frac{\sigma_p^2}{2} = \sum_{i=1}^{N} u_i f_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} f_i f_j \sigma_{ij}. \tag{A12}$$

The variance of the growth can be calculated by plugging Eq. (A12) in its definition [Eq. (A2)], resulting in

$$\mathrm{Var}[g_p(t)] = \frac{1}{t^2} E\{\ln[W(t)^2]\} - E[g_p(t)]^2 = \frac{1}{t}\sigma_p^2 = \frac{1}{t}\sum_{i=1}^{N}\sum_{j}^{N} f_i f_j \sigma_{ij}. \tag{A13}$$

APPENDIX B

Proof of proposition 2:

The correlation matrix is defined using Pearson's formula. The expected growth of the portfolio is given by Eq. (13), hence, for identically distributed assets ($f_i = \frac{1}{N}$), we get

$$E[g_p(t)] = u - \frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{ij}. \tag{B1}$$

For the uncorrelated portfolio (i.e., when $\sigma_{ij} = 0$ for $i \neq j$) this becomes

$$E[g_p(t)] = u - \frac{\sigma^2}{2N}. \tag{B2}$$

We can rewrite Eq. (B1) by splitting it to correlated and uncorrelated parts:

$$E[g_p(t)] = u - \frac{\sigma^2}{2N} - \frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\rho_{ij}\sigma^2. \tag{B3}$$

The last term represents the change due to correlation. To find N_{eff}, we compare the expected growth of a correlated portfolio to an uncorrelated one:

$$u - \frac{\sigma^2}{2N} - \frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j}^{N}\sigma_{ij} = u - \frac{\sigma^2}{2N_{\mathrm{eff}}}, \tag{B4}$$

which yields

$$N_{\mathrm{eff}} = \frac{N^2}{N + \sum_{i=1}^{N}\sum_{j\neq i}^{N}\rho_{ij}}. \tag{B5}$$

In the simple case of uniform correlation coefficient, where $\rho_{ij} = \rho\ \forall i \neq j$, we get

$$N_{\mathrm{eff}} = \frac{N}{1 + (N - 1)\rho}. \tag{B6}$$

APPENDIX C

Proof of Proposition 3 Eq. (24):

$$E[g_p(t)] = \frac{1}{t}\int_0^t \left\{u - \frac{\sigma^2}{2}\left[1 - \sqrt{\frac{\ln(N)}{t\sigma}}\right]\right\}dt = u - \frac{\sigma^2}{2} + \sigma^2\sqrt{\frac{\ln(N)}{t\sigma}}. \tag{C1}$$

Proposition: For i.i.d. assets,

$$E[g_p(t)] = \frac{E\{\ln[\mathrm{W(t)}]\}}{t} = \frac{1}{t}\int_0^t \left[u - \frac{\sigma^2}{2}\sum_{i=1}^{N} f_i(t)^2\right]dt. \tag{C2}$$

Proof: Looking at the time frame, $[t,t + dt]$ where $dt \to 0$ with initial wealth of $W(t)$ and initial fractions of $f_i(t) i \in (1,N)$ and applying the same theoretical analysis as in Eqs. (A1)–(A12), we get

$$\frac{E\left\{\ln\left[\frac{W(t+dt)}{W(t)}\right]\right\}}{dt} = u - \frac{\sigma^2}{2}\sum_{i=1}^{N} f_i(t)^2. \tag{C3}$$

Therefore,

$$\frac{E\{\ln[W(t+dt)]\}}{dt} = \frac{E\{\ln[W(t)]\}}{dt} + u - \frac{\sigma^2}{2}\sum_{i=1}^{N} f_i(t)^2. \tag{C4}$$

Expanding it further by taking steps of size $dt \to 0$, we get

$$\frac{E\{\ln[W(t+dt)]\}}{dt} = \sum_{t=0}^{\frac{t}{dt}} u - \frac{\sigma^2}{2}\sum_{i=1}^{N} f_i(t)^2. \tag{C5}$$

Thus, in the limit of $dt \to 0$, we get

$$\frac{E\{\ln[W(t)]\}}{t} = \frac{1}{t}\int_0^t \left[u - \frac{\sigma^2}{2}\sum_{i=1}^{N} f_i(t)^2\right] dt\blacksquare. \tag{C6}$$

Next, we get numerical results for $\sum_{i=1}^{N} f_i(t)^2$ and by fitting a functional form to the numerical results, we get

$$\sum_{i=1}^{N} f_i(t)^2 = 1 - \sqrt{\frac{\ln(N)}{t\sigma}}. \tag{C7}$$

We can now plug it into the integral of Eq. (C6) and get

$$E[g_p(t)] = u - \frac{\sigma^2}{2} + \sigma^2\sqrt{\frac{\ln(N)}{t\sigma}}\blacksquare, \tag{C8}$$

as requested in Eq. (C1)

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