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Unified path integral approach to theories of diffusion-influenced reactions

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Building on mathematical similarities between quantum mechanics and theories of diffusion-influenced reactions, we develop a general approach for computational modeling of diffusion-influenced reactions that is capable of capturing not only the classical Smoluchowski picture but also alternative theories, as is here exemplified by a volume reactivity model. In particular, we prove the path decomposition expansion of various Green's functions describing the irreversible and reversible reaction of an isolated pair of molecules. To this end, we exploit a connection between boundary value and interaction potential problems with δ- and δ'-function perturbation. We employ a known path-integral-based summation of a perturbation series to derive a number of exact identities relating propagators and survival probabilities satisfying different boundary conditions in a unified and systematic manner. Furthermore, we show how the path decomposition expansion represents the propagator as a product of three factors in the Laplace domain that correspond to quantities figuring prominently in stochastic spatially resolved simulation algorithms. This analysis will thus be useful for the interpretation of current and the design of future algorithms. Finally, we discuss the relation between the general approach and the theory of Brownian functionals and calculate the mean residence time for the case of irreversible and reversible reactions.

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I. INTRODUCTION

Stochastic processes, like Brownian motion, and quantum phenomena can be described by quite similar mathematical structures [1–3]. A case in point is the transition probability density function (PDF) $p(x,t|x_0,t_0)$, a central quantity in the theory of Brownian motion that yields the probability to find a Brownian walker at x at time t, provided it started at x_0 at $t_0 < t$. The transition PDF can be calculated in terms of a path integral that may formally be written as

$$p(x,t|x_0,t_0) \sim \sum_{x(\tau):x_0\to x} e^{-\frac{1}{\hbar}S_E[x(\tau)]}, \tag{1}$$

where the sum includes all paths $x(\tau)$ that satisfy $x(t_0)=x_0$, $x(t)=x$ and $S_E[x(\tau)]$ denotes a certain functional of $x(\tau)$. Quite analogously, the Feynmann transition *amplitude* $K(x,t|x_0,t_0)$ is also given by a path integral

$$K(x,t|x_0,t_0) \sim \sum_{x(\tau):x_0\to x} e^{\frac{i}{\hbar}S[x(\tau)]}, \tag{2}$$

where $S[x(\tau)]$ and $\hbar$ refer to the classical action functional and the reduced Planck's constant, respectively. Although the physics described by Eq. (1) is quite different from the one represented by Eq. (2), the path integrals can in general be transformed into each other by means of a Wick rotation [1–3] and the functional $S_E[x(\tau)]$ in Eq. (1) is the Euclidean version of the classical action $S[x(\tau)]$.

In the theory of diffusion-influenced reactions (DIR) [4–6], the transition PDF $p(x,t|x_0)$ traditionally appears as Green's function (GF) of the diffusion equation supplemented by suitable boundary conditions (BC) at the encounter distance $x = a$ that account for the actual reaction [4–11].

The GF plays a pivotal role in the theory of DIR because its knowledge permits obtaining the solution of the diffusion equation for any given initial distribution and calculating other important observable quantities, notably the survival probability and the time-dependent reaction rate coefficient [9–12]. Furthermore, analytical expressions for GF of an isolated pair serve as an essential ingredient of a number of stochastic simulation algorithms that aim at resolving single-particle behavior, which is necessary for accurately representing spatially heterogeneous biochemical systems [13]. In addition, GF can be utilized to help validate the correctness of newly devised stochastic simulation algorithms [14].

Its mathematical formulation as a boundary value problem makes the Smoluchowski-Collins-Kimball (SCK) [4–8] approach an example of a so-called contact-reactivity (CR) model [15]. While the Smoluchowski picture has successfully been applied to a great many problems, imposing BC at the encounter distance is not the only way to implement reactions [15–23]. In fact, the distinct role of the encounter radius has raised criticisms and it has been argued that a CR model may not be an appropriate or a too narrow description for certain problems. In the context of cell biology, for example, receptor-ligand interactions on two opposing cell membranes may not be adequately represented by the CR framework. A viable alternative is to abandon BC and instead add sink terms to the diffusion equation. It has been discussed that, under certain conditions, the addition of sink terms is equivalent to a BC formulation. However, this formulation allows for a more general and flexible approach than the traditional one [15–23] and is capable of including, in addition, non-CR approaches, like Doi-type models that are also referred to as volume (or area) reactivity (VR) theories [15,20–22]. Another advantage is that the diffusion equation with sink terms assumes a mathematical form that resembles the Schroedinger equation. In fact, it is well-known that both equations are related by a

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Wick-transformation, that is by $t \rightarrow -it$ [1–3], analogously to the corresponding path integrals. The formal resemblance opens the opportunity to borrow powerful methods from quantum mechanics to facilitate the study of DIR.

A case in point is the path decomposition expansion (PDX), which was originally employed to study various quantum-mechanical problems [3,24–28]. Here, we present a straightforward construction of the PDX of the propagators that play an important role in theories of DIR. The derivation is based on another technique imported from quantum mechanics. In a series of papers [29–31], Grosche discussed the implementation of Dirichlet and Neumann BC in terms of interaction potentials that are δ- and δ'-function perturbed. By using an exact summation of the expansion of the corresponding path integral, an exact convolution relation between the perturbed and unperturbed GF can be established. Because Dirichlet and Neumann BC correspond to absorbing (completely reactive) and reflecting (non-reactive) BC in the theory of diffusion-influenced reactions, the Wick-rotated (Euclidean) version of Grosche's results can be immediately applied to the case of an isolated pair. By choosing different unperturbed potentials, we are able to derive several important relations between the GF corresponding to various BC and potential functions, which are far from obvious from a differential equation point of view [32]. With these relations at our disposal, the actual proof of the PDX becomes rather accessible. As an application, we will show that the GF of the irreversible and reversible reaction can be factorized in the Laplace domain as a product of the first, last passage and rebinding time probability densities [33]. These three quantities, as well as the full propagator, figure prominently in the theory of stochastic processes and play an important role in stochastic simulation algorithms [34–36]. Typically, they can be more easily calculated than the full propagator, but the PDX shows that knowledge of them is sufficient to construct the full propagator.

The presented approach is quite general and permits to treat CR models, like the classical Smoluchowski-Collins-Kimball approach, as well as VR models, in a unified and systematic fashion. Also, although we mainly use a one-dimensional (1D) specific notation, we point out that the obtained results are valid in 1D, 2D, and 3D, and in most cases only small changes to prefactors have to be made to adapt the results to higher dimensions.

We emphasize that the study of the VR model is not only motivated by purely theoretical and mathematical reasons discussed above. For certain potential applications to cell-biological systems, for instance, the classical SCK approach does not appear to accurately reflect biological reality. A paradigmatic example is provided by the ligand-receptor interactions of two cells while being in contact. An effectively 2D description of the ligands' and receptors' diffusive motion on the two opposing cell membranes [Fig. 1(a)] needs to account for the possibility that ligands' trajectories may cross the reaction area established by the receptors. Also, a realistic model should yield a nonvanishing probability to find the ligand located unbound within the reaction area [Fig. 1(b)]. However, those requirements are at odds with the SCK model that does not have the ability to account for such space-time behavior, because it asserts that a ligand's trajectory cannot enter the area bounded by a receptor's encounter radius. Note, however, that for a different system, where ligands and receptors diffuse and interact on the same cell membrane, the SCK model remains a valid candidate for a realistic description [Figs. 1(c) and 1(d)].

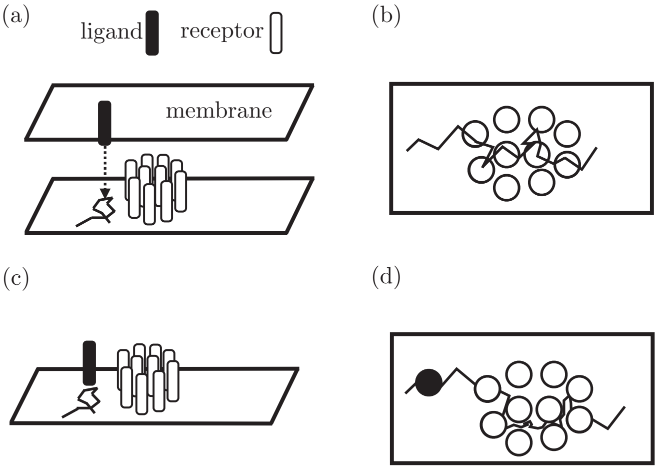

FIG. 1. Schematic comparison between the SCK and VR model of ligand-receptor interactions. (a) Ligands and receptors undergo a diffusive motion on opposing membranes of two cells in contact. (b) A ligand's trajectory may cross the reaction area established by the receptors. The SCK approach cannot account for such a behavior. (c) Ligands and receptors diffuse and bind on the same membrane. (d) Trajectories cannot pass through the areas bounded by the encounter radius, as asserted by the SCK model.

Furthermore, VR models can serve as an effective description of many receptors in a mean field sense. Taking again a ligand-receptor system on two opposing membranes as an example, one may refrain from resolving the interactions on the level of individual receptors. Instead, the effect of several receptors may be modeled by postulating a uniform reactivity across the corresponding region, naturally described in terms of a VR model whose reaction area now represents the whole region populated by receptors instead of the domain occupied by a single receptor. Such a coarse-grained description will also be employed in the final part of the present paper, where we will discuss the relation between the path integral approach to DIR and the theory of Brownian functionals [37,38]. The specific potentials that define the CR and VR models are intimately connected to the so-called local T_δ and occupation time T_Θ, respectively, which are random variables that play an important role in the theory of stochastic processes. In particular, we will discuss how to generalize the notion of the mean residence time [39] for the case of irreversible and reversible reactions.

We point out that some of the relations that we will derive have been obtained before by using the general form of the GF of the Smoluchowski equation [32]. Alternative derivations of the PDX have been given in Refs. [24–26,40]. The application of quantum field theoretical methods to diffusion-limited reactions and path integral techniques to stochastic differential equation are reviewed in Refs. [41] and [42], respectively. The manuscript is structured as follows. First, we will describe the traditional SCK approach and introduce the alternative formulation given in terms of sink potentials. Then, we will briefly recall the PDX and summarize Grosche's results. Equipped with those, we will derive various convolution relations, from which the path decomposition expansion of the Green's functions describing the irreversible and reversible reaction in the context of three different models follows. Our results will enable us to rederive important convolution relations connecting different survival probabilities. Furthermore, we provide two examples of the PDX and finally, we will discuss Brownian functionals.

II. BACKGROUND: THEORIES OF DIFFUSION-INFLUENCED REACTIONS

A. Irreversible reaction

We consider an isolated pair of irreversibly reacting molecules A and B that diffuse with diffusion constants D_A and D_B, respectively. This system may equivalently be described by a single molecule diffusing with diffusion constant $D = D_A + D_B$ either in the presence of a reactive boundary (SCK) at $x = a$, a reactive sink (δ-function potential) at $x = a$, or a reactive domain (Θ-function potential) specified by $x \leqslant a$. Without loss of generality, we may assume that $a = 0$. First, we will focus on the SCK approach.

1. Contact reactivity: Smoluchowski-Collins-Kimball

The theory of DIR [4–6] is traditionally formulated in terms of the diffusion equation subject to a suitable BC that implements the physics at the encounter distance. The conditional PDF $p(x,t|x_0)$ that yields the probability to find the molecule at position x at time t, given that it was initially ($t = 0$) localized at x_0, is the GF of the diffusion equation,

$$\frac{\partial p(x,t|x_0)}{\partial t} = D\frac{\partial^2 p(x,t|x_0)}{\partial x^2}, \tag{3}$$

satisfying the initial condition,

$$p(x,t=0|x_0) = \delta(x - x_0). \tag{4}$$

Henceforth, we will use a subscript to indicate the specific BC satisfied by the corresponding GF. The free-space GF $p_{\mathrm{free}}(x,t|x_0)$ is subject to the following BC at infinity,

$$p_{\mathrm{free}}(x,t|x_0) \underset{x\to\pm\infty}{\longrightarrow} 0. \tag{5}$$

To incorporate the actual binding reaction the SCK approach [4–7] imposes a radiation BC at the encounter distance $x = a$ [5–7,9–11], thus replacing the BC for $x \to -\infty$ [Eq. (5)],

$$D\left.\frac{\partial p_{\mathrm{rad}}(x,t|x_0)}{\partial x}\right|_{x=a} = \kappa_a p_{\mathrm{rad}}(x=a,t|x_0). \tag{6}$$

In the limit $\kappa_a \to \infty$, the radiation BC transforms into the classical Smoluchowski, or completely absorbing BC,

$$p_{\mathrm{abs}}(a,t|x_0) = 0, \tag{7}$$

describing a molecule that instantaneously reacts when reaching the boundary. The opposite limit $\kappa_a \to 0$ leads to the reflective BC,

$$\left.\frac{\partial p_{\mathrm{ref}}(x,t|x_0)}{\partial x}\right|_{x=a} = 0, \tag{8}$$

accounting for a nonreactive molecule that never binds upon encounter with the boundary. Clearly, $p_{\mathrm{rad}}(x,t|x_0)$ and hence its limiting cases $p_{\mathrm{abs}}(x,t|x_0)$ and $p_{\mathrm{ref}}(x,t|x_0)$ are only defined on the semi-infinite interval, that is for $x,x_0 \geqslant a$ (or alternatively $x,x_0 \leqslant a$), in contrast to $p_{\mathrm{free}}(x,t|x_0)$. The GF permits to derive other important quantities. Notably, the survival probability [12], providing the probability that the molecule has not reacted by time t, can be calculated from the GF according to

$$S_{\mathrm{rad}}(t|x_0) = \int_a^\infty p_{\mathrm{rad}}(x,t|x_0)dx. \tag{9}$$

Making use of the diffusion equation [Eq. (3)] and the definition of the survival probability [Eq. (9)], we obtain

$$-\frac{\partial S_{\mathrm{rad}}(t|x_0)}{\partial t} = D\left.\frac{\partial p_{\mathrm{rad}}(x,t|x_0)}{\partial x}\right|_{x=a}. \tag{10}$$

Note that this relation holds as a consequence of the diffusion equation, independent of any imposed BC at the encounter distance. However, we may combine the SCK boundary condition [Eq. (6)] and Eq. (10) to arrive at

$$f_{\mathrm{reb}}(t) \equiv -\frac{\partial S_{\mathrm{rad}}(t|x_0=a)}{\partial t} = \kappa_a p_{\mathrm{rad}}(a,t|a). \tag{11}$$

We see that the quantity $\kappa_a p_{\mathrm{rad}}(a,t|a)$ yields the rebinding time PDF $f_{\mathrm{reb}}(t)$ that plays an essential role in understanding spatial stochastic fluctuations [35,43].

Furthermore, we recall that also in the limit of an absorbing BC the negative time derivative of the corresponding survival probability bears a special meaning. It is the first passage time PDF [44],

$$f_{\mathrm{FP}}(t|x_0) = -\frac{\partial S_{\mathrm{abs}}(t|x_0)}{\partial t} = D\left.\frac{\partial p_{\mathrm{abs}}(x,t|x_0)}{\partial x}\right|_{x=a}. \tag{12}$$

Finally, it will turn out to be convenient to consider the PDF $f_{\mathrm{LR}}(\tau|x,t)$ of the last reflection time τ before t, given that the molecule is located at x at time t. This quantity is closely related to the first passage time PDF via

$$f_{\mathrm{LR}}(\tau|x,t) = f_{\mathrm{FP}}(t-\tau|x). \tag{13}$$

2. Contact reactivity: δ-function potential

An alternative and more general approach that abandons the requirement of a BC at the encounter distance implements the chemical reaction by adding appropriate sink terms to the equation of motion [16]. More precisely, one considers a diffusion equation featuring a nonvanishing potential,

$$\frac{\partial p_V(x,t|x_0)}{\partial t} = D\frac{\partial^2 p_V(x,t|x_0)}{\partial x^2} - V(x)p_V(x,t|x_0). \tag{14}$$

In general, unless stated otherwise, the GF solution $p_V(x,t|x_0)$ is subject to the BC given by Eq. (5).

Clearly, the form of the potential $V(x)$ specifies which class of models is being considered. The choice

$$V(x) = V_\delta(x) = \gamma_a\delta(x-a), \tag{15}$$

defines a CR model. We note that also the SCK approach belongs to the class of CR models. However, we emphasize that the GF solution $p_\delta(x,t|x_0)$ of the potential problem specified by Eqs. (14), (15), and (5) is not identical to the GF $p_{\mathrm{rad}}(x,t|x_0)$

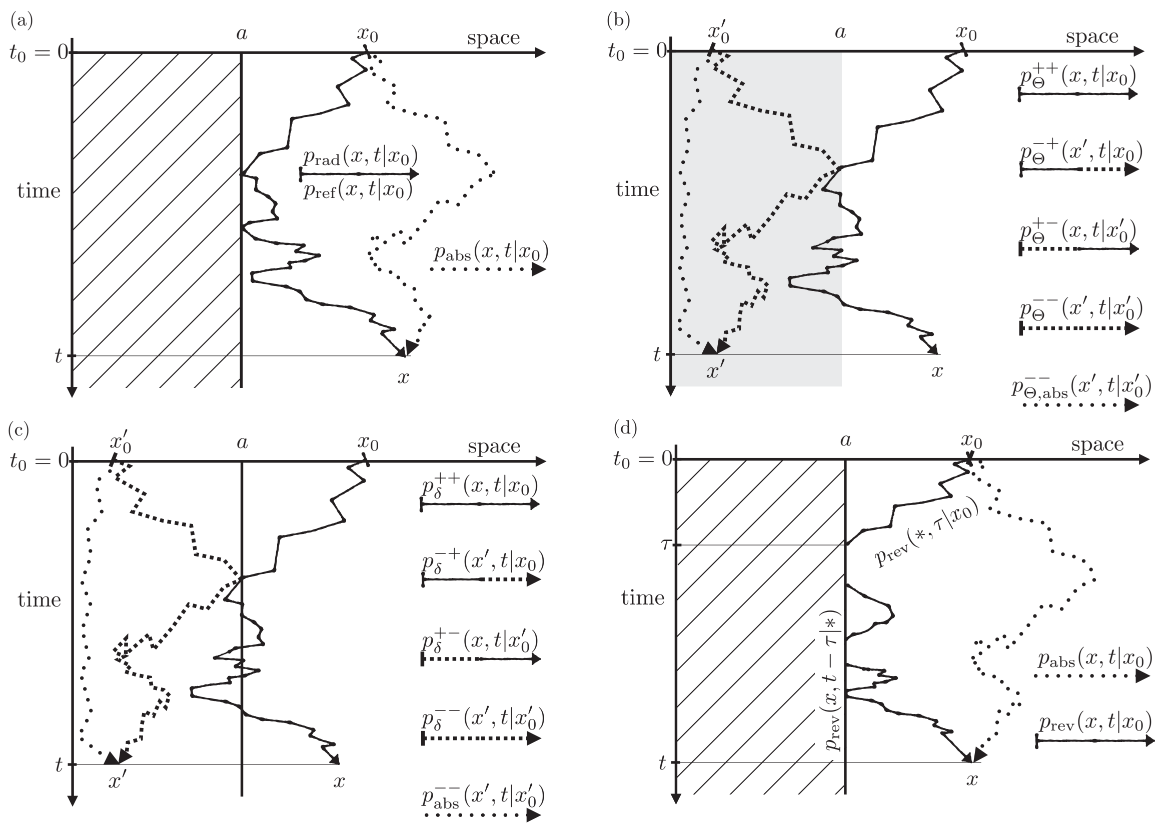

FIG. 2. Schematic overview of the various Green's functions describing diffusion-influenced irreversible and reversible reactions. (a) The propagator $p_{\mathrm{rad}}(x,t|x_0)$ accounts for irreversible binding according to the classical SCK model, while $p_{\mathrm{ref}}(x,t|x_0)$ describes purely nonreactive encounters. Those GF do not exist for $x,x_0 < a$, meaning that the particle can reach but not cross the boundary. (b) The propagator corresponding to a Θ-function sink describing irreversible binding in the context of an VR model. The "components" of the GF solution that yield the probability to find the particle located outside ($x > a$) the reactive domain are referred to as $p_\Theta^{++}(x,t|x_0)$ and $p_\Theta^{+-}(x,t|x_0')$, depending on the location of the initial position. The GF $p_\Theta^{-+}(x',t|x_0)$ and $p_\Theta^{--}(x',t|x_0')$ are defined analogously. (c) The propagator corresponding to a δ-function sink describing irreversible binding. The notation is completely analogous to the one used for the Θ-function potential propagators. (d) In the case of reversible binding, here shown for the SCK model, the GF includes additional components that account for the possibility of an initial or final bound state, denoted as $*$. The GF $p_{\mathrm{abs}}(x,t|x_0)$ that satisfies an absorbing boundary condition describes only paths that never reach the boundary. This GF appears in all models and plays an essential role in the formulation of the various PDX. For the δ- and Θ-potential model there is a version of this GF defined on the domain $x,x_0 < a$, denoted by $p_{\mathrm{abs}}^{--}(x',t|x_0')$ and $p_{\Theta,\mathrm{abs}}^{--}(x',t|x_0')$, respectively.

of the radiation BC problem [45] and, in particular, does not satisfy the radiation BC [Eq. (6)] at the encounter distance $x = a$, unless one makes additional assumptions [16]. One obvious difference is that $p_\delta(x,t|x_0)$ is also defined for $x < a$ and $x_0 < a$, in contrast to $p_{\mathrm{rad}}(x,t|x_0)$. Henceforth, we will use superscripts p_δ^{++}, p_δ^{+-}, p_δ^{-+}, p_δ^{--} to refer to the components of the GF solution of Eqs. (14) and (15) for different domains defined by $x,x_0 > a; x > a > x_0; x_0 > a > x$ and $x,x_0 < a$, respectively (see Fig. 2). Note that even p_δ^{++} is not equal to p_{rad}, although both are defined on the same domain $x,x_0 > a$ [45].

Another difference between $p_\delta(x,t|x_0)$ and $p_{\mathrm{rad}}(x,t|x_0)$ lies in their behavior in the non-reactive limit $\gamma_a,\kappa_a \to 0$. While $p_\delta(x,t|x_0)$ becomes the free-space GF, one has $p_{\mathrm{rad}}(x,t|x_0) \to p_{\mathrm{ref}}(x,t|x_0)$.

The survival probability is now defined by

$$S_\delta(t|x_0) = \int_{-\infty}^{\infty} p_\delta(x,t|x_0)dx. \tag{16}$$

Using this definition and integrating Eq. (14) over the whole infinite interval it follows that

$$-\frac{\partial S_\delta(t|x_0)}{\partial t} = \gamma_a p_\delta(a,t|x_0), \tag{17}$$

It is convenient to introduce two additional survival probabilities,

$$S_\delta^{+}(t|x_0) = \int_a^{\infty} p_\delta(x,t|x_0)dx, \tag{18}$$

$$S_\delta^{-}(t|x_0) = \int_{-\infty}^{a} p_\delta(x,t|x_0)dx, \tag{19}$$

which provide the probability that the molecule has not reacted by time t and that it is located in the domain $x \geqslant a$ and $x \leqslant a$, respectively. Employing again the diffusion equation [Eq. (14)] leads to

$$\mp\frac{\partial S_\delta^{\pm}(t|x_0)}{\partial t} = D\frac{\partial p_\delta(x,t|x_0)}{\partial x}\bigg|_{x\to a^{\pm}}. \tag{20}$$

We point out that the SCK model can be recovered by imposing [16]

$$-\frac{\partial S_\delta^-(t|x_0)}{\partial t} = 0, \tag{21}$$

so that, combining Eqs. (20) and (17), the radiation BC is recovered.

Finally, we point out that GF satisfying an absorbing or reflective BC also make sense in the context of the δ-function potential problem. The only difference to the SCK case lies in the fact that there are now two versions of those GF: one defined on the domain given by $x,x_0 \geqslant a$ (known from the SCK model), the other defined on $x,x_0 \leqslant a$, which we will denote by $p_{\mathrm{abs,ref}}^{--}$. Note that neither GF can describe a molecule path crossing the location of the sink $x = a$.

3. *Volume reactivity: Θ-function potential*

Turning to VR theories, we assume that the potential in Eq. (14) takes the form

$$V(x) = V_\Theta(x) = \kappa_r \Theta(a - x), \tag{22}$$

where $\Theta(x)$ refers to the Heaviside step-function that gives $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$. For this choice of $V(x)$, Eq. (14) is also known as Feynman-Kac equation (FKE) [15,20–22].

The survival probability $S_\Theta(t|x_0)$ is defined analogously to the δ-function case

$$S_\Theta(t|x_0) = \int_{-\infty}^{\infty} p_\Theta(x,t|x_0)dx, \tag{23}$$

Then, the equation of motion [Eqs. (14) and (22)] yields

$$-\frac{\partial S_\Theta(t|x_0)}{\partial t} = \kappa_r S_\Theta^-(t|x_0), \tag{24}$$

where we introduced the survival probabilities

$$S_\Theta^\pm(t|x_0) = \pm \int_a^{\pm\infty} p_\Theta(x,t|x_0)dx. \tag{25}$$

Making use of these definitions and Eqs. (14) and (22), we are led to

$$-\frac{\partial S_\Theta^+(t|x_0)}{\partial t} = D\frac{\partial p_\Theta(x,t|x_0)}{\partial x}\bigg|_{x\to a^+}, \tag{26}$$

$$-\frac{\partial S_\Theta^-(t|x_0)}{\partial t} = -D\frac{\partial p_\Theta(x,t|x_0)}{\partial x}\bigg|_{x\to a^-} + \kappa_r S_\Theta^-(t|x_0). \tag{27}$$

As in the case of δ-function sink, we will use superscripts p_Θ^{++}, p_Θ^{+-}, p_Θ^{-+}, p_Θ^{--} to refer to the components of the GF solution of Eqs. (14) and (22) for the corresponding domains; see Fig. 2.

Furthermore, one may in the present context also use the concept of a GF satisfying absorbing boundary conditions. For $x,x_0 > a$, the corresponding GF is identical to their CK counterparts and hence we will use the same notation. However, in contrast to the CR models considered before, the absorbing boundary condition GF $p_{\Theta,\mathrm{abs}}^{--}(x,t|x_0)$ defined on $x,x_0 \leqslant a$ is not governed by the free diffusion equation [Eq. (3)] but obeys

$$\frac{\partial p_{\Theta,\mathrm{abs}}^{--}(x,t|x_0)}{\partial t} = D\frac{\partial^2 p_{\Theta,\mathrm{abs}}^{--}(x,t|x_0)}{\partial x^2} - \kappa_r p_{\Theta,\mathrm{abs}}^{--}(x,t|x_0) \tag{28}$$

and is subject to the absorbing BC,

$$p_{\Theta,\mathrm{abs}}^{--}(x = a,t|x_0) = 0. \tag{29}$$

Quite analogous remarks apply to the free-space and reflecting BC GF also.

B. Reversible reaction

Turning to reversible reactions, the irreversible models considered so far have to be extended to account for two new aspects: The dissociation of the bound state and an additional possible initial condition describing the bound molecule. We will again consider the SCK approach first.

1. *Contact reactivity: Smoluchowski-Collins-Kimball*

To incorporate dissociation of the bound state within the SCK framework, one generalizes the SCK boundary condition [Eq. (6)] to a backreaction BC [8–11],

$$D\frac{\partial p_{\mathrm{rev}}(x,t|x_0)}{\partial x}\bigg|_{x=a} = \kappa_a p_{\mathrm{rev}}(a,t|x_0) - \kappa_d[1 - S_{\mathrm{rev}}(t|x_0)], \tag{30}$$

where κ_d denotes the intrinsic dissociation constant. Because Eq. (10) remains valid for $S_{\mathrm{rev}}(t|x_0)$, it follows that

$$\frac{\partial S_{\mathrm{rev}}(t|x_0)}{\partial t} = -\kappa_a p_{\mathrm{rev}}(a,t|x_0) + \kappa_d[1 - S_{\mathrm{rev}}(t|x_0)], \tag{31}$$

generalizing Eq. (11). We now seek to eliminate the survival probability from the backreaction BC [Eq. (30)] as follows. Using the Laplace transform $\tilde{f}(s)$ of a function $f(t)$, defined by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)dt, \tag{32}$$

as well as Eq. (31) together with the initial condition $S_{\mathrm{rev}}(t = 0|x_0) = 1$, we arrive at

$$\tilde{S}_{\mathrm{rev}}(s|x_0) = \frac{1}{s} - \frac{\kappa_a}{\kappa_d}\tilde{\psi}(s)\tilde{p}_{\mathrm{rev}}(a,s|x_0), \tag{33}$$

where we have introduced the Laplace-transform of the residence time density of the bound state,

$$\tilde{\psi}(s) = \frac{\kappa_d}{s + \kappa_d}. \tag{34}$$

Usually, one assumes a Poissonian form of $\psi(t) = \kappa_d \exp(-\kappa_d t)$ but the formalism is flexible enough to accommodate also non-Markovian generalizations [46]. From Eq. (33) it follows that

$$\kappa_d[1 - S_{\mathrm{rev}}(t|x_0)] = \kappa_a \int_0^t \psi(t-t')p_{\mathrm{rev}}(a,t'|x_0)dt'. \tag{35}$$

This result permits to eliminate the survival probability from the backreaction BC [Eq. (30)]. To this end, it will turn out to be convenient to introduce a convolution operator $\mathcal{O}_\psi(t)$, acting on functions $f(t)$ as

$$\mathcal{O}_\psi(t) : f(t) \to \mathcal{O}_\psi(t)f(t) = \int_0^t \psi(t - t')f(t')dt'. \tag{36}$$

Then, the backreaction BC takes the form

$$D\left.\frac{\partial p_{\mathrm{rev}}(x,t|x_0)}{\partial x}\right|_{x=a} = \kappa_a \mathcal{O}_{\delta-\psi}(t)p_{\mathrm{rev}}(a,t|x_0), \tag{37}$$

$$D\left.\frac{\partial \tilde{p}_{\mathrm{rev}}(x,s|x_0)}{\partial x}\right|_{x=a} = \frac{\kappa_a s}{s+\kappa_d}\tilde{p}_{\mathrm{rev}}(a,s|x_0), \tag{38}$$

in the time and Laplace domain, respectively. Note that the notation δ appearing in the subscript of the convolution operator $\mathcal{O}_{\delta-\psi}(t)$ in Eq. (37) indicates the δ function of time $\delta(t)$.

Next, we turn to the bound state, denoted by $*$, as a possible initial condition [10–12]. The corresponding GF $p_{\mathrm{rev}}(x,t|*)$ yields the probability to find the particle separated at a distance x at time t, given that initially it was in the bound state. This GF satisfies the same BC [Eq. (30)], but the IC now reads

$$p_{\mathrm{rev}}(x,t=0|*) = 0. \tag{39}$$

For later use we recall [10–12,46] that the GF describing the unbound and bound state as initial condition are related by

$$\tilde{p}_{\mathrm{rev}}(x,s|*) = \tilde{p}_{\mathrm{rev}}(x,s|a)\tilde{\psi}(s). \tag{40}$$

2. *Contact reactivity: δ-function potential*

In this CR model, the dissociation of the bound state is incorporated by generalizing the potential problem given by Eqs. (14) and (15) as follows [15]:

$$\frac{\partial p_{\delta,\mathrm{rev}}(x,t|x_0)}{\partial t} = D\frac{\partial^2 p_{\delta,\mathrm{rev}}(x,t|x_0)}{\partial x^2} - \gamma_a\delta(x-a)p_{\delta,\mathrm{rev}}(a,t|x_0) + \gamma_d\delta(x-a)[1-S_{\delta,\mathrm{rev}}(t|x_0)]. \tag{41}$$

However, the appearance of the survival probability renders this form of the equation of motion unsuitable for a direct application of the approach we will discuss in Sec. II D. Therefore, similar to the reversible SCK approach, we seek to eliminate $S_{\delta,\mathrm{rev}}(t|x_0)$ in terms of $p_{\delta,\mathrm{rev}}(x,t|x_0)$. To this end, we integrate Eq. (41) over the infinite interval to arrive at

$$\frac{\partial S_{\delta,\mathrm{rev}}(t|x_0)}{\partial t} = -\gamma_a p_{\delta,\mathrm{rev}}(a,t|x_0) + \gamma_d[1-S_{\delta,\mathrm{rev}}(t|x_0)], \tag{42}$$

which is completely analogous to its SCK counterpart Eq. (31). We can thus again employ the convolution operator defined by Eq. (36) to recast Eq. (41) as

$$\frac{\partial p_{\delta,\mathrm{rev}}(x,t|x_0)}{\partial t} = D\frac{\partial^2 p_{\delta,\mathrm{rev}}(x,t|x_0)}{\partial x^2} - V_{\delta,\mathrm{rev}}(x,t)p_{\delta,\mathrm{rev}}(x,t|x_0), \tag{43}$$

where we introduced the potential

$$\begin{aligned} V_{\delta,\mathrm{rev}}(x,t) &= V_\delta(x) - \gamma_a\delta(x-a)\mathcal{O}_\psi(t) \\ &= \gamma_a\delta(x-a)\mathcal{O}_{\delta-\psi}(t), \end{aligned} \tag{44}$$

noting that here $\psi(t) = \gamma_d\exp(-\gamma_d t)$, slightly abusing notation. Note that in the Laplace domain we have the relation

$$\tilde{V}_{\delta,\mathrm{rev}}(x,s,\gamma_a) = \gamma_a\frac{s}{s+\gamma_d}\delta(x-a) = \tilde{V}_\delta\left(x,\gamma_a\frac{s}{s+\gamma_d}\right). \tag{45}$$

Finally, we point out that Eq. (40) remains valid in the present case, relating $\tilde{p}_{\delta,\mathrm{rev}}(x,s|a)$ with $\tilde{p}_{\delta,\mathrm{rev}}(x,s|*)$.

3. *Volume reactivity: Θ-function potential*

The VR Θ-potential model can be treated quite analogously to its δ-function counterpart. However, there is one important difference: The bound state is infinitely degenerate because it can assume its position anywhere in $x < a$. To account for that, one introduces an additional PDF $q_{\Theta,\mathrm{rev}}(x,t|x_0)$ that gives the probability to find the molecule bound at x at time t, provided it was not bound at x_0 at time $t=0$. As a result, the dynamics is captured by two coupled equations of motion [15,20–22]:

$$\begin{aligned} \frac{\partial p_{\Theta,\mathrm{rev}}(x,t|x_0)}{\partial t} &= D\frac{\partial^2 p_{\Theta,\mathrm{rev}}(x,t|x_0)}{\partial x^2} \\ &\quad - \kappa_r\Theta(a-x)p_{\Theta,\mathrm{rev}}(x,t|x_0) \\ &\quad + \kappa_d q_{\Theta,\mathrm{rev}}(x,t|x_0), \end{aligned} \tag{46}$$

$$\begin{aligned} \frac{\partial q_{\Theta,\mathrm{rev}}(x,t|x_0)}{\partial t} &= \kappa_r\Theta(a-x)p_{\Theta,\mathrm{rev}}(x,t|x_0) \\ &\quad - \kappa_d q_{\Theta,\mathrm{rev}}(x,t|x_0). \end{aligned} \tag{47}$$

Again, we have to transform the equations of motion to a form amenable to the path integral approach we will later on employ. We eliminate $q_{\Theta,\mathrm{rev}}(x,t|x_0)$ in terms of $p_{\Theta,\mathrm{rev}}(x,t|x_0)$, using the initial conditions

$$p_{\Theta,\mathrm{rev}}(x,t=0|x_0) = \delta(x-a) \quad \text{and} \quad q_{\Theta,\mathrm{rev}}(x,t=0|x_0) = 0, \tag{48}$$

as well as Eq. (47) to find

$$q_{\Theta,\mathrm{rev}}(x,t|x_0) = \frac{\kappa_r}{\kappa_d}\Theta(a-x)\int_0^t \psi(t-t')p_{\Theta,\mathrm{rev}}(x,t'|x_0)dt'. \tag{49}$$

By virtue of this equation, we can decouple Eqs. (46) and (47) and observe that the resulting equation of motion of $p_{\Theta,\mathrm{rev}}(x,t|x_0)$ can be rewritten in the general form of Eq. (14). The potential is now given by

$$\begin{aligned} V_{\Theta,\mathrm{rev}}(x,t) &= V_\Theta(x) - \kappa_r\Theta(a-x)\mathcal{O}_\psi(t) \\ &= \kappa_r\Theta(a-x)\mathcal{O}_{\delta-\psi}(t), \end{aligned} \tag{50}$$

$$\tilde{V}_{\Theta,\mathrm{rev}}(x,s) = \kappa_r\frac{s}{s+\kappa_d}\Theta(a-x), \tag{51}$$

in the time and Laplace domain, again quite reminiscent of the δ-function potential case. Finally, the GF describing the initially unbound and bound state state satisfy

$$\tilde{p}_{\Theta,\mathrm{rev}}(x,s|x_0,*) = \tilde{\psi}(s)\tilde{p}_{\theta,\mathrm{rev}}(x,s|x_0). \tag{52}$$

Note that the GF corresponding to an initial bound state depends on an additional spatial variable x_0 due to the aforementioned bound state's degeneracy.

C. The path decomposition expansion

The PDX was developed in Ref. [24] to study quantum tunneling. Subsequently, it has been used for other applications as well, see Refs. [3,25]. In Refs. [27,28], this technique was used to calculate the Feynmann propagator for a δ- and Θ-function potential in an elegant and simple way. The PDX allows to break space into different regions and to study the individual contributions from the propagators restricted to one region to the complete propagator across all regions. While the decomposition of space may be chosen arbitrarily, the existence of a non-vanishing potential or a reactive boundary may suggest a natural decomposition. This is in particular true for a δ- and Heaviside step-function potential. These potentials are of special interest in the theory of diffusion-influenced reactions, because they correspond to the CR and VR model, as we have discussed in Sec. II.

To be specific, we consider a quantum mechanical particle of mass m in 1D. Then, the propagator satisfies the following convolution relation [3,24,27,28]:

$$K(x,t|x_0) = \frac{i\hbar}{2m}\int_0^t K(x,t|a,\tau)\left.\frac{\partial K^r(\xi,\tau|x_0,0)}{\partial \xi}\right|_{\xi=a} d\tau. \quad (53)$$

This relation means that every path can be split into a restricted part (represented by K^r) that starts at x_0 at time $t_0 = 0$, never crosses the boundary at $x = a = 0$, but that ends at the boundary at time τ, and an unrestricted part (represented by K) that starts at the boundary at time τ and ends at x at time t. Similarly, one has

$$K(x,t|x_0) = -\frac{i\hbar}{2m}\int_0^t \left.\frac{\partial K^r(x,t|\xi,\tau)}{\partial \xi}\right|_{\xi=a} K(a,\tau|x_0,0)d\tau, \quad (54)$$

which corresponds to a split into into an unrestricted part that starts at x_0 at time $t_0 = 0$, may cross the boundary $x = a = 0$ many times, and finally ends at the boundary after time τ and a restricted part that starts at the boundary at time τ, never crosses the boundary after that and ends at x at time t. Combining these two formulas [Eqs. (53) and (54)], one obtains for the Feynmann propagator [27,28],

$$K(x,t|x_0) = \frac{\hbar^2}{4m^2}\int_0^t dT \int_0^T d\tau \left.\frac{\partial K^r(x,t|\xi,T)}{\partial \xi}\right|_{\xi=a} \times K(a,T|a,\tau)\left.\frac{\partial K^r(\xi,\tau|x_0,0)}{\partial \xi}\right|_{\xi=a}, \quad (55)$$

or equivalently in the Laplace domain,

$$\tilde{K}(x,s|x_0) = \frac{\hbar^2}{4m^2}\left.\frac{\partial \tilde{K}^r(x,s|\xi)}{\partial \xi}\right|_{\xi=a}\tilde{K}(a,s|a)\left.\frac{\partial \tilde{K}^r(\xi,s|x_0)}{\partial \xi}\right|_{\xi=a}. \quad (56)$$

Note that for $x,x_0 > a$ and $x,x_0 < a$ one has to add K^r to the right-hand side of Eqs. (55) and (56).

In the context of theories of diffusion-influenced reactions, we will derive the following form of the PDX [47]:

$$\tilde{p}(x,s|x_0) = \tilde{p}_{\mathrm{abs}}(x,s|x_0) + D^2\left.\frac{\partial \tilde{p}_{\mathrm{abs}}(x,s|\xi)}{\partial \xi}\right|_{\xi=a} \times \tilde{p}(a,s|a)\left.\frac{\partial \tilde{p}_{\mathrm{abs}}(\xi,s|x_0)}{\partial \xi}\right|_{\xi=a}, \quad (57)$$

where $\tilde{p}(x,s|x_0)$ refers to different Greens' functions describing irreversible and reversible reactions according to the CR and VR model, as for instance $\tilde{p}_\delta(x,s|x_0)$ [Eq. (108)], $\tilde{p}_\Theta(x,s|x_0)$ [Eq. (113)], $\tilde{p}_{\mathrm{ref}}(x,s|x_0)$ [Eq. (89)], $\tilde{p}_{\mathrm{rad}}(x,s|x_0)$ [Eq. (95)], and $\tilde{p}_{\mathrm{rev}}(x,s|x_0)$ [Eq. (116)]. The restricted propagator K^r appearing in Eq. (55) vanishes at the boundary $K^r(x = a,t|x_0) = 0$. Hence, K^r corresponds to the GF satisfying absorbing boundary conditions that describes totally diffusion-controlled reactions. However, of course, $p_{\mathrm{abs}}(x,t|x_0)$ is a PDF and therefore corresponds to a directly observable quantity. The same is true for its derivative at the boundary that is the first-passage time PDF [Eq. (12)]. In contrast, neither $K^r(x,t|x_0)$, being a probability amplitude, nor its derivative directly represent an observable. We will later see that the $\tilde{p}_{\mathrm{abs}}(x,s|x_0)$ term on the right-hand side of Eq. (57) vanishes if x and x_0 denote positions on domains separated by $x = a$, because this means that all paths contributing to $\tilde{p}(x,s|x_0)$ have to cross the boundary, reminiscent to its quantum-mechanical counterpart [Eq. (55)].

The derivation of Eq. (57) that we will present is based on an exactly summable perturbative expansion of a path integral that gives the propagator associated with δ- and δ'-function perturbed interaction potentials, which we will recall now.

D. Path integral representation of δ- and δ'-function perturbed potentials

Following Grosche [29–31], we consider a potential $W(x)$ composed of an arbitrary potential $V(x)$ and a δ-function perturbation,

$$W(x) = V(x) + V_{\mathrm{pert}}(x) = V(x) - \gamma\delta(x-a). \quad (58)$$

The corresponding GF enjoys the following path integral representation

$$p_W(x,t|x_0) = \int \mathcal{D}x(\tau)e^{-\int_0^t[\frac{1}{4D}\dot{x}^2(\tau)+W(x)]d\tau}, \quad (59)$$

Assuming that the GF for the unperturbed problem

$$p_V(x,t|x_0) = \int \mathcal{D}x(\tau)e^{-\int_0^t[\frac{1}{4D}\dot{x}^2(\tau)+V(x)]d\tau} \quad (60)$$

is known, the path integral for the full problem [Eq. (59)] can be expanded in a perturbation series in terms of $p_V(x,t|x_0)$ [29–31],

$$p_W(x,t|x_0) = p_V(x,t|x_0) + \sum_{n=1}^{\infty}\gamma^n\int_0^t d\tau^{(n)}\int_0^{\tau^{(n)}} d\tau^{(n-1)} \dots \int_0^{\tau^{(2)}} d\tau^{(1)} p_V(a,\tau^{(1)}|x_0)\dots p_V(a,\tau^{(n)} - \tau^{(n-1)}|a)p_V(x,t-\tau^{(n)}|a). \quad (61)$$

Then, by virtue of the convolution theorem of the Laplace transform, one obtains in the Laplace domain the exact relation,

$$\tilde{p}_W(x,s|x_0) = \tilde{p}_V(x,s|x_0) + \sum_{n=1}^{\infty}\gamma^n\tilde{p}_V(x,s|a)\tilde{p}_V(a,s|x_0) \times [\tilde{p}_V(a,s|a)]^{n-1} = \tilde{p}_V(x,s|x_0) + \gamma\frac{\tilde{p}_V(x,s|a)\tilde{p}_V(a,s|x_0)}{1-\gamma\tilde{p}_V(a,s|a)}. \quad (62)$$

We note that Eq. (62) immediately leads to the relations

$$\tilde{p}_W(x,s|a) = \frac{\tilde{p}_V(x,s|a)}{1-\gamma\,\tilde{p}_V(a,s|a)}, \tag{63}$$

$$\frac{\tilde{p}_W(x,s|a)}{\tilde{p}_W(a,s|a)} = \frac{\tilde{p}_V(x,s|a)}{\tilde{p}_V(a,s|a)}, \tag{64}$$

which will prove useful in the following.

Instead of a δ function, one may study potentials perturbed by the derivative of the δ function [31],

$$W(x) = V(x) - \beta\delta'(x-a). \tag{65}$$

This class of potentials allows for incorporating reflective BC. The derivation of an expansion analog to Eq. (62) is possible, but more complicated than in the δ-function case [31]. The result is

$$\tilde{p}_W(x,s|x_0) = \tilde{p}_V(x,s|x_0) - \beta\frac{\frac{\partial\tilde{p}_V(x,s|\xi)}{\partial\xi}\Big|_{\xi=a}\frac{\partial\tilde{p}_V(\xi,s|x_0)}{\partial\xi}\Big|_{\xi=a}}{1+\beta\frac{\partial^2\tilde{p}_V(\xi,s|\xi_0)}{\partial\xi\,\partial\xi_0}\Big|_{\xi=\xi_0=a}}. \tag{66}$$

The Eqs. (62) and (66) are of key importance and starting point of all subsequent derivations.

We emphasize that Eqs. (62) and (66) are valid for arbitrary unperturbed potentials V. Therefore, they also apply to the VR model by choosing $V(x) = \kappa_r\Theta(a-x)$. Still, Eqs. (62) and (66) will turn out to be somewhat more powerful for CR, because here both the unperturbed potential as well as the perturbation are given by either a δ- or δ'- function, giving rise to more derivable relations. A case in point is Eq. (115), connecting the irreversible and reversible case. Such a relation cannot be straightforwardly derived in the VR case, because the perturbation term that gives rise to the reversible reaction cannot be written as δ function, rendering Eqs. (62) and (66) inapplicable.

III. RESULTS

In this section, we will derive the path decomposition expansion of the Green's functions describing the irreversible and reversible reaction in the context of the three different models. Along the way, we will obtain a number of other relations and discuss commonalities and differences between the considered models. First, as a prerequisite, we will derive a number of identities that relate p_{free}, p_{abs}, and p_{ref}. We start by choosing

$$V(x) = 0, \tag{67}$$

$$V_{\text{pert}}(x) = -\lim_{\gamma\to-\infty}\gamma\delta(x-a), \tag{68}$$

that is, we choose the free particle and an infinitely repulsive δ function as unperturbed and perturbed problem. It has been shown that an infinitely repulsive δ function gives rise to an absorbing BC [30]. Therefore, application of Eq. (62) to the potential problem specified by Eqs. (67) and (68) yields

$$\tilde{p}_{\text{abs}}(x,s|x_0) = \tilde{p}_{\text{free}}(x,s|x_0) - \frac{\tilde{p}_{\text{free}}(x,s|a)\tilde{p}_{\text{free}}(a,s|x_0)}{\tilde{p}_{\text{free}}(a,s|a)}. \tag{69}$$

Note that $p_{\text{abs}}(x,t|a) = p_{\text{abs}}(a,t|x_0) = 0$ immediately follows from that representation. Another consequence of Eq. (69) is

$$\frac{\partial\tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\Big|_{\xi=a} = \frac{\partial\tilde{p}_{\text{free}}(\xi,s|x_0)}{\partial\xi}\Big|_{\xi=a} - \frac{\partial\tilde{p}_{\text{free}}(\xi,s|a)}{\partial\xi}\Big|_{\xi\to a} \times\frac{\tilde{p}_{\text{free}}(a,s|x_0)}{\tilde{p}_{\text{free}}(a,s|a)}. \tag{70}$$

Combining this equation with an identity that results from properties of the free-space GF only [Eq. (A5)], we obtain for $x_0 > a$ the relations

$$D\frac{\partial\tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\Big|_{\xi=a} = \frac{\tilde{p}_{\text{free}}(a,s|x_0)}{\tilde{p}_{\text{free}}(a,s|a)} \tag{71}$$

$$= \frac{D\partial_\xi\tilde{p}_{\text{free}}(\xi,s|x_0)|_{\xi=a}}{1+D\partial_\xi\tilde{p}_{\text{free}}(\xi,s|a)|_{\xi\to a}} \tag{72}$$

$$= \frac{\partial_\xi\tilde{p}_{\text{free}}(\xi,s|x_0)|_{\xi=a}}{\partial_\xi\tilde{p}_{\text{free}}(\xi,s|x_0\to a)|_{\xi=a}}, \tag{73}$$

where we introduced the notation $\partial_\xi \equiv \partial/\partial\xi$. In turn, Eq. (71) gives rise to

$$D\frac{\partial^2\tilde{p}_{\text{abs}}(\xi,s|\xi_0)}{\partial\xi\,\partial\xi_0}\Big|_{\xi=\xi_0=a} = \frac{\partial_\xi\partial_{\xi_0}\tilde{p}_{\text{free}}(\xi,s|\xi_0)|_{\xi=\xi_0=a}}{\partial_\xi\tilde{p}_{\text{free}}(\xi,s|x_0\to a)|_{\xi=a}}$$

$$= \frac{\partial_\xi\tilde{p}_{\text{free}}(\xi,s|a)|_{\xi\to a}}{\tilde{p}_{\text{free}}(a,s|a)}. \tag{74}$$

Applying Eq. (71) to Eq. (69) results in the PDX of the free-space GF

$$\tilde{p}_{\text{free}}(x,s|x_0) = \tilde{p}_{\text{abs}}(x,s|x_0) + D^2\frac{\partial\tilde{p}_{\text{abs}}(x,s|\xi)}{\partial\xi}\Big|_{\xi=a} \times\tilde{p}_{\text{free}}(a,s|a)\frac{\partial\tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\Big|_{\xi=a}. \tag{75}$$

We emphasize that Eq. (69) is valid for all $-\infty < x,x_0 < \infty$. Obviously, the free-space GF $p_{\text{free}}(x,t|x_0)$ is defined on the whole infinite interval. However, as described in Sec. II A 2, there are two GF satisfying an absorbing BC at $x = a$: $p_{\text{abs}}(x,t|x_0)$ and $p^{--}_{\text{abs}}(x,t|x_0)$ defined throughout $x,x_0 \geqslant a$ and $x,x_0 \leqslant a$, respectively. Now, for $x_0 < a$, Eq. (71) is changed to

$$D\frac{\partial\tilde{p}^{--}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\Big|_{\xi=a} = -\frac{\tilde{p}_{\text{free}}(a,s|x_0)}{\tilde{p}_{\text{free}}(a,s|a)}, \tag{76}$$

as we can see from Eq. (A9). This sign change becomes apparent when we rewrite the PDX for $x,x_0 < a$,

$$\tilde{p}_{\text{free}}(x,s|x_0) = \tilde{p}^{--}_{\text{abs}}(x,s|x_0) + D^2\frac{\partial\tilde{p}^{--}_{\text{abs}}(x,s|\xi)}{\partial\xi}\Big|_{\xi=a} \times\tilde{p}_{\text{free}}(a,s|a)\frac{\partial\tilde{p}^{--}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\Big|_{\xi=a}, \tag{77}$$

as follows:

$$\tilde{p}_{\text{free}}(x,s|x_0) = \tilde{p}^{--}_{\text{abs}}(x,s|x_0) - D\,\tilde{p}_{\text{free}}(x,s|a)\frac{\partial\tilde{p}^{--}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\Big|_{\xi=a}. \tag{78}$$

The minus sign on the right-hand side appears for $x,x_0 < a$ only.

Neither of the two absorbing boundary condition GF can describe any path crossing $x = a$, for instance any path connecting a $x < a$ and a $x_0 > a$. Therefore, the corresponding PDX for $p_{\mathrm{free}}(x,t|x_0)$ should not contain p_{abs} itself but only derivatives thereof. This expectation is met as one can explicitly confirm. Again, by virtue of Eq. (69), we find that the PDX of p_{free} assumes for $x < a$ and $x_0 > a$ the form

$$\tilde{p}_{\mathrm{free}}(x,s|x_0) = -D^2 \frac{\partial \tilde{p}^{--}_{\mathrm{abs}}(x,s|\xi)}{\partial \xi}\bigg|_{\xi=a} \times \tilde{p}_{\mathrm{free}}(a,s|a) \frac{\partial \tilde{p}_{\mathrm{abs}}(\xi,s|x_0)}{\partial \xi}\bigg|_{\xi=a}, \tag{79}$$

where we also used Eqs. (71) and (76). A corresponding formula describes the case $x > a, x_0 < a$.

We point out that in the context of the VR model the free-space GF and $p^{--}_{\Theta,\mathrm{abs}}$ satisfy Eqs. (69), (78), and (79) as well. However, note that the free-space GF p_{free} means here the GF solution of Eqs. (28) and (5) that in the present context corresponds to a unperturbed potential $V(x) = \kappa_r$ instead of $V(x) = 0$ [Eq. (67)].

Next, we turn to the GF describing a purely reflecting BC. Because we will apply later the relations of p_{ref} to the SCK approach only, we limit our considerations to $x,x_0 > a$ for this boundary condition. We proceed as in the absorbing BC case and make use of the fact that an infinitely repulsive δ' potential implements a reflecting boundary [31]. Hence, we consider the free particle as unperturbed problem and take the limit of an infinitely repulsive δ' perturbation,

$$V(x) = 0,$$
$$V_{\mathrm{pert}}(x) = -\lim_{\beta \to -\infty} \beta \delta'(x-a). \tag{80}$$

Then, one obtains from Eq. (66),

$$\tilde{p}_{\mathrm{ref}}(x,s|x_0) = \tilde{p}_{\mathrm{free}}(x,s|x_0) - \frac{\partial_\xi \tilde{p}_{\mathrm{free}}(x,s|\xi)|_{\xi=a} \partial_\xi \tilde{p}_{\mathrm{free}}(\xi,s|x_0)|_{\xi=a}}{\partial_\xi \partial_{\xi_0} \tilde{p}_{\mathrm{free}}(\xi,s|\xi_0)|_{\xi=\xi_0=a}}. \tag{81}$$

Employing Eqs. (A8) and (A5), we conclude from Eq. (81) that

$$\tilde{p}_{\mathrm{ref}}(a,s|x_0) = -\frac{1}{D} \frac{\tilde{p}_{\mathrm{free}}(a,s|x_0)}{\partial_\xi \tilde{p}_{\mathrm{free}}(a,s|\xi)|_{\xi \to a}}, \tag{82}$$

and hence, using Eq. (71), we arrive at

$$\frac{\tilde{p}_{\mathrm{ref}}(a,s|x_0)}{\tilde{p}_{\mathrm{ref}}(a,s|a)} = \frac{\tilde{p}_{\mathrm{free}}(a,s|x_0)}{\tilde{p}_{\mathrm{free}}(a,s|a)} = D \frac{\partial \tilde{p}_{\mathrm{abs}}(\xi,s|x_0)}{\partial \xi}\bigg|_{\xi=a} = \frac{\partial_\xi \tilde{p}_{\mathrm{free}}(\xi,s|x_0)|_{\xi=a}}{\partial_\xi \tilde{p}_{\mathrm{free}}(\xi,s|x_0 \to a)|_{\xi=a}}. \tag{83}$$

In addition, Eq. (82) tells us that

$$\partial_\xi \tilde{p}_{\mathrm{ref}}(a,s|\xi)|_{\xi \to a} = -\frac{1}{D}. \tag{84}$$

Another consequence of Eq. (81) is the relation

$$\tilde{p}_{\mathrm{ref}}(a,s|a) = -\frac{1}{D^2} \frac{1}{\partial_\xi \partial_{\xi_0} \tilde{p}_{\mathrm{abs}}(\xi,s|\xi_0)|_{\xi=\xi_0=a}}, \tag{85}$$

where we have used Eqs. (82) and (74). We note that Eqs. (84) and (85) and the relation

$$\frac{\tilde{p}_{\mathrm{ref}}(a,s|x_0)}{\tilde{p}_{\mathrm{ref}}(a,s|a)} = D \frac{\partial \tilde{p}_{\mathrm{abs}}(\xi,s|x_0)}{\partial \xi}\bigg|_{\xi=a} \tag{86}$$

[see Eq. (83)] have also been obtained by other means [32]. Next, we consider a potential of the form

$$W(x) = V(x) - \gamma_{\mathrm{ref}} \delta'(x-a), \tag{87}$$

where the unperturbed potential is given by a δ function, that is, $V(x) = -\gamma_{\mathrm{abs}} \delta(x-a)$. Then, we take the limit of an infinitely repulsive δ and δ' potential. More precisely, we consider $\lim_{\gamma_{\mathrm{abs}},\gamma_{\mathrm{ref}} \to -\infty} W(x)$ and obtain

$$\tilde{p}_{\mathrm{ref}}(x,s|x_0) = \tilde{p}_{\mathrm{abs}}(x,s|x_0) - \frac{\partial_\xi \tilde{p}_{\mathrm{abs}}(x,s|\xi)|_{\xi=a} \partial_\xi \tilde{p}_{\mathrm{abs}}(\xi,s|x_0)|_{\xi=a}}{\partial_\xi \partial_{\xi_0} \tilde{p}_{\mathrm{abs}}(\xi,s|\xi_0)|_{\xi=\xi_0=a}}. \tag{88}$$

Then, by making use of Eq. (85), we obtain the PDX for $\tilde{p}_{\mathrm{ref}}(x,s|x_0)$,

$$\tilde{p}_{\mathrm{ref}}(x,s|x_0) = \tilde{p}_{\mathrm{abs}}(x,s|x_0) + D^2 \partial_\xi \tilde{p}_{\mathrm{abs}}(x,s|\xi)|_{\xi=a} \times \tilde{p}_{\mathrm{ref}}(a,s|a) \partial_\xi \tilde{p}_{\mathrm{abs}}(\xi,s|x_0)|_{\xi=a}. \tag{89}$$

We note that the considered potential [Eq. (87)] also gives rise to the relation

$$\tilde{p}_{\mathrm{abs}}(x,s|x_0) = \tilde{p}_{\mathrm{ref}}(x,s|x_0) - \frac{\tilde{p}_{\mathrm{ref}}(x,s|a)\tilde{p}_{\mathrm{ref}}(a,s|x_0)}{\tilde{p}_{\mathrm{ref}}(a,s|a)}, \tag{90}$$

if we consider $V(x)$ as perturbation and $\gamma_{\mathrm{ref}} \delta'(x-a)$ as the unperturbed potential.

A. Irreversible reaction

1. Contact reactivity: Smoluchowski-Collins-Kimball

Before turning to the case of an irreversible reaction described by a radiation BC, we will prove a more general statement:

Assume that the GF of the perturbed $\tilde{p}_W(x,s|x_0)$ and unperturbed $\tilde{p}_V(x,s|x_0)$ system satisfy Eq. (62). In addition, the propagator of the unperturbed problem $\tilde{p}_V(x,s|x_0)$ can be expressed in terms of a PDX. Then, $\tilde{p}_W(x,s|x_0)$ fulfills a PDX as well.

The proof is straightforward. First, note that the PDX of $\tilde{p}_V$ in particular means that

$$\tilde{p}_V(x,s|a) = D \frac{\partial \tilde{p}_{\mathrm{abs}}(x,s|\xi)}{\partial \xi}\bigg|_{\xi=a} \tilde{p}_V(a,s|a). \tag{91}$$

This result, together with Eqs. (63) and the PDX of $\tilde{p}_V(x,s|x_0)$, leads to the PDX of $\tilde{p}_W(x,s|x_0)$.

It has been shown that the GF $\tilde{p}_{\mathrm{rad}}(x,s|x_0)$ corresponding to a radiation BC enjoys a path integral representation, provided one considers a δ-function potential in the presence of a Neumann (=reflecting) BC [48,49]. Put differently, this means that $\tilde{p}_{\mathrm{rad}}(x,s|x_0)$ is obtained from the potential $W(x) = V(x) + V_{\mathrm{pert}}(x)$, where the unperturbed potential and the perturbation are given by

$$V(x) = -\lim_{\beta \to -\infty} \beta \delta'(x-a), \tag{92}$$

$$V_{\mathrm{pert}}(x) = \kappa_a \delta(x-a), \tag{93}$$

respectively. Using once again the general convolution formula Eq. (62), we arrive at

$$\tilde{p}_{\rm rad}(x,s|x_0) = \tilde{p}_{\rm ref}(x,s|x_0) - \kappa_a \frac{\tilde{p}_{\rm ref}(x,s|a)\tilde{p}_{\rm ref}(a,s|x_0)}{1+\kappa_a \tilde{p}_{\rm ref}(a,s|a)}. \tag{94}$$

We point out that Eq. (94) has been derived before [32,50] in a different way.

Because $\tilde{p}_{\rm ref}(x,s|x_0)$ has been shown to satisfy a PDX, we can conclude that this holds true for $\tilde{p}_{\rm rad}(x,s|x_0)$ as well,

$$\begin{aligned}\tilde{p}_{\rm rad}(x,s|x_0) = \tilde{p}_{\rm abs}(x,s|x_0) + D^2 \partial_\xi \tilde{p}_{\rm abs}(x,s|\xi)|_{\xi=a}\\ \times \tilde{p}_{\rm rad}(a,s|a)\partial_\xi \tilde{p}_{\rm abs}(\xi,s|x_0)|_{\xi=a}.\end{aligned} \tag{95}$$

Now we can write the propagator satisfying a radiation BC as a product of three important waiting time probability densities. In fact, using Eqs. (11)–(13) as well as Eq. (95), we arrive at

$$\begin{aligned}&\tilde{p}_{\rm rad}(x,s|x_0) - \tilde{p}_{\rm abs}(x,s|x_0)\\ &\quad = \kappa_a^{-1} \tilde{f}_{\rm LR}(s|x)\tilde{f}_{\rm reb}(s)\tilde{f}_{\rm FP}(s|x_0) &(96)\\ &\quad = -\kappa_a^{-1} \frac{\partial \widetilde{S_{\rm abs}}}{\partial t}(s|x)\frac{\partial \widetilde{S_{\rm rad}}}{\partial t}(s|a)\frac{\partial \widetilde{S_{\rm abs}}}{\partial t}(s|x_0). &(97)\end{aligned}$$

This form of the PDX makes its physical content quite clear: The trajectory of a molecule that encounters the reactive boundary at least once can be decomposed in three qualitatively different paths that are characterized by the first-passage, the rebinding, and last-passage time; see also Fig. 3. Next, we note that, due to

$$1 - \frac{D^2}{\kappa_a}\partial_\xi \partial_{\xi_0} \tilde{p}_{\rm abs}(\xi,s|\xi_0)|_{\xi=\xi_0=a} = \frac{1}{\kappa_a \tilde{p}_{\rm rad}(a,s|a)}, \tag{98}$$

see Eqs. (85), (91), and (94), the PDX for $\tilde{p}_{\rm rad}$ gives rise to a representation of $\tilde{p}_{\rm rad}$ in terms of $\tilde{p}_{\rm abs}$ only,

$$\begin{aligned}\tilde{p}_{\rm rad}(x,s|x_0) &= \tilde{p}_{\rm abs}(x,s|x_0)\\ &\quad + \frac{D^2}{\kappa_a}\frac{\partial_\xi \tilde{p}_{\rm abs}(x,s|\xi)|_{\xi=a}\partial_{\xi_0}\tilde{p}_{\rm abs}(\xi_0,s|x_0)|_{\xi_0=a}}{1-\frac{D^2}{\kappa_a}\partial_\xi\partial_{\xi_0}\tilde{p}_{\rm abs}(\xi,s|\xi_0)|_{\xi=\xi_0=a}}.\end{aligned} \tag{99}$$

2. *Contact reactivity: δ-function potential*

To analyze the δ-function potential model, we consider the free-particle case, that is $V(x)=0$, perturbed by $V_{\rm pert}(x) = \gamma_a \delta(x-a), 0 < \gamma_a < \infty$. Then, by virtue of Eq. (62), it follows that

$$\tilde{p}_\delta(x,s|x_0) = \tilde{p}_{\rm free}(x,s|x_0) - \gamma_a \frac{\tilde{p}_{\rm free}(x,s|a)\tilde{p}_{\rm free}(a,s|x_0)}{1+\gamma_a \tilde{p}_{\rm free}(a,s|a)}. \tag{100}$$

Note that Eq. (100) is valid for all components $\tilde{p}_\delta^{++}$, $\tilde{p}_\delta^{+-}$, $\tilde{p}_\delta^{-+}$, $\tilde{p}_\delta^{--}$. As a consequence, together with Eqs. (63) and (71), we arrive at

$$\frac{\tilde{p}_\delta(a,s|x_0)}{\tilde{p}_\delta(a,s|a)} = \frac{\tilde{p}_{\rm free}(a,s|x_0)}{\tilde{p}_{\rm free}(a,s|a)} = D\frac{\partial \tilde{p}_{\rm abs}(\xi,s|x_0)}{\partial \xi}\bigg|_{\xi=a}, \quad x_0 > a, \tag{101}$$

$$-\frac{\tilde{p}_\delta(a,s|x_0)}{\tilde{p}_\delta(a,s|a)} = D\frac{\partial \tilde{p}_{\rm abs}^{--}(\xi,s|x_0)}{\partial \xi}\bigg|_{\xi=a}, \quad x_0 < a. \tag{102}$$

Also, note that continuity requirements impose

$$\tilde{p}_\delta^{++}(x,s|a) = \tilde{p}_\delta^{+-}(x,s|a), \quad x > a, \tag{103}$$

$$\tilde{p}_\delta^{--}(x,s|a) = \tilde{p}_\delta^{-+}(x,s|a), \quad x < a, \tag{104}$$

$$\tilde{p}_\delta^{++}(a,s|a) = \tilde{p}_\delta^{+-}(a,s|a) = \tilde{p}_\delta^{--}(a,s|a) = \tilde{p}_\delta^{-+}(a,s|a). \tag{105}$$

Now, we are sufficiently prepared to turn to the actual proof of the PDX version that relates $\tilde{p}_{\rm abs}$ with $\tilde{p}_\delta^{++}$. To this end, we suggest the following construction. Starting point is the potential $W(x) = V(x) - \gamma_{\rm abs}\delta(x-a_{\rm abs})$, where $V(x) = \gamma_a \delta(x-a)$, that is, we have a δ-function potential perturbed by a second δ-function potential localized at a different position. We assume that $a_{\rm abs} < a$. Then, the propagators $\tilde{p}_W(x,s|x_0)$

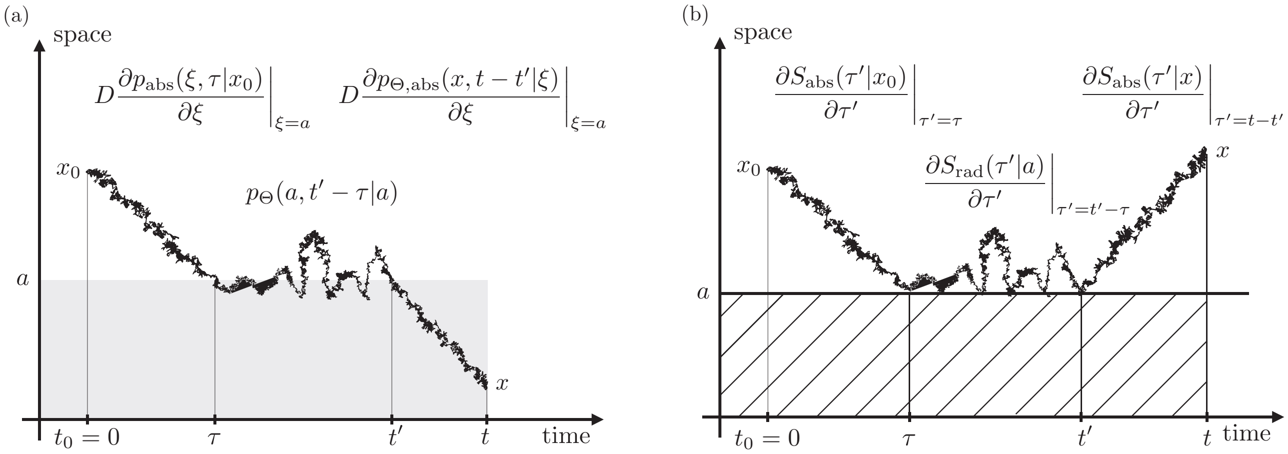

FIG. 3. Symbolic depiction of the path decomposition expansion. (a) A path contributing to the propagator $p_\Theta(x,t|x_0)$ and the path's dissection into different parts corresponding to the terms defining the PDX [Eq. (114)]. (b) In the case of the SCK model, the individual expressions allow for an interpretation as first-, last-passage and rebinding time probability densities [Eqs. (96) and (97)]. Here, trajectories can reach, but not cross the boundary.

and $\tilde{p}_V(x,s|x_0)$ satisfy

$$\tilde{p}_W(x,s|x_0) = \tilde{p}_V(x,s|x_0) + \gamma_{\text{abs}} \frac{\tilde{p}_V(x,s|a_{\text{abs}})\tilde{p}_V(a_{\text{abs}},s|x_0)}{1-\gamma_{\text{abs}}\tilde{p}_V(a_{\text{abs}},s|a_{\text{abs}})}. \tag{106}$$

Note that we have for the unperturbed GF $\tilde{p}_V(x,s|x_0) = \tilde{p}_\delta^{++}(x,s|x_0)$, while $\tilde{p}_W(x,s|x_0)$ is not equal to any of the GF we introduced before. However, as a next step, we consider the limit $a_{\text{abs}} \to a$ and $\gamma_{\text{abs}} \to -\infty$. In this limit, the potential $W(x)$ assumes the form corresponding to a purely absorbing BC, that is $W(x) = -\lim_{\gamma\to-\infty}\gamma\delta(x-a)$. Therefore, Eq. (106) becomes

$$\tilde{p}_{\text{abs}}(x,s|x_0) = \tilde{p}_\delta^{++}(x,s|x_0) - \frac{\tilde{p}_\delta^{++}(x,s|a)\tilde{p}_\delta^{++}(a,s|x_0)}{\tilde{p}_\delta^{++}(a,s|a)}. \tag{107}$$

Now, using Eq. (101), the PDX formula for $\tilde{p}_\delta^{++}$ immediately results from the just established convolution relation

$$\tilde{p}_\delta^{++}(x,s|x_0) = \tilde{p}_{\text{abs}}(x,s|x_0) + D^2 \frac{\partial \tilde{p}_{\text{abs}}(x,s|\xi)}{\partial\xi}\bigg|_{\xi=a} \times \tilde{p}_\delta^{++}(a,s|a)\frac{\partial \tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\bigg|_{\xi=a}. \tag{108}$$

The PDX of p_δ^{--} has the same form, up to the subsititution $p_{\text{abs}} \to p_{\text{abs}}^{--}$. We turn now to the case that $x<a$, $x_0>a$. Invoking Eq. (102) and using the fact that p_{abs} has to vanish in this case, we find

$$\tilde{p}_\delta^{-+}(x,s|x_0) = -D^2 \frac{\partial \tilde{p}_{\text{abs}}^{--}(x,s|\xi)}{\partial\xi}\bigg|_{\xi=a} \times \tilde{p}_\delta(a,s|a)\frac{\partial \tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\bigg|_{\xi=a}, \tag{109}$$

resembling Eq. (79). Finally, we can relate p_{rad} with p_δ. In fact, nothing prevents us from considering the δ-function term [Eq. (93)] as unperturbed potential and the δ' term [Eq. (92)] as perturbation. Then, we obtain

$$\tilde{p}_{\text{rad}}(x,s|x_0) = \tilde{p}_\delta^{++}(x,s|x_0) - \frac{\partial_{\xi_0}\tilde{p}_\delta^{+-}(x,s|\xi_0)|_{\xi_0=a}\,\partial_\xi\tilde{p}_\delta^{-+}(\xi,s|x_0)|_{\xi=a}}{\partial_\xi\partial_{\xi_0}\tilde{p}_\delta^{+-}(\xi,s|\xi_0)|_{\xi=\xi_0=a}}. \tag{110}$$

3. *Volume reactivity: Θ-function potential*

Having discussed two different types of CR theories, we now consider the VR model where the potential assumes the form of a Heaviside-step function $\Theta(a-x)$. In fact, our analysis in this case can be performed analogously to the case of p_δ, in particular, we will obtain verbatim the Eqs. (107), (108), and (109) for p_Θ. However, there are also significant differences. For VR models, the approach is somewhat less powerful than for CR models. Those potential problems studied so far, where the reactive sink term plays the role of the perturbation, cannot be extended to the VR case, because it relies on the assumption of a δ-function perturbation. A case in point is Eq. (100), a corresponding identity cannot be derived for the VR model. On the other hand, all identities that rest on the assumption that the potential describing the reaction appears as the unperturbed part of the potential, can analogously derived for the VR case. For instance, using the same line of reasoning as for the CR δ-function model, Eq. (106) remains valid, if one assumes an unperturbed potential of the form $V = \kappa_r\Theta(a-x)$. Thus, we arrive at

$$\tilde{p}_{\text{abs}}(x,s|x_0) = \tilde{p}_\Theta^{++}(x,s|x_0) - \frac{\tilde{p}_\Theta^{++}(x,s|a)\tilde{p}_\Theta^{++}(a,s|x_0)}{\tilde{p}_\Theta^{++}(a,s|a)}. \tag{111}$$

As a result of Eq. (111), we obtain

$$D\tilde{p}_\Theta^{++}(a,s|a)\frac{\partial \tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\bigg|_{\xi=a} = \tilde{p}_\Theta^{++}(a,s|x_0), \tag{112}$$

where we also made use of the fact that Eq. (A5) applies to p_Θ as well. Equipped with that identity, the PDX immediately follows from Eq. (111)

$$\tilde{p}_\Theta^{++}(x,s|x_0) = \tilde{p}_{\text{abs}}(x,s|x_0) + D^2 \frac{\partial \tilde{p}_{\text{abs}}(x,s|\xi)}{\partial\xi}\bigg|_{\xi=a} \times \tilde{p}_\Theta^{++}(a,s|a)\frac{\partial \tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\bigg|_{\xi=a}. \tag{113}$$

Similarly, and completely analogously to the δ-function potential model, we can derive the PDX for p_Θ^{--}, p_Θ^{-+}, p_Θ^{+-}. Again we find that for $x<a,x_0>a$ and $x>a,x_0<a$ the PDX assumes a somewhat different form. For instance, one finds

$$\tilde{p}_\Theta^{-+}(x,s|x_0) = -D^2 \frac{\partial \tilde{p}_{\Theta,\text{abs}}^{--}(x,s|\xi)}{\partial\xi}\bigg|_{\xi=a} \times \tilde{p}_\Theta(a,s|a)\frac{\partial \tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\bigg|_{\xi=a}, \tag{114}$$

again quite reminiscent of the CR model [Eq. (109)]. Note also that p_Θ obeys the same continuity requirements as p_δ [Eqs. (103)–(105)].

B. Reversible reaction

1. *Contact reactivity: Smoluchowski-Collins-Kimball*

We can now apply the general formula [Eq. (62)] to the solution of Eq. (43) and find

$$\tilde{p}_{\text{rev}}(x,s|x_0) = \tilde{p}_{\text{rad}}(x,s|x_0) + g\frac{\tilde{p}_{\text{rad}}(x,s|a)\tilde{p}_{\text{rad}}(a,s|x_0)}{1-g\tilde{p}_{\text{rad}}(a,s|a)}, \tag{115}$$

where we have introduced $g = g(s) = \kappa_a\tilde{\psi}(s)$; see Eq. (34). Next, because $\tilde{p}_{\text{rad}}(x,s|x_0)$ satisfies a PDX, the general result obtained before tells us immediately that $\tilde{p}_{\text{rev}}(x,s|x_0)$ satisfies a PDX expansion as well,

$$\tilde{p}_{\text{rev}}(x,s|x_0) = \tilde{p}_{\text{abs}}(x,s|x_0) + D^2 \frac{\partial \tilde{p}_{\text{abs}}(x,s|\xi)}{\partial\xi}\bigg|_{\xi=a} \times \tilde{p}_{\text{rev}}(a,s|a)\frac{\partial \tilde{p}_{\text{abs}}(\xi,s|x_0)}{\partial\xi}\bigg|_{\xi=a}. \tag{116}$$

Alternatively, we may interpret the complete potential $V_{\mathrm{rev}}(x,t)$ as perturbation to find

$$\tilde{p}_{\mathrm{rev}}(x,s|x_0) = \tilde{p}_{\mathrm{ref}}(x,s|x_0) - h\frac{\tilde{p}_{\mathrm{ref}}(x,s|a)\tilde{p}_{\mathrm{ref}}(a,s|x_0)}{1+h\tilde{p}_{\mathrm{ref}}(a,s|a)}, \tag{117}$$

where we have introduced $h = h(s) = \kappa_a s/(s+\kappa_d)$. Equation (117) has been obtained before by other means [11,51]. Again, we can immediately conclude that $\tilde{p}_{\mathrm{rev}}(x,s|x_0)$ fulfills a PDX because we know that $\tilde{p}_{\mathrm{ref}}(x,s|x_0)$ satisfies it as well.

Turning to the bound state as initial state, and using Eq. (40) together with the PDX for $\tilde{p}_{\mathrm{rev}}(x,s|x_0)$ [Eq. (116)] shows that also $\tilde{p}_{\mathrm{rev}}(x,s|*)$ satisfies a PDX of the form

$$\tilde{p}_{\mathrm{rev}}(x,s|*) = D\frac{\partial\tilde{p}_{\mathrm{abs}}(x,s|\xi)}{\partial\xi}\bigg|_{\xi=a}\tilde{p}_{\mathrm{rev}}(a,s|a)\tilde{\psi}(s). \tag{118}$$

The obtained result intuitively makes sense: The FP time process describing the transition from $x_0 \to a$ is replaced in the case of an initial bound state by the first dissociation time, characterizing the transition from $* \to a$. Finally, we note that Eq. (118) immediately results in the known [12] relation

$$\tilde{S}_{\mathrm{rev}}(s|*) = \tilde{\psi}(s)\tilde{S}_{\mathrm{rev}}(s|a). \tag{119}$$

We have seen that the developed approach allows to derive a number of important relations between GF in a unified and systematic manner. Here, we will provide another example of the versatility of the approach and prove convolution relations involving $S_{\mathrm{rev}}(t|x_0)$, $S_{\mathrm{rev}}(t|*)$, and $S_{\mathrm{rad}}(t|x_0)$. More precisely, we will derive the following convolution relations [12,52]:

$$S_{\mathrm{rev}}(t|x_0) = S_{\mathrm{rad}}(t|x_0) + \kappa_d\int_0^t d\tau\, S_{\mathrm{rad}}(t-\tau|a)[1-S_{\mathrm{rev}}(\tau|x_0)], \tag{120}$$

$$S_{\mathrm{rev}}(t|*) = \kappa_d\int_0^t d\tau\, S_{\mathrm{rad}}(t-\tau|a)[1-S_{\mathrm{rev}}(\tau|*)], \tag{121}$$

$$S_{\mathrm{rev}}(t|x_0) = S_{\mathrm{rad}}(t|x_0) - \int_0^t d\tau\, S_{\mathrm{rev}}(t-\tau|*)\frac{\partial S_{\mathrm{rad}}(\tau|x_0)}{\partial\tau}. \tag{122}$$

Starting point is once again Eq. (115). Using Eqs. (63) and (35), we readily arrive at

$$p_{\mathrm{rev}}(x,t|x_0) = p_{\mathrm{rad}}(x,t|x_0) + \kappa_d\int_0^t d\tau\, p_{\mathrm{rad}}(x,t-\tau|a) \times [1-S_{\mathrm{rev}}(\tau|x_0)]. \tag{123}$$

Now, we merely have to read off the convolution relation Eq. (120). Note that Eqs. (123) and (120) have been derived in Ref. [52] in a more complicated way.

Next, to prove Eq. (121), we multiply the Laplace transform of Eq. (123) by $\tilde{\psi}(s)$ and use Eqs. (40) as well as $\tilde{\psi}(s)(1+\kappa_d/s) = \kappa_d/s$ to arrive at

$$p_{\mathrm{rev}}(x,t|*) = \kappa_d\int_0^t d\tau\, p_{\mathrm{rad}}(x,t-\tau|a)[1-S_{\mathrm{rev}}(\tau|*)]. \tag{124}$$

From this, we can read off Eq. (121).

Finally, we are using Eqs. (115), (63), and (40) to conclude

$$\tilde{p}_{\mathrm{rev}}(x,s|x_0) = \tilde{p}_{\mathrm{rad}}(x,s|x_0) + \kappa_a\tilde{p}_{\mathrm{rev}}(x,s|*)\tilde{p}_{\mathrm{rad}}(a,s|x_0). \tag{125}$$

From that, we obtain

$$p_{\mathrm{rev}}(x,t|x_0) = p_{\mathrm{rad}}(x,t|x_0) - \int_0^t d\tau\, p_{\mathrm{rev}}(x,t-\tau|*)\frac{S_{\mathrm{rad}}(\tau|x_0)}{\partial\tau}, \tag{126}$$

and Eq. (122) follows.

2. *Contact reactivity: δ-function potential*

For reversible reactions described by δ-function potentials, we can proceed analogously to the SCK model. Provided we make the replacements $p_{\mathrm{rad}}(x,t|x_0) \leftrightarrow p_\delta(x,t|x_0)$ as well as $p_{\mathrm{rev}}(x,t|r_0) \leftrightarrow p_{\delta,\mathrm{rev}}(x,t|r_0)$ and correspondingly for the survival probabilities, all equations from Eqs. (115) to (126) remain valid. The only exception is Eq. (117) and the reason why it is not compatible with the δ-function model is easily understood: in the limit of no associations and dissociations, Eq. (117) tells us that p_{rev} becomes p_{ref}, which is correct in the SCK model, but not consistent with the δ-function model, where $p_{\delta,\mathrm{rev}} \to p_{\mathrm{free}}$ in the non-reactive limit, as already discussed in Sec. II A 2.

3. *Volume reactivity: Θ-function potential*

As for the irreversible reaction, the approach applied to a Θ-function model does not result in as many relations as in the CR case. In the reversible case, the difference results from the fact that no relation equivalent to Eq. (115) can be derived for the VR model. Again, the underlying reason is that Eq. (115) relies on a potential function purely composed of δ functions.

Nevertheless, the PDX can still be derived for $p_{\Theta,\mathrm{rev}}$. Following the same route that gave rise to the PDX of p_Θ described in Sec. III A 3, we find that $p_{\Theta,\mathrm{rev}}$ satisfies verbatim the PDX Eqs. (113) and (114) of its irreversible counterpart.

Using the PDX and Eq. (52), we also conclude that the GF describing the bound state as initial state enjoys the following representation for $x_0 < a$:

$$\begin{aligned}\tilde{p}_{\Theta,\mathrm{rev}}(x,s|x_0,*) &= -D^2\frac{\partial\tilde{p}_{\mathrm{abs}}(x,s|\xi)}{\partial\xi}\bigg|_{\xi=a}\tilde{p}_{\Theta,\mathrm{rev}}(a,s|a)\frac{\partial\tilde{p}^{--}_{\Theta,\mathrm{abs}}(\xi,s|x_0)}{\partial\xi}\bigg|_{\xi=a} \\ &\quad\times\tilde{\psi}(s),\quad x>a, \qquad (127)\\ &= \tilde{p}^{--}_{\Theta,\mathrm{abs}}(x,s|x_0)\tilde{\psi}(s) + D^2\frac{\partial\tilde{p}^{--}_{\mathrm{abs}}(x,s|\xi)}{\partial\xi}\bigg|_{\xi=a}\tilde{p}_{\mathrm{rev}}(a,s|a) \\ &\quad\times\frac{\partial\tilde{p}^{--}_{\mathrm{abs}}(\xi,s|x_0)}{\partial\xi}\bigg|_{\xi=a}\tilde{\psi}(s),\quad x<a. \qquad (128)\end{aligned}$$

Note that the appearance of the absorbing boundary condition GF for the case $x < a, x_0 < a$ is due to the fact that in the VR model the just dissociated molecule is not necessarily located at $x_0 = a$, and hence $p^{--}_{\Theta,\mathrm{abs}}$ does not vanish.

IV. EXAMPLES

A. Contact-reactivity models in one dimension

All relations that have been derived so far may also be verified explicitly. To check Eq. (96), we note that $\tilde{p}_{\mathrm{rad}}(x,s|x_0)$

assumes the form [53]

$$\tilde{p}_{\mathrm{rad}}(x,s|x_0) = \frac{1}{2Dq}e^{-q|x-x_0|} + \frac{q-\kappa_D}{2Dq[q+\kappa_D]}e^{-q(x+x_0)}, \tag{129}$$

where $q = \sqrt{s/D}$ and $\kappa_D = \kappa_a/D$. The GF satisfying absorbing BC is given by

$$\tilde{p}_{\mathrm{abs}}(x,s|x_0) = \frac{1}{2Dq}[e^{-q|x-x_0|} - e^{-q(x+x_0)}]. \tag{130}$$

Then, it follows that

$$\tilde{f}_{\mathrm{FP}}(s|x_0) = e^{-\sqrt{s}\frac{x_0}{\sqrt{D}}}, \tag{131}$$

$$\tilde{p}_{\mathrm{rad}}(x,s|x_0) - \tilde{p}_{\mathrm{abs}}(x,s|x_0) = \frac{1}{D[q+\kappa_D]}e^{-q(x+x_0)}, \tag{132}$$

confirming that Eq. (96) does indeed hold.

The next example we consider is Eq. (110). To proceed, we consider the Laplace-transform of the solution given in Ref. [45],

$$\tilde{p}_{\delta}^{\pm\pm}(x,s|x_0) = \frac{1}{2Dq}e^{-q|x-x_0|} - \frac{\kappa_D}{2Dq[q+\kappa_D]}e^{-q(\pm x \pm x_0)}, \tag{133}$$

$$\tilde{p}_{\delta}^{\mp\pm}(x,s|x_0) = \frac{1}{2D[q+\kappa_D]}e^{-q(\mp x \pm x_0)}. \tag{134}$$

Using these explicit expressions, we find

$$\begin{aligned}
&\tilde{p}_{\mathrm{rad}}(x,s|x_0) - \tilde{p}_{\delta}^{++}(x,s|x_0)\\
&\quad= \frac{1}{2D[q+\kappa_D]}e^{-q(x+x_0)}\\
&\quad= -\frac{\partial_{\xi_0}\tilde{p}_{\delta}^{+-}(x,s|\xi_0)|_{\xi_0=a}\,\partial_{\xi}\tilde{p}_{\delta}^{-+}(\xi,s|x_0)|_{\xi=a}}{\partial_{\xi}\partial_{\xi_0}\tilde{p}_{\delta}^{+-}(\xi,s|\xi_0)|_{\xi=\xi_0=a}}.
\end{aligned} \tag{135}$$

B. Volume reactivity model in two dimensions

The second example that we discuss is the volume reactivity model in 2D [15,21,22], also referred to as Doi model in the case of an irreversible reaction [17,18,23]. The equation of motion reads

$$\frac{\partial p_{\Theta}(r,t|r_0)}{\partial t} = \left[D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - \kappa_r\Theta(a-r)\right]p_{\Theta}(r,t|r_0). \tag{136}$$

In the following, considering the case $r_0 > a$, $r < a$, we will show that

$$\begin{aligned}
\tilde{p}_{\Theta}^{-+}(r,s|r_0) = {}& -4\pi^2a^2D^2\left.\frac{\partial\tilde{p}_{\mathrm{abs}}^{--}(r,s|\xi)}{\partial\xi}\right|_{\xi=a}\\
&\times \tilde{p}_{\Theta}(a,s|a)\left.\frac{\partial\tilde{p}_{\mathrm{abs}}(\xi,s|r_0)}{\partial\xi}\right|_{\xi=a};
\end{aligned} \tag{137}$$

see Eq. (114). Furthermore, we emphasize that, although we consider here explicitly the irreversible reaction, Eq. (137) remains valid for the reversible case. In fact, one only has to substitute $w = q\sqrt{(s+\kappa_r)/s} \to q\sqrt{(s+\kappa_r+\kappa_d)/(s+\kappa_d)}$; see Ref. [22].

$\tilde{p}_{\Theta}^{-+}(r,s|r_0)$ assumes the following form [21,22]:

$$\tilde{p}_{\Theta}^{-+}(r,s|r_0) = \frac{I_0(wr)K_0(qr_0)}{2\pi aD\mathcal{N}}, \tag{138}$$

$$\mathcal{N} = vI_0(wa)K_1(qa) + wI_1(wa)K_0(qa), \tag{139}$$

where $I_{0,1}, K_{0,1}$ refer to the modified Bessel function of first and second kind, respectively, and zeroth and first order [54, Sec. 9.6].

Let us now turn to the GF $p_{\mathrm{abs}}(r,t|r_0)$ satisfying absorbing BC. As mentioned before, one has to take into account two different GF p_{abs}, corresponding to the domains $r,r_0 > a$ and $r,r_0 < a$. For $r > a, r_0 > a$, one has in the Laplace domain [Eq. (69)],

$$\tilde{p}_{\mathrm{abs}}(r,s|r_0) = \tilde{p}_{\mathrm{free}}(r,q|r_0) - \frac{1}{2\pi D}K_0(qr)K_0(qr_0)\frac{I_0(qa)}{K_0(qa)}, \tag{140}$$

where

$$\tilde{p}_{\mathrm{free}}(r,s|r_0) = \frac{1}{2\pi D}\begin{cases} I_0(qr_0)K_0(qr), & r > r_0\\ I_0(qr)K_0(qr_0), & r < r_0. \end{cases} \tag{141}$$

It follows that

$$2\pi aD\left.\frac{\partial\tilde{p}_{\mathrm{abs}}(\xi,s|r_0)}{\partial\xi}\right|_{\xi=a} = \frac{K_0(qr_0)}{K_0(qa)}. \tag{142}$$

Next, using again Eq. (69), one has for $r < a, r_0 < a$,

$$\tilde{p}_{\Theta,\mathrm{abs}}^{--}(r,s|r_0) = \tilde{p}_{\mathrm{free}}(r,w|r_0) - \frac{1}{2\pi D}I_0(wr)I_0(wr_0)\frac{K_0(wa)}{I_0(wa)}. \tag{143}$$

Note that the appearance of w instead of q is due to the fact that within the domain $r < a$ the molecule moves in the presence of a constant potential given by $-\kappa_r$. Thus, one obtains

$$2\pi aD\left.\frac{\partial\tilde{p}_{\Theta,\mathrm{abs}}^{--}(r,s|\xi)}{\partial\xi}\right|_{\xi=a} = -\frac{I_0(wr)}{I_0(wa)}. \tag{144}$$

Finally, using Eqs. (138), (142), and (144), we can verify that indeed Eq. (137) does hold true. Similarly, one can show that Eq. (137) is fulfilled by the GF corresponding to $r,r_0 > a$ and $r,r_0 < a$, as well as by the GF describing the reversible reaction.

Finally, note that if we used for the case $r < a$ the expression $-\partial S_{\Theta,\mathrm{abs}}^{-}(t|r)/\partial t$, instead of the flux given by Eq. (144) we would not recover the full propagator, because in the VR model one has [22]

$$\frac{\partial S_{\Theta,\mathrm{abs}}^{-}(t|r_0)}{\partial t} = 2\pi aD\left.\frac{\partial p_{\Theta,\mathrm{abs}}^{--}(r,t|r_0)}{\partial r}\right|_{r=a} - \kappa_r S_{\Theta,\mathrm{abs}}^{-}(t|r_0); \tag{145}$$

see also Eq. (27).

V. BROWNIAN FUNCTIONALS

The representation of DIR in terms of path integrals and diffusion equations with nonvanishing potential paves the way for applications beyond the reaction of a pair of isolated molecules. In fact, highlighting the essential role of the

δ- and Θ-function potentials that appear in the path integral in the form $e^{-\int_0^t d\tau \delta[x(\tau)-a]}$ and $e^{-\int_0^t d\tau \Theta[a-x(\tau)]}$, respectively, suggests to consider these DIR models from the perspective of a certain class of random variables that are well-known as Brownian functionals [37,38]. Vice versa, approaches to problems initially framed in the context of Brownian functionals may benefit from the close connection to path integrals and hence techniques related to thereof. This intimate connection is established via the Feynman-Kac formula, as we will briefly recall now.

A Brownian functional T is defined as follows [37,38],

$$T = \int_0^t d\tau V[x(\tau)], \tag{146}$$

where $x(\tau)$ refers to a Brownian path, with fixed initial position, leading from x_0 to x at time t. Every realization of the Brownian path induces a different value of T and hence, T itself is a random variable along with an associated PDF, denoted by $P(T,t|x_0)$. Assuming that the potential V has positive support allows one to consider the Laplace transform of the PDF $P(T,t|x_0)$ with respect to T,

$$S(k,t,|x_0) = \int_0^\infty e^{-kT} P(T,t|x_0) dT = \langle e^{-kT} \rangle, \tag{147}$$

where $\langle\,\rangle$ indicates averaging over all Brownian paths starting at x_0 at time 0 and propagating for a time t. The function $S(k,t,|x_0)$ is the characteristic function of the PDF $P(T,t|x_0)$, that is, it generates all moments of T

$$S(k,t,|x_0) = 1 - k\langle T(t|x_0)\rangle + \frac{k\langle T^2(t|x_0)\rangle}{2!} - \cdots, \tag{148}$$

that can be obtained by taking the nth derivative of $S(k,t,|x_0)$ with respect to k,

$$\langle T^n(t|x_0)\rangle = (-1)^n \frac{\partial S(k,t|x_0)}{\partial k}\bigg|_{k=0}. \tag{149}$$

The link to the models of DIR that we have considered so far is established when one considers specifically the Brownian functionals,

$$T_\delta = \int_0^t d\tau \delta[x(\tau) - a], \tag{150}$$

$$T_\Theta = \int_0^t d\tau \Theta[a - x(\tau)], \tag{151}$$

because, as is well-known, their moment generating function $S(k,t,|x_0)$ is exactly the survival probability of a diffusing particle whose irreversible reaction is described by a CR and VR model with k corresponding to γ_a and κ_r, respectively. The random variables T_δ and T_Θ are well-known in the theory of random walks as local and occupation (also referred to as residence) time [37–39] and represent a measure characterizing the number of times the Brownian walker encounters the boundary and how much time it spends below $x = a$, respectively.

Now, the Feynman-Kac formula [37] expresses the expectation value in Eq. (147) in terms of a path integral,

$$\left\langle e^{-k\int_0^t V[x(\tau)]d\tau} \right\rangle = \int_{-\infty}^{\infty} dx \int_{x(0)}^{x(t)} \mathcal{D}x(\tau) \times e^{-\frac{1}{4D}\int_0^t \dot{x}(\tau)^2 d\tau - k\int_0^t V[x(\tau)]d\tau}. \tag{152}$$

Importantly, the path integral represents [see Eq. (60)] the GF of a diffusing particle in the presence of a δ- or a Θ-function sink. Thus, it follows that Eq. (147) is indeed, at least formally, the survival probability, even in those cases where T_δ and T_Θ model phenomena completely unrelated to actual irreversible binding.

Henceforth, we will focus on the residence time and briefly discuss its prototypical application example, single molecule fluorescent experiments [39]. The mean residence time (MRT) is a measure for the count of photons emitted by a single fluorescent dye that diffuses in a domain illuminated by a laser spot. As we have recalled above, the MRT can be calculated via the Feynman-Kac route, amounting to the following steps [37,39]. First, calculate the corresponding GF, then, the survival probability to determine in turn the PDF $P(T,t|x_0)$ via the inverted Laplace transform with respect to k [Eq. (147)]. Finally, obtain the MRT by invoking Eq. (149).

Alternatively, in case that besides the first all higher moments are of no interest, one can derive and solve the equation of motion of the MRT directly [39].

Here, we will address the question of how the presence of a true irreversible binding site influences the residence time. This modified residence time corresponds to a fluorescent molecule that may also undergo a binding reaction within the region illuminated by the laser and that ceases to fluoresce as soon as it becomes bound. Note that the theoretical description of this situation involves two Θ-function sinks concurrently: one that corresponds to the actual residence time (and hence to a purely fictitious sink) and one that accounts for the actual binding reaction (and hence represents a genuine sink).

The binding reaction itself is described by the VR model. In the present context we do not consider it as a model for the binding of two isolated molecules though, rather we use the VR formalism in a coarse-grained sense describing a reactive site in which the dye molecule can move in and out. Such a description may represent a region populated densely by multiple receptors, however, the individual existence of the receptors is not resolved but their effect is accounted for uniformly over the whole domain of interest.

Furthermore, we only consider the special case that the region illuminated by the laser spot coincides with the reactive site and that both are given by $\Theta(a - x)$.

To proceed, it is instructive to briefly recall the line of reasoning in the standard case that corresponds to no binding. The GF of the free backward diffusion equation [39],

$$\frac{\partial p_{\text{free}}(x,t|x_0)}{\partial t} = D\nabla^2_{x_0} p_{\text{free}}(x,t|x_0), \tag{153}$$

permits to calculate the MRT via

$$\langle T(t|x_0)\rangle = \int_0^t dt' \int_{-\infty}^{\infty} dx \Theta(a - x) p_{\text{free}}(x,t'|x_0). \tag{154}$$

Then, taking into account the initial condition of the GF [Eq. (4)], one arrives at the equation of motion for $\langle T(t|x_0)\rangle$ [39],

$$\frac{\partial\langle T(t|x_0)\rangle}{\partial t} = D\nabla^2_{x_0}\langle T(t|x_0)\rangle + \Theta(a - x_0). \tag{155}$$

Now, we seek to generalize the previous considerations by including irreversible binding. We start with the backward equation

$$\frac{\partial p_\Theta(x,t|x_0)}{\partial t} = D\nabla^2_{x_0} p_\Theta(x,t|x_0) - \kappa_r \Theta(a-x_0) p_\Theta(x,t|x_0), \tag{156}$$

leading immediately to the equation of motion for the generalized MRT $\langle T_{\rm irr}(t|x_0)\rangle$,

$$\frac{\partial \langle T_{\rm irr}(t|x_0)\rangle}{\partial t} = D\nabla^2_{x_0}\langle T_{\rm irr}(t|x_0)\rangle + \Theta(a-x_0) - \kappa_r\Theta(a-x_0)\langle T_{\rm irr}(t|x_0)\rangle. \tag{157}$$

This equation of motion is consistent with an alternative approach. As discussed in Sec. II A 3, the Feynman-Kac equation for $p_\Theta(x,t|x_0)$ results in the relation

$$\frac{\partial S_\Theta(t|x_0)}{\partial t} = -\kappa_r S^-_\Theta(t|x_0),$$

where $S^-_\Theta(t|x_0) = \int_{-\infty}^{\infty} dx\,\Theta(a-x)p_\Theta(x,t|x_0) = \int_{-\infty}^{a} dx\,p^{-\pm}_\Theta(x,t|x_0)$. Comparing the definition of $S^-_\Theta(t|x_0)$ with the definition of the MRT [Eq. (154)], we see that

$$S^-_\Theta(t|x_0) = \frac{\partial\langle T_{\rm irr}(t|x_0)\rangle}{\partial t}, \tag{158}$$

and hence we have

$$\langle T_{\rm irr}(t|x_0)\rangle = \frac{1}{\kappa_r} - \frac{1}{\kappa_r}S_\Theta(t|x_0). \tag{159}$$

Combining this relation between S_Θ and $\langle T_{\rm irr}(t|x_0)\rangle$ with the backward equation of motion for the survival probability [12,39],

$$\frac{\partial S_\Theta(t|x_0)}{\partial t} = D\nabla^2_{x_0} S_\Theta(t|x_0) - \kappa_r\Theta(a-x_0)S_\Theta(t|x_0), \tag{160}$$

we recover the equation of motion for the MRT in the presence of a reactive sink [Eq. (157)].

At this point, the question naturally arises whether we can obtain the generalized MRT $\langle T_{\rm irr}(t|x_0)\rangle$ via the Feynman-Kac route also. Evidently, $\langle T_{\rm irr}(t|x_0)\rangle$ cannot be obtained by taking the derivative of S_Θ; see Eq. (159). However, we heuristically assume there is a generating function for the moments of the residence time $T_{\rm irr}$ in the presence of an irreversibly reactive sink and let $Q(T_{\rm irr},t|x_0)$ denote the corresponding PDF. Then, Eqs. (147) and (159) tell us that

$$\int_0^\infty T_{\rm irr} Q(T_{\rm irr},t|x_0)dT_{\rm irr} = \frac{1}{\kappa_r}\int_0^\infty (1-e^{-\kappa_r T})P(T,t|x_0)dT. \tag{161}$$

This relation suggests defining

$$T_{\rm irr} = \frac{1-e^{-\kappa_r T}}{\kappa_r}, \tag{162}$$

and thus, we obtain

$$Q(T_{\rm irr},t|x_0) = \frac{1}{1-\kappa_r T_{\rm irr}} P\big(-\kappa_r^{-1}\ln(1-\kappa_r T_{\rm irr}),t|x_0\big)\,\Theta\big(\kappa_r^{-1}-T_{\rm irr}\big). \tag{163}$$

This means, in particular, that knowledge of the classical residence time PDF $P(T,t|x_0)$ is sufficient to construct the generalized residence time PDF that includes the possibility of irreversible reactions.

We now apply the theory developed so far to obtain a concrete expression for the generalized MRT $\langle T_{\rm irr}(t|x_0)\rangle$ and the corresponding PDF $Q(T_{\rm irr},t|x_0)$ for the special case $x_0 = a = 0$. We start with the known expression [37]

$$\tilde{S}(k,s|x_0=0) = \frac{1}{\sqrt{s}\sqrt{s+k}}. \tag{164}$$

We now exploit the dual role played by the survival probability and interpret the expression given in Eq. (164) as the survival probability of an actual irreversible reaction with $k = \kappa_r$. The inverted Laplace transform with respect to s yields

$$S_\Theta(t|x_0=0) = e^{-\frac{1}{2}t\kappa_r} I_0\big(\tfrac{1}{2}t\kappa_r\big). \tag{165}$$

We can now use Eq. (159) to immediately obtain an explicit expression for the generalized MRT,

$$\langle T_{\rm irr}(t|x_0=0)\rangle = \frac{1}{\kappa_r} - \frac{1}{\kappa_r}e^{-\frac{1}{2}t\kappa_r} I_0\left(\frac{1}{2}t\kappa_r\right). \tag{166}$$

Next, to obtain $P(T,t|x_0=0)$ we interpret the survival probability [Eq. (164)] in the classical MRT sense. Applying the inverted Laplace transform twice with respect to s and k yields [37]

$$P(T,t|x_0=0) = \frac{1}{\pi}\frac{1}{\sqrt{T(t-T)}}, \tag{167}$$

allowing us to arrive at the expression for $Q(T_{\rm irr},t|x_0=0)$, according to Eq. (163),

$$Q(T_{\rm irr},t|x_0=0) = \frac{1}{\pi}\frac{\Theta\big(\kappa_r^{-1}-T_{\rm irr}\big)}{\sqrt{-\kappa_r^{-1}\ln(1-\kappa_r T_{\rm irr})\big(t+\kappa_r^{-1}\ln(1-\kappa_r T_{\rm irr})\big)}} \times \frac{1}{1-\kappa_r T_{\rm irr}}. \tag{168}$$

Finally, we now allow for dissociation of the bound state to account for reversible binding. Evidently, in this case, a binding reaction does not necessarily cause the residence time stop increasing. Following an analogous line of reasoning that led to Eq. (159), we arrive at an identity that relates the MRT in the presence of reversible binding to the reversible survival probability,

$$\langle \tilde{T}_{\rm rev}(s|x_0)\rangle = \frac{1}{s\kappa(s)} - \frac{1}{\kappa(s)}\tilde{S}_{\Theta,\rm rev}(s|x_0), \tag{169}$$

where we have defined $\kappa(s) = \kappa_r s/(s+\kappa_d)$. Thus, we find that knowledge of the MRT in the presence of irreversible binding is sufficient to obtain $\langle\tilde{T}_{\rm rev}(s|x_0)\rangle$, the only necessary change is to perform the substitution $\kappa_r \to \kappa_r s/(s+\kappa_d)$ (that also yields $\tilde{S}_\Theta \to \tilde{S}_{\Theta,\rm rev}$). In terms of the classical residence time PDF, one has

$$\langle\tilde{T}_{\rm rev}(s|x_0)\rangle = \frac{s+\kappa_d}{\kappa_r s}\int_0^\infty (1-e^{-\kappa(s)T})\tilde{P}(T,s|x_0)dT. \tag{170}$$

Note that the term

$$\int_0^\infty e^{-\kappa(s)T}\tilde{P}(T,s|x_0)dT$$

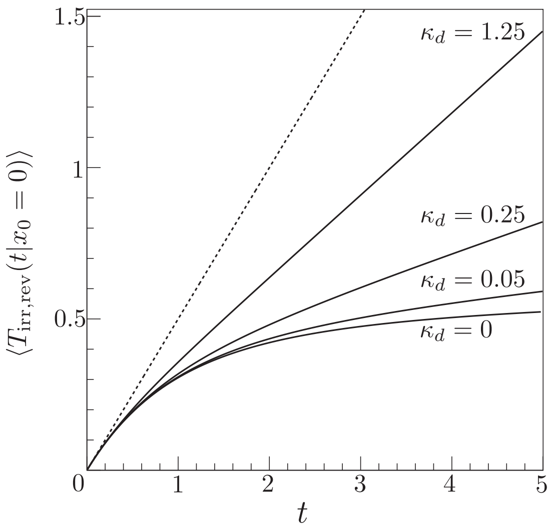

FIG. 4. The generalized mean residence time for the case of irreversible and reversible binding. The figure shows the dependence of $\langle T_{\mathrm{irr}}(t|x_0=0)\rangle$ and $\langle T_{\mathrm{rev}}(t|x_0=0)\rangle$ on the observed time t. The dashed line refers to the classical MRT without any binding ($\kappa_r = \kappa_d = 0$). The solid lines are obtained from Eqs. (166) and (171) for various values of the dissociation constant $\kappa_d = 0.0,\ 0.05,\ 0.25,\ 1.25$. The association constant is for all solid curves $\kappa_r = 1.5$.

corresponds exactly to the path-integral expression for the GF $p_{\Theta,\mathrm{rev}}$ describing reversible binding; see Eqs. (51), (60), and (152). With Eqs. (164) and (169) at our disposal, we find the following expression for the MRT in the presence of reversible binding,

$$\begin{aligned}\langle T_{\mathrm{rev}}(t|x_0=0)\rangle &\\ = \frac{1}{\kappa_r} + \frac{\kappa_d}{\kappa_r}t &\\ -\frac{1}{\kappa_r}\left[m(t)+2\kappa_d\int_0^t m(t')dt' + \kappa_d^2\int_0^t (t-t')m(t')dt'\right]&,\end{aligned} \tag{171}$$

where we have defined

$$m(t) = e^{-\frac{1}{2}t\kappa_r-\kappa_d t}I_0\left(\tfrac{1}{2}t\kappa_r\right). \tag{172}$$

Figure 4 shows the generalized MRT as a function of the observation time t. For comparison, we plotted the classical MRT without binding given by $t/2$. As expected, we find that in the irreversible case the MRT approaches the value κ_r^{-1} for $t \to \infty$. As soon as one allows for dissociations the resulting mean residence times "interpolate" with varying κ_d between the two limiting cases *no binding* versus *irreversible binding*. For $\kappa_d \to \infty$ one has $\langle T_{\mathrm{rev}}(t|x_0)\rangle = \langle T(t|x_0)\rangle$, as it should be.

In Fig. 5, we plotted the residence time PDF in the presence of irreversible binding dependent on the residence time T_{irr} and for different values of the intrinsic binding constant κ_r. Evidently, with increasing κ_r the PDF becomes more and more narrow, because the peak of the classical MRT given by $t = T$ is shifted to $t = -\kappa_r^{-1}\ln(1-\kappa_r T_{\mathrm{irr}})$.

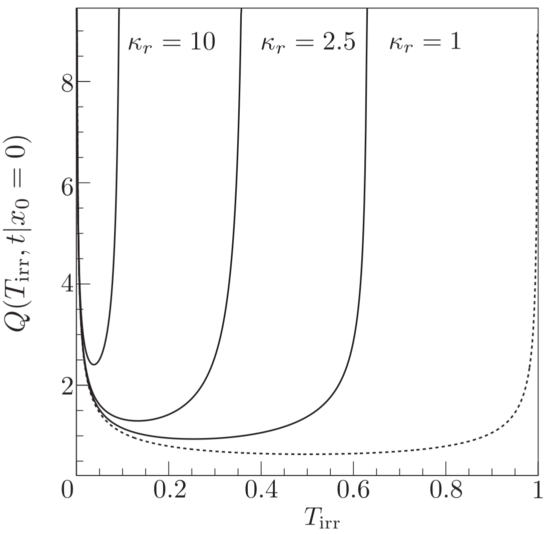

FIG. 5. The probability density function of the residence time for the case of irreversible binding. The figure shows the T_{irr} dependence of $Q(T_{\mathrm{irr}},t|x_0=0)$ for various values of the association rate $\kappa_r = 1.0,\ 2.5,\ 10.0$. The observation time is $t = 1.0$. The dashed line refers to the probability density $P(T,t|x_0=0)$ [Eq. (167)]. The solid lines are obtained from Eq. (168). As expected, $Q(T_{\mathrm{irr}},t|x_0=0)$ assumes its second peak for smaller T_{irr} with increasing κ_r.

VI. CONCLUSIONS

In this work, borrowing techniques from quantum mechanics, we have discussed a path-integral approach that encompasses not only the classical SCK model, but also alternative theories, notably the VR model in a unified and systematic fashion. The salient idea is to utilize a powerful relation between boundary value problems and systems involving δ- and δ'-function perturbations. We have showed that a number of important identities establishing deep links between different GF can be ultimately traced back to two known path-integral based exact summation formulas of the corresponding perturbation series [Eqs. (62) and (66)]. In turn, these identities on the level of GF give rise to relations between coarse-grained quantities, as exemplified here for different survival probabilities that have played an important role in the development of the theory of DIR.

The PDX plays a key role in the presented approach. In fact, other relations that have previously been derived by employing various differential equation techniques appear here as a direct result of the PDX. Also, while the PDX can be proven for all considered models, there are a couple of resulting relations that make sense only for the SCK model. This can be easily seen from the path-integral point of view: the SCK propagators $p_{\mathrm{rad}}(x,t|x_0)$ and $p_{\mathrm{rev}}(x,t|x_0)$ correspond to a system involving only the δ-function and its derivative as interaction potentials, thus enabling the repeated application of the convolution relation. By contrast, VR theories are based on Θ-function potentials, rendering the crucial exact perturbative expansion somewhat less powerful in this case.

Besides the conceptual advances, the derived relations between propagators and survival probabilities provide a way to compute complicated GF and from thereof derived quantities without the need to solve the corresponding partial differential equations or path integrals. Instead, they may be constructed from quantities that typically can be more easily obtained than the full propagator. In fact, this strategy has been employed for quantum-mechanical systems, where the PDX has been used to calculate complicated propagators [27].

Furthermore, a substantial body of work has been devoted to study quantum mechanical δ- and Θ-function problems [29–31]. Notably, many systems involving δ-function potentials allow for a path-integral solution, leading to the expectation that at least some of the results can be carried over in a DIR context.

Moreover, we expect that the discussed approach will help answer to what extent and for which parameter ranges the alternative models of DIR yield significantly different predictions [15,21].

Finally, we emphasize the approach's versatility, making it potentially applicable beyond theories of DIR, as we have exemplified here by discussing the relation to the theory of Brownian functionals. Obviously, it is desirable to extend the chosen toy example to make it more amenable for a realistic application. The corresponding analytical expressions of the VR propagators in 3D and 2D, respectively, have been worked out already [20,21] and should provide a suitable starting point to develop a framework of generalized residence times for more realistic systems along the lines discussed here.

In conclusion, we anticipate that our obtained results provide multiple essential theoretical tools that help create more powerful and accurate computational models of reaction diffusion processes, in particular in the field of cell biological signaling phenomena.

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APPENDIX

Let us define

$$u(x,s) \equiv \tilde{p}_{\text{free}}(x,s|x_0), \tag{A1}$$

$$v(x,s) \equiv \tilde{p}_{\text{free}}(x,s|a). \tag{A2}$$

Then, $u(x,s)$ and $v(x,s)$ satisfy the free-space diffusion equation,

$$su(x,s) - \delta(x - x_0) = D\frac{\partial^2 u(x,s)}{\partial x^2}, \tag{A3}$$

$$sv(x,s) - \delta(x - a) = D\frac{\partial^2 v(x,s)}{\partial x^2}. \tag{A4}$$

Next, assuming $x,x_0 > a$, multiplying Eq. (A3) with $v(x,s)$ and Eq. (A4) with $u(x,s)$, as well as substracting the resulting equations and integrating the outcome from a to ∞ leads to

$$\tilde{p}_{\text{free}}(a,s|x_0) = D\left[\tilde{p}_{\text{free}}(a,s|a)\frac{\partial \tilde{p}_{\text{free}}(\xi,s|x_0)}{\partial \xi}\bigg|_{\xi=a} - \tilde{p}_{\text{free}}(a,s|x_0)\frac{\partial \tilde{p}_{\text{free}}(\xi,s|a)}{\partial \xi}\bigg|_{\xi\to a}\right], \tag{A5}$$

where we have used the definitions Eqs. (A1) and (A2), the BC at infinity, $\lim_{x\to\infty} \tilde{p}_{\text{free}}(x,s|x_0) \to 0$, and $\tilde{p}_{\text{free}}(x,s|x_0) = \tilde{p}_{\text{free}}(x_0,s|x)$. Taking the limit $x_0 \to a$, it follows that

$$\partial_\xi \tilde{p}_{\text{free}}(\xi,s|x_0 \to a)|_{\xi=a} - \partial_\xi \tilde{p}_{\text{free}}(\xi,s|a)|_{\xi\to a} = D^{-1}. \tag{A6}$$

Furthermore, dividing Eq. (A5) by $\tilde{p}_{\text{free}}(a,s|a)$ and using Eq. (A6), we obtain the chain of identities

$$\frac{\tilde{p}_{\text{free}}(a,s|x_0)}{\tilde{p}_{\text{free}}(a,s|a)} = \frac{D\partial_\xi \tilde{p}_{\text{free}}(\xi,s|x_0)|_{\xi=a}}{1 + D\partial_\xi \tilde{p}_{\text{free}}(\xi,s|a)|_{\xi\to a}} = \frac{\partial_\xi \tilde{p}_{\text{free}}(\xi,s|x_0)|_{\xi=a}}{\partial_\xi \tilde{p}_{\text{free}}(\xi,s|x_0 \to a)|_{\xi=a}}. \tag{A7}$$

Finally, as a consequence of Eq. (A7), we arrive at

$$\frac{\partial_\xi \partial_{\xi_0} \tilde{p}_{\text{free}}(\xi,s|\xi_0)|_{\xi=\xi_0=a}}{\partial_\xi \tilde{p}_{\text{free}}(\xi,s|x_0 \to a)|_{\xi=a}} = \frac{\partial_\xi \tilde{p}_{\text{free}}(\xi,s|a)|_{\xi\to a}}{\tilde{p}_{\text{free}}(a,s|a)}. \tag{A8}$$

Finally, when we assume that $x,x_0 < a$, we obtain the same result as for $x,x_0 > a$ up to a minus sign

$$-\tilde{p}_{\text{free}}(a,s|x_0) = D\left[\tilde{p}_{\text{free}}(a,s|a)\frac{\partial \tilde{p}_{\text{free}}(\xi,s|x_0)}{\partial \xi}\bigg|_{\xi=a} - \tilde{p}_{\text{free}}(a,s|x_0)\frac{\partial \tilde{p}_{\text{free}}(\xi,s|a)}{\partial \xi}\bigg|_{\xi\to a}\right]. \tag{A9}$$

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