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# Passive scalar transport by a non-Gaussian turbulent flow in the Batchelor regime

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We analyze passive scalar advection by a turbulent flow in the Batchelor regime. No restrictions on the velocity statistics of the flow are assumed. The properties of the scalar are derived from the statistical properties of velocity; analytic expressions for the moments of scalar density are obtained. We show that the scalar statistics can differ significantly from that obtained in the frames of the Kraichnan model.

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## I. INTRODUCTION

The problem of passive scalar advection by a turbulent flow has been widely discussed for several decades (see, e.g., Ref. [1]). Most of the analytic results have been obtained in the frame of the Kraichnan model [2], velocity statistics assuming to be Gaussian and $\delta$ correlated in time. The model appeared to be very productive and revealed many interesting properties (see, e.g., Ref. [3]). However, in real turbulent flows, the velocity statistics differs significantly from that in the Kraichnan model. Non-Gaussianity is all the more important in that the averages of scalar density moments can be reduced to the averages of $\exp(\int X dt)$, where $X$ is a stationary random process; though $\int X dt$ itself behaves as a Gaussian random value (as a result of the central limit theorem), its exponential is essentially not log normal.

As opposed to the moments of an integral (where the Gaussian term dominates), the averages of the exponential contain contributions of connected correlation functions of all orders equally, so the Gaussian distribution is no longer a good approximation. (This is a consequence of the theory of large deviations [4]; see an illustration of the fact in the Appendix.)

To study the case of a non-Gaussian velocity field, one uses the Batchelor limit [5]: The velocity field is assumed to be smooth, and the Lagrangian formalism can be developed based on the velocity strain tensor statistics. This regime corresponds to the scales below the viscous length, and implies that the diffusivity of the advected quantity is small as compared to the fluid viscosity. Qualitative results in this field were obtained in Ref. [6].

Another generalization of the Kraichnan model employs the renorm-group methods [7] and allows one to calculate Eulerian multipoint statistics for the scales inside the inertial range (and also in the beginning of the viscous range, where viscosity is no longer negligible and exceeds the diffusivity). In particular, the finite correlation time for the Gaussian velocity statistics was taken into account in Refs. [8,9], and the possibility of generalization for the non-Gaussian case was discussed in Ref. [10]. However, the multipoint correlation functions calculated within the renorm-group framework have a necessarily scaling behavior and diverge in the ultraviolet limit, and one-point correlations can only be calculated in the essentially viscous range.

In this paper, we calculate exact expressions for Lagrangian one-point statistical moments of a passive scalar and analyze the dependence of their exponents $\gamma_\alpha$ on the moment order $\alpha$. The important step as compared to previous articles on the subject is that we derive all the statistical properties from velocity statistics, using the results of our previous papers [11,12]. In particular, it appears that the increment of the second eigenvalue of the inertia tensor ($\lambda_2$) is determined entirely by the non-Gaussian part of the cumulant function (or probability density), and one cannot take $\lambda_2 \neq 0$ while assuming the Gaussian approximation.

Generally, non-Gaussianity cannot be neglected in the calculation of any of the exponents $\gamma_\alpha$, though the qualitative behavior of the function $\gamma_\alpha(\alpha)$ remains the same as that for the Gaussian distribution. For example, saturation takes place for orders more than some critical value. Moreover, this critical value appears to be universal, but for exponents of lower-order moments the function may differ significantly from the parabola corresponding to the Gaussian.

The paper is organized as follows. In Sec. II we discuss the problem statement and express the passive scalar moments in terms of convenient variables associated with the evolution matrix. In Sec. III we recall the results of our previous papers to derive the relation between velocity and $X$ statistics. In Sec. IV we calculate the averages, and Sec. V provides a significant simplification by making use of the three dimensionality of space.

In Sec. VI we consider some exactly solvable particular cases: These are the Gaussian case, a small cubic deviation from the Gaussian, and one more exact solution corresponding to the exponential probability density of the velocity strain tensor. The results are summarized in Sec. VII.

## II. PASSIVE SCALAR DYNAMICS

The transport of a passive scalar $\theta(\mathbf{r},t)$ in the velocity field $\mathbf{v}$ obeys the equation

$$\frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta = \varkappa \Delta \theta. \qquad (1)$$

We now pass on to the frame comoving with a chosen particle (quasi-Lagrangian frame). In a smooth velocity field, for small

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enough lengths, we can introduce the velocity strain tensor

$$A_{ij} = \frac{\partial v_i}{\partial r_j}$$

and rewrite (1) as

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial r_i} A_{ij} r_j = \varkappa \Delta \theta. \quad (2)$$

The incompressibility condition results in $A_{ii} = 0$.

The problem statement is as follows. In a turbulent flow, $A_{ij}$ is random; its statistics is assumed to be given, and the correlation time $t_c$ is assumed to be small as compared with the time $T$ of observation. On the other hand, to remain in the linear regime, the blob's size should stay much smaller than the viscous scale. Then $\theta$, and in particular $\theta(t) \equiv \theta(\mathbf{0},t)$, is also random. Without loss of generality, the initial distribution $\theta(\mathbf{r},0)$ can be set Gaussian. Our aim is to find the time evolution of the moments $\langle \theta^\alpha(t) \rangle$.

To this purpose, we make the Fourier transform of Eq. (2). For $\theta(\mathbf{k},t)$ we get

$$\frac{\partial \theta}{\partial t} - k_i A_{ij} \frac{\partial \theta}{\partial k_j} = -\varkappa k^2 \theta.$$

We now introduce the evolution matrix $Q(t)$ defined by the equation

$$\dot{Q} = -QA, \quad Q(0) = I.$$

The formal solution to this equation can be written in terms of the antichronological exponential $Q = \overset{+}{T} e^{-\int A dt} = \sum_n \frac{1}{n!} \int_0^t d\tau_1 \cdots d\tau_n \overset{+}{T} (A(\tau_1) \cdots A(\tau_n))$,[1] and the properties of $Q$ will be discussed later.

Choosing the new variable $p_m$ according to

$$k_n = p_m Q_{mn},$$

we get

$$\frac{\partial}{\partial t} \theta(\mathbf{p},t) = -\varkappa p_m p_n \left( Q Q^T \right)_{mn} \theta.$$

The solution to this equation is

$$\theta(\mathbf{p},t) = e^{-\varkappa p_m p_n \int (Q Q^T)_{mn}(t') dt'} \theta(\mathbf{p},0).$$

We now make the inverse Fourier transform. From incompressibility it follows $\det Q = 1$, hence $d^3 k = d^3 p$. Thus,

$$\theta(t) = \int \theta(\mathbf{p},0) e^{-\varkappa p_m p_n \int (Q Q^T)_{mn}(t') dt'} d^3 p.$$

Generally, the initial distribution $\theta(\mathbf{p},0)$ is parabolic in the vicinity of the center and has some characteristic scale $l$. We take

$$\theta(\mathbf{p},0) = e^{-l^2 \mathbf{p}^2}. \quad (3)$$

Since the density is reversely proportional to the volume of a blob, any other distribution with the same scale would give the same behavior of the moments; only the preexponents would be different. We will see later that the exponents are also independent on the scale $l$.

Then

$$\theta(t) = \int e^{-p_m p_n D_{mn}} d^3 p = (\det D)^{-1/2},$$

where

$$D_{mn} = \varkappa \int (Q Q^T)_{mn}(t') dt' + l^2 \delta_{mn}. \quad (4)$$

Thus, we are interested in

$$\langle \theta^\alpha \rangle = \langle (\det D)^{-\alpha/2} \rangle. \quad (5)$$

Equation (5) describes the evolution of central concentration of a drop captured by a turbulent flow. Alternatively, one can consider a random statistically homogeneous initial condition with a characteristic length $l$, as in Ref. [1]. This results in a change of the exponent in (5): In this statement,

$$\langle \theta^\alpha \rangle = \langle (\det D)^{-\alpha/4} \rangle.$$

Hereafter we will adhere to the statement of a single drop and, consequently, will use Eq. (5). To go over to the problem with random homogeneous initial conditions, one only has to divide $\alpha$ by 2 in all eventual equations.

## III. STATISTICAL PROPERTIES OF $Q$ AND $D$

### A. Evolution matrix

To calculate the determinant, we have to learn more about the $D$ and $Q$ matrices.[2] Following Refs. [13,14] and others, we make the Iwasawa decomposition of $Q$,

$$Q = Z d R, \quad (6)$$

where $Z$ is an upper triangular matrix with unit diagonal elements, $d$ is a diagonal matrix with positive elements $d_{ii} \equiv d_i > 0$, $d_1 d_2 \cdots d_N = 1$ because of incompressibility, and $R \in \mathrm{SO}(N)$ is a rotation matrix. In what follows, we only need the symmetric combination

$$Q Q^T = Z d^2 Z^T, \quad (7)$$

so we are not interested in $R$. The generalization of the enlarged law of large numbers (see Ref. [15] for a survey) states that with unitary probability there exists the limit

$$\lambda_i = \lim_{t \to \infty} \frac{\ln d_i(t)}{t}, \quad \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N. \quad (8)$$

The set of constants $\lambda_i$ is called the Lyapunov spectrum (LS) [16]. It is an important statistical characteristic of the process $A(t)$, and it does not depend on the realization. The LS is a natural tool in many other physical applications [17].

So, the diagonal part of the Iwasawa decomposition demonstrates a systematic exponential growth. The triangular matrix, to the contrary, stabilizes as $t \to \infty$ [1,15]: With unitary probability there exists the limit $Z(t) \to Z_\infty$, where $Z_\infty$ is not universal for a given process and depends on the realization. The matrix $R$ does not stabilize and continues

[1]This definition of $Q$ differs slightly from those used in Refs. [11,12].

[2]So far we do not restrict ourselves by the three-dimensional space, so let them be $N \times N$ matrices.

doing a random walk in SO(3). Luckily, this randomness does not contribute to (7).

Thus, for a large enough time $t$, the integrals containing the growing $d_j$ in (4) are proportional to the integrands, while the integrals with decreasing $d_j$ are constants. Since $\det Z = 1$, we can factor $Z$ and $Z^T$ out of the determinant (we are interested in exponentials, so constant summands are not important)

$$\det D = \det\left(\varkappa \int d_{mn}^2(t')dt' + Z_{mk}^{-1}(l^2\delta_{kl} + \varkappa c_{kl})(Z^T)_{ln}^{-1}\right),$$

where $c_{kl}$ are some constants. At least one of the diagonal elements grows exponentially, and no elements decrease (because of the second term). Hence, the main summand of the determinant with unitary probability is proportional to the product of all the growing $d_j^2$,

$$\det D \propto d_1^{2\mu_1} \dots d_N^{2\mu_N}, \quad \mu_i = (\operatorname{sgn} \dot{d}_i + 1)/2. \tag{9}$$

### B. Cumulant function and $X$ variables

It is convenient to describe the random process $A(t)$ by the cumulant functional $W_A[\eta(t)]$,

$$e^{W_A[\eta(t)]} = \langle e^{i\int \operatorname{tr}\eta(t)A(t)dt}\rangle. \tag{10}$$

Here, $\eta(t)$ is an $N \times N$ matrix function of time. All statistical moments can then be expressed in terms of derivatives of $W$ (see the Appendix for details), and the probability density functional is related to $e^W$ via a Fourier transform. For functions $\eta(t)$ that change much slower than $t_c$, $W[\eta]$ can be reduced to the cumulant function $w(k)$ [12],

$$W[\eta(t)] = \int w(\eta(t))dt. \tag{11}$$

The argument of $w$ is a (nonrandom) matrix. One also often uses the Kramer function (rate function), which is the Legendre transform of $w$.

We restrict our consideration by isotropic processes; they obey the relation

$$w(O^T k O) = w(k) \quad \forall O \in \mathrm{SO}(N),$$

and all their averages are independent of the orientation of the reference frame. The cumulant function of the isotropic processes depends on invariant combinations of $k_{ij}$ only,

$$w(k) = w(\operatorname{tr} k, \operatorname{tr} k^2, \operatorname{tr} kk^T, \operatorname{tr} k^3).$$

In Ref. [11] we introduced the $X$ variables defined by

$$X = -RAR^T = RQ^{-1}\dot{Q}R^T.$$

Substituting (6) for $Q$ shows that, in particular, the diagonal part of the $X$ matrix is

$$X_{nn} \equiv \rho_n = d_n^{-1}\dot{d}_n = \frac{d}{dt}\ln d_n.$$

Hence,

$$d_n(T) = e^{\int_0^T \rho_n dt}. \tag{12}$$

One of the main results of Ref. [11] is that for isotropic processes the cumulant function of $X$ variables is related to that of $A$ variables in a simple way,[3]

$$w_X(k) = w_A(ik_0 - k) - w_A(ik_0), \tag{13}$$

where

$$(k_0)_{ij} = \frac{2j - 1 - N}{2}\delta_{ij}. \tag{14}$$

We see from (9) and (12) that only the diagonal part of the $X$ matrix contributes to the averages we need. So, one can simplify the problem averaging over all the nondiagonal components [11]. The corresponding cumulant function can be obtained from the complete function $w_X$ by setting all the nondiagonal elements equal to zero,

$$w_\rho(k_1, \dots, k_N) = w_X(k)\Big|_{k_{ij} = \begin{cases} k_i, & i = j, \\ 0, & i \neq j. \end{cases}}$$

So, for the diagonal components $\rho_n$ of $X$ we eventually get

$$\begin{aligned} w_\rho(k_1, \dots, k_N) &= w_A(i(k_0)_1 - k_1, \dots, i(k_0)_N - k_N) \\ &\quad - w_A(i(k_0)_1, \dots, i(k_0)_N), \\ (k_0)_j &= \left\{-\frac{N-1}{2}, -\frac{N-3}{2}, \dots, \frac{N-1}{2}\right\}. \end{aligned} \tag{15}$$

We note that in the case of an imaginary argument, the function $w_A$, and hence $w_X$ and $w_\rho$, are concave; also, in accordance with the definitions (10) and (11), $w_A(0) = w_X(0) = 0$.

## IV. AVERAGING THE DETERMINANT

We now proceed to calculate the average (5); according to (9),

$$\langle \theta^\alpha \rangle = \langle (\det D)^{-\alpha/2} \rangle \propto \langle d_1^{-\alpha\mu_1} \cdots d_N^{-\alpha\mu_N} \rangle.$$

To this purpose, we introduce a new set of variables $z = \{z_i\}$,

$$z_i = \frac{1}{T}\int_0^T \rho_i dt, \quad \ln d_i = z_i T.$$

Treating $z_i$ as random quantities, not processes (i.e., regarding $T$ as a parameter), we see that

$$e^{W_z(k)} \equiv \langle e^{ik_j z_j} \rangle = \langle e^{i\frac{k_j}{T}\int_0^T \rho_j dt} \rangle = e^{w_\rho(\frac{k}{T})T}.$$

Thus, $W_z(k) = w_\rho(k/T)T$. The probability density of a random variable is related to the exponential of its cumulant function by a Fourier transform, so

$$P(z) = \int d^3\tilde{k} e^{-i\tilde{k}z + W_z(\tilde{k})} \propto \int d^3k e^{[-ikz + w_\rho(k)]T}.$$

(The coefficient is proportional to $T^3$ but we are only interested in the exponents.) As $T \to \infty$, we can use the saddle-point approximation to calculate this integral,

$$P(z) \propto e^{[-ik_j^* z_j + w_\rho(k^*)]T + o(T)},$$

[3]Notations here differ slightly from those in Ref. [11].

where $k^*(z)$ obeys the condition of maximum for the integrand,

$$-iz_j + \left.\frac{\partial w_\rho}{\partial k_j}\right|_{k^*} = 0. \tag{16}$$

Now, the average of the determinant takes the form

$$\langle(\det D)^{-\alpha/2}\rangle = \int P(z)(\det D)^{-\alpha/2}dz = \int e^{\phi(z)T}dz,$$

where $dz = dz_1 \cdots dz_N$,

$$\phi(z) = -ik_j^* z_j + w_\rho(k^*(z)) - \frac{\alpha}{2}\ln(\det D)/T, \quad k^* = k^*(z).$$

This integral can also be calculated by means of the saddle-point method. Substituting (9) for det $D$, we see that

$$\phi(z) = -ik_j^* z_j + w_\rho(k^*(z)) - \alpha\sum_{j=1}^{N}\mu_j z_j \tag{17}$$

does not depend on $T$. Thus, for large $T$,

$$\langle\theta^\alpha\rangle = \langle(\det D)^{-\alpha/2}\rangle = e^{\gamma_\alpha T}, \quad \gamma_\alpha = \max_z \phi(z). \tag{18}$$

We note that $\phi(z)$ is not analytic because of the steplike functions $\mu_j(z)$.[4] So, the whole space $(z_1,z_2,\ldots,z_n)$ is divided into regions (sectors) where $\mu_j$ are constants. If the maximum of $\phi$ is situated inside one of these regions, then the conditions $\partial\phi/\partial z_j = 0$ give

$$k_j^* = i\alpha\mu_j. \tag{19}$$

Substituting this into (16), one can find the optimal $z_j$ that makes the crucial contribution to the average. But the maximum can be attained as well at one of the boundary planes, then $z_n = 0$ for some $n$. The position of the maximum is then determined by the conditions

$$\frac{\partial w_\rho}{\partial k_n} = 0, \quad k_{j\neq n}^* = i\alpha\mu_j. \tag{20}$$

In both cases (19) and (20), from (18) we get

$$\langle\theta^\alpha\rangle = e^{\gamma_\alpha T}, \quad \gamma_\alpha = w_\rho(k^*). \tag{21}$$

## V. THREE-DIMENSIONAL CASE

In the case of three dimensions, the results can be illustrated more explicitly. First, it is convenient to regard $-ik_j^*$ as independent variables instead of $z_j$; then $z_j(k^*)$ are determined by (16). Second, to take advantage of the incompressibility, we choose the variables

$$\xi = i(k_1^* - k_2^*), \quad \eta = i(k_1^* - k_3^*), \quad \psi = i(k_1^* + k_2^* + k_3^*).$$

The incompressibility condition $\partial w/\partial k_1 + \partial w/\partial k_2 + \partial w/\partial k_3 = 0$ then becomes

$$\left.\frac{\partial w}{\partial\psi}\right|_{\xi,\eta} = 0, \quad w = w(\xi,\eta),$$

and $w$ becomes a function of two parameters. Equation (15), relating $w_A$ and $w_\rho$, can be written in a simple form,

$$w_\rho(\xi,\eta) = w_A(1-\xi,2-\eta) - w_A(1,2). \tag{22}$$

The variables $z_j$ are expressed via $\xi,\eta$ by

$$z_1 = \frac{\partial w_\rho}{\partial\xi} + \frac{\partial w_\rho}{\partial\eta}, \quad z_2 = -\frac{\partial w_\rho}{\partial\xi}, \quad z_3 = -\frac{\partial w_\rho}{\partial\eta}, \tag{23}$$

and $\phi$ takes the form

$$\phi(\xi,\eta) = -\xi\frac{\partial w_\rho}{\partial\xi} - \eta\frac{\partial w_\rho}{\partial\eta} + w_\rho(\xi,\eta) + \alpha\left((\mu_2-\mu_1)\frac{\partial w_\rho}{\partial\xi} + (\mu_3-\mu_1)\frac{\partial w_\rho}{\partial\eta}\right). \tag{24}$$

Now we have to find the maximum of this function for any $\alpha$. We recall that $\mu_j$ are functions of $k^*$ and hence of $\xi,\eta$,

$$2\mu_j - 1 = \operatorname{sgn}\dot{d} = \operatorname{sgn} z_j(\xi,\eta). \tag{25}$$

The condition (19) now reads as

$$\xi^* = \alpha(\mu_2-\mu_1), \quad \eta^* = \alpha(\mu_3-\mu_1), \tag{26}$$

and (21) transforms into

$$\langle\theta^\alpha\rangle = e^{w_\rho(\xi^*,\eta^*)T}. \tag{27}$$

In the case if the maximum is attained at a boundary, one of the conditions (26) is no longer valid; in particular, if $z_2^* = 0$ (which is, as we will see below, the case for any reasonable function), (20) gives

$$\frac{\partial w_\rho}{\partial\xi}(\xi^*,\eta^*) = 0, \quad \eta^* = \alpha(\mu_3-\mu_1). \tag{28}$$

The process $A(t)$ is isotropic and traceless; hence, $\langle A_{ij}\rangle = 0$. In terms of a cumulant function this formulates as $\frac{\partial w_A}{\partial k_i}\big|_0 = 0$, hence

$$\frac{\partial w_A}{\partial\xi}(0,0) = \frac{\partial w_A}{\partial\eta}(0,0) = 0,$$

and $\xi = \eta = 0$ is the point of the minimum of $w_A$. Accordingly, the minimum of $w_\rho$ appears to be at $\xi = 1, \eta = 2$.

Our "starting point" for $\alpha = 0$ is $\xi^* = \eta^* = 0$. The condition $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$, $\lambda_i = \langle\rho_i\rangle$ [see (8)] requires

$$z_1(0,0) \leqslant z_2(0,0) \leqslant z_3(0,0).$$

So, at the beginning, the point $(\xi^*,\eta^*)$ is situated in the sector $z_1 < 0, z_3 > 0$. The sign of $z_2$ is arbitrary. Depending on it, the point of extremum shifts in the $(\xi,\eta)$ plane according to (26): $\eta^* = \alpha, \xi^* = \alpha$, or $\xi^* = 0$. In both cases, at some $\alpha$ it reaches the boundary $z_2 = 0$.[5] As $\alpha$ increases further, the maximum

[4] Physically, the sharp edges are of course smoothed by other summands of the determinant, but this does not affect the exponents.

[5] The boundaries $z_i = 0$ all transect at $\xi = 1$, $\eta = 2$ and have no additional transections (because of the convexity of $w$). They are ordered clockwise, and if $\mu_2 = 1$, the boundary $z_2 = 0$ is located to the right of (0,0); the extremum point also shifts to the right as $\alpha$ increases. The position of $z_3 = 0$ is strictly to the left from the diagonal straight line $\eta = \xi + 1$, $\xi \leqslant 1$; so our extremum point cannot meet it. Analogously, if $\mu_2 = 0$, the boundary $z_2 = 0$ is "up and to the left" from the zero point, and it lies on the extremum's trajectory as it shifts up.

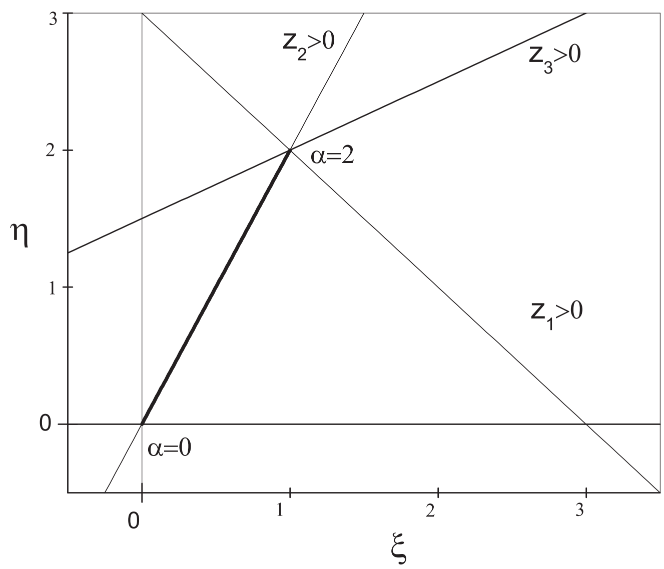


FIG. 1. Position of the maximum in the $(\xi,\eta)$ plane as a function of $\alpha$: Gaussian statistics. Straight lines correspond to the boundaries $z_i = 0$.

point continues shifting along the line in accordance with (28), and as $\alpha = 2$, it coincides with the minimum of $w_\rho$: $\xi = 1$, $\eta = 2$.

Eventually, we note that [as a consequence of (23) and (25)] the expression multiplied by $\alpha$ in (24) (which is $-\mu_1 z_1 - \mu_2 z_2 - \mu_3 z_3$) cannot be positive. So, for large enough values of $\alpha$ ($\alpha \geqslant 2$), the maximal $\phi$ is always achieved in the point where this term is equal to zero, i.e., in the minimum of $w_\rho$. This is why the dependence $\gamma_\alpha(\alpha)$ is saturated at large $\alpha$ and is equal to the minimal value of $w_\rho$.

## VI. EXAMPLES

To illustrate the properties discussed above, we consider several examples.

### A. Gaussian statistics

Let $A$ be an isotropic traceless Gaussian process. Then [11]

$$w_A^G = D\left(\frac{1}{3}(k_1 + k_2 + k_3)^2 - \left(k_1^2 + k_2^2 + k_3^2\right)\right) = \frac{2}{3}D(\xi^2 + \eta^2 - \xi\eta),$$

where $D$ is proportional to the dispersion, $\langle A_i^2 \rangle = \frac{4}{3}D$. The corresponding $w_\rho$ is

$$w_\rho^G = \tfrac{2}{3}D((\xi - 1)^2 + (\eta - 2)^2 - (\xi - 1)(\eta - 2) - 3).$$

The boundaries separating the regions $\mu_j = 0$ and $\mu_j = 1$ are defined by the conditions

$$\xi + \eta - 3 = 0 \quad (z_1 = 0), \quad 2\xi - \eta = 0 \quad (z_2 = 0),$$
$$2\eta - \xi - 3 = 0 \quad (z_3 = 0)$$

(see Fig. 1). For $\alpha = 0$, the maximum of $\phi$ is evidently attained at $\xi^* = \eta^* = 0$. At these $\xi$ and $\eta$ in the Gaussian case we have $z_1 < 0$, $z_2 = 0$, $z_3 > 0$. So, the point of extremum is from the very beginning situated on the boundary $z_2 = 0$. As $\alpha$ increases, it moves in the $\xi,\eta$ plane along the straight line $z_2 = 0$; one can check that there is no extremum either to the right or to the left from it. Taking the derivative along this direction, we find that, in accordance with (28),

$$\xi^* = \alpha/2, \quad \eta^* = \alpha, \tag{29}$$

and the corresponding maximal value is

$$\gamma_\alpha^G = \phi(\xi^*,\eta^*) = w_\rho(\alpha/2,\alpha) = \frac{D}{2}\alpha(\alpha - 4), \quad \alpha \leqslant 2. \tag{30}$$

As $\alpha = 2$, the extremum point coincides with the minimum of $w_\rho$. At this point all the $z_j$ are zero, so further growth of $\alpha$ no longer changes the maximum. The dependence $\gamma_\alpha(\alpha)$ is saturated,

$$\gamma_\alpha^G = \begin{cases} \frac{D}{2}\alpha(\alpha - 4)\,, & \alpha \leqslant 2, \\ -2D\,, & \alpha > 2. \end{cases} \tag{31}$$

This result coincides with the results of Refs. [1,6].

### B. Small deviation from Gauss

We now consider a small cubic addition to the Gaussian cumulant function. The conditions of isotropy and zero trace require [11]

$$w_A = \tfrac{2}{3}D(\xi^2 + \eta^2 - \xi\eta) - \tfrac{2}{9}F(\xi^3 + \eta^3) + \tfrac{1}{3}F\xi\eta(\xi + \eta).$$

The constant $F$ must be small enough in order not to break the condition of positive definiteness of $w_A(\xi,\eta)$ at least up to $|\xi| = 1, |\eta| = 2$, otherwise we have to take the next summands into account. Proceeding from $w_A$ to $w_\rho$ in accordance with (22) and calculating the derivatives, we see that the boundaries $z_j = 0$ are shifted and curved as compared to the Gaussian case (Fig. 2). In particular, the condition $z_2 = 0$ now gives

$$\xi_b(\eta) = \frac{\eta}{2} + \vartheta(\eta), \quad \vartheta = \frac{\sqrt{1 + \frac{3}{4}\left[\frac{F}{D}(\eta - 2)\right]^2} - 1}{F/D}. \tag{32}$$

So, if $F > 0$, the point $(0,0) = (\xi^*,\eta^*)(\alpha = 0)$ is inside the region $z_2 > 0$, $\mu_2 = 1$; otherwise, $z_2(0,0) < 0$ and $\mu_2 = 0$. Two other boundaries remain far from the "starting" point (0,0).

So, the behavior of $\gamma_\alpha$ at small $\alpha$ depends on the sign of $F$. Consider first $F < 0$. Then, as $\alpha$ increases, the extremum point shifts in the $(\xi,\eta)$ plane along the $\eta$ axis: $\xi^* = 0$, $\eta^* = \alpha$ [in accordance with (26)]; hence,

$$\gamma_\alpha = w_\rho(0,\alpha) = w_A(1,2 - \alpha) - w_A(1,2).$$

In the case $F > 0$, the coordinates of the extremum point depend on $\alpha$ as $\xi^* = \alpha$, $\eta^* = \alpha$. Correspondingly, $\gamma_\alpha = w_\rho(\alpha,\alpha)$. In both cases we get

$$\gamma_\alpha = \alpha(\alpha - 3)\frac{2}{3}\left[D + \frac{1}{3}|F|\left(\frac{3}{2} - \alpha\right)\right], \quad \alpha < \alpha_c.$$

In both cases, the extremum point reaches the boundary (32) as $\xi_b(\alpha) = \mu_2\alpha$, i.e., $\alpha_c = 3|F/D| + O((F/D)^3)$.

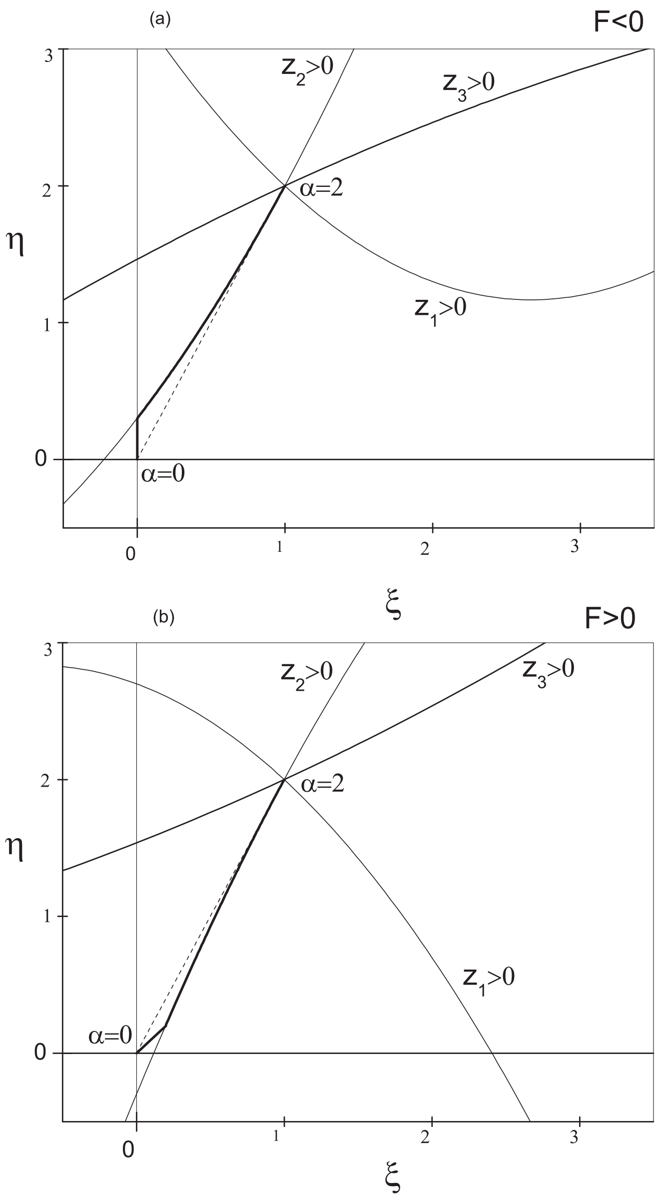


FIG. 2. Position of the maximum in the $(\xi,\eta)$ plane as a function of $\alpha$: Cubic deviation from the Gaussian statistics, (a) $F=-0.1$, (b) $F=0.1$.

While $\alpha$ increases further, it remains on the boundary $z_2=0$ and moves up to the point (1,2). The position of the maximum in this range of $\alpha$ is determined by (28)

$$\eta=\alpha, \quad \xi=\xi_b(\alpha) \quad (\alpha>\alpha_c).$$

The dependence $\gamma_\alpha$ for these $\alpha$ is

$$\begin{aligned}\gamma_\alpha &= w_\rho(\xi_b(\alpha),\alpha)\\ &= -2D+\frac{2}{3}D(\alpha-2)^2\\ &\quad -\frac{4}{9}D\frac{\vartheta}{F/D}-\frac{1}{3}F\vartheta(\alpha-2)^2, \quad \alpha>\alpha_c.\end{aligned}$$

Up to the second order in $F/D$, we obtain

$$\gamma_\alpha = -2D+\frac{1}{2}D(\alpha-2)^2-\frac{3}{32}D\left(\frac{F}{D}\right)^2(\alpha-2)^4 + O((F/D)^4).$$

We note that the first order does not contribute to the result; independently on the sign of $F$, the exponent is smaller than that in the Gaussian case.

The value $\gamma_\alpha$ coincides with that of the Gaussian case both at $\alpha=0$, $\gamma_0=0$, and at $\alpha\geqslant 2$, $\gamma_{\geqslant 2}=w_\rho(1,2)=-w_A(1,2)$. So, the difference is most significant in the range of $\alpha$ near the "turning point" of the extremum, $\alpha\simeq 3|F/D|$.

### C. Exponential statistics

To see how the result can possibly differ from the Gaussian, we consider one more analytically solvable model. Let the probability distribution function for diagonal elements of $A$ be

$$f(A_1,A_2,A_3)=f_0e^{-c\sqrt{A_1^2+A_2^2+A_3^2}}\delta(A_1+A_2+A_3). \quad (33)$$

This is a three-dimensional analog to the exponential distribution. Exponential decay at high $A$ provides the finiteness of the moments of all orders. The $\delta$ function is introduced to satisfy the incompressibility condition. Making the Fourier transform, we get

$$\begin{aligned}e^{w_A(k_1,k_2,k_3)} &\propto \int e^{-\frac{c}{2}\rho}e^{-i\rho(k_1'\cos\phi+k_2'\sin\phi)}\rho d\rho d\phi\\ &= \int_0^{2\pi}\frac{d\phi}{[\frac{c}{2}+i(k_1'\cos\phi+k_2'\sin\phi)]^2},\end{aligned}$$

where $k_1'=\frac{1}{2\sqrt{3}}(2k_1-k_2-k_3)$, $k_2'=\frac{1}{2}(k_3-k_2)$. The integral converges for all real $k'$. For imaginary arguments (which are of interest for us, since $k^*$ is imaginary) the cumulant function exists if $|k_1'^2+k_2'^2|\leqslant c^2/4$. Then

$$w_A=-\tfrac{1}{2}\ln[3c^2-4(\xi^2+\eta^2-\xi\eta)]+\tfrac{1}{2}\ln(3c^2),$$

where, as in previous sections, $\xi=i(k_1-k_2)$ and $\eta=i(k_1-k_3)$.

As usual, we proceed to $w_\rho$ according to (22). The equation for the boundary $\partial w_\rho/\partial\xi=0$ is again $\eta=2\xi$, just as in the Gaussian case. This is because no asymmetry has been introduced in the cumulant function. So, the extremum of $\phi$ shifts in the $(\xi,\eta)$ plane as in (29). However, the value of the extremum differs from (30),

$$\gamma_\alpha=\begin{cases}\phi(\alpha/2,\alpha)=\frac{1}{2}\ln\frac{c^2-4}{c^2-(\alpha-2)^2}, & \alpha\leqslant 2,\\ \phi(1,2)=\frac{1}{2}\ln\frac{c^2-4}{c^2}, & \alpha>2.\end{cases}$$

The difference between this function and the parabola that corresponds to the Gaussian case is illustrated in Fig. 3. The graph is compared to that for a Gaussian process with the same value in the saturated regime. The difference is essential as the parameter $c$ becomes close to 2. We note that these two dependencies, though with the same saturations, are produced by initial distributions of $A$ with significantly different dispersions: For $c=2.2$ they differ by more than two times.

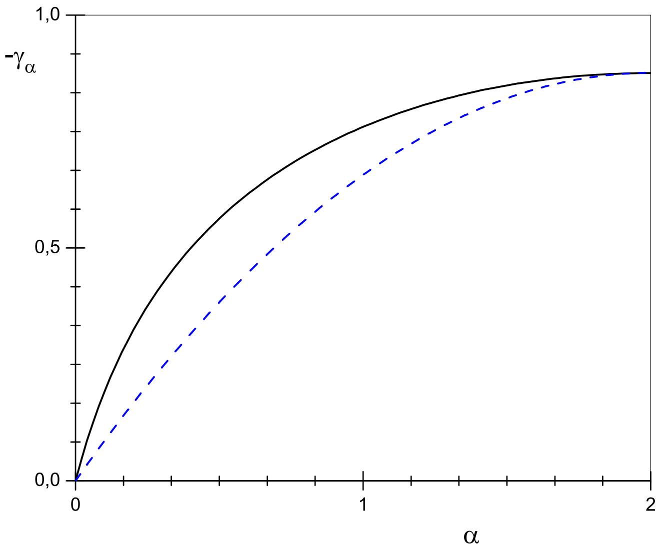


FIG. 3. The increment $-\gamma_\alpha$ as a function of the order for the model 33 with $c = 2.2$. The dashed curve is the result for the Gaussian distribution which produces the same saturated $\gamma_\alpha$.

Evidently, the velocity distributions with the same dispersions would produce scalar advection with significantly different saturations of the exponent, so the Gaussian is not a good approximation for this (and hence arbitrary) model.

## VII. CONCLUSION

We analyze passive scalar advection in a turbulent flow. We restrict our consideration to the regime where the viscosity of the fluid is much bigger than the scalar diffusivity, which allows one to consider a linear approximation for velocity space distribution (Batchelor regime). The velocity distribution is assumed to be statistically isotropic and homogeneous, and we consider the passive scalar evolution at times much bigger than the velocity correlation time. This is the only restriction for the correlation time. We also consider arbitrary (non-Gaussian) velocity statistics.

We trace the evolution of the concentration in a drop advected by a flow, and derive the exponential behavior of the moments of all orders $\alpha$. An exact expression for the exponents $\gamma_\alpha$ is obtained for any given velocity strain tensor statistics and expressed in terms of the strain tensor cumulant function. This is done, in particular, by taking advantage of the isotropy of the velocity distribution.

We show that there is a universal saturation of the exponents at $\alpha = 2$. (The only exclusion is a distribution for which the moments do not exist for $\alpha < 2$.) The shape of the curve $\gamma_\alpha(\alpha)$ and the level of saturation depend on the velocity statistics and can differ significantly from those for the Gaussian distribution with the same dispersion.

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## APPENDIX: AVERAGES, CUMULANT FUNCTIONALS, AND CUMULANT FUNCTIONS

In this Appendix we recall some basic properties of cumulant functionals and their relation to statistical moments, and we illustrate the peculiarity of the averages of exponentials.

Let a one-dimensional (for simplicity) random process $A(t)$ be defined by its cumulant functional (10)

$$e^{W[\eta(t)]} = \left\langle \exp\left[ i \int \eta(t) A(t) dt \right] \right\rangle, \tag{A1}$$

with a normalization condition $W[0] = 0$. Then the statistical moments of $A$ can be derived by

$$\langle A(t_1) \cdots A(t_n) \rangle = (-i)^n \frac{\delta}{\delta\eta(t_1)} \cdots \frac{\delta}{\delta\eta(t_n)} e^{W[\eta(t)]} \bigg|_{\eta=0}. \tag{A2}$$

The functional can be expanded into a Taylor series; from the time of homogeneity it follows that the coefficients can depend on time differences only,

$$W[\eta] = \sum_n \frac{1}{n!} \int W^{(n)}(t_1 - t_2, \ldots, t_1 - t_n) \eta(t_1) \cdots \times \eta(t_n) dt_1 \cdots dt_n. \tag{A3}$$

In particular, $W^{(1)}$ does not depend on time. The coefficients are related to the connected correlation functions of $A$ according to

$$W^{(n)}(t_1 - t_2, \ldots, t_1 - t_n) = i^n \langle A(t_1) \cdots A(t_n) \rangle_c. \tag{A4}$$

The coefficient $W^{(1)}$ has the meaning of the average $\langle A \rangle$ and $W^{(2)}(t_1 - t_2)$ is the pair correlation function. In the case of a Gaussian random process, $W(\eta)$ contains only these two terms.

Consider the average

$$\left\langle \left( \int_0^T A(t) dt \right)^n \right\rangle = \int_0^T dt_1 \cdots dt_n \langle A(t_1) \cdots A(t_n) \rangle. \tag{A5}$$

Substituting (A2) and (A3), we get

$$\begin{aligned} &\left\langle \left( i \int A(t) dt \right)^n \right\rangle \\ &= \int dt_1 \cdots dt_n \bigg[ W^{(n)}(t_1 - t_2, \ldots, t_1 - t_n) \\ &\quad + C_n^2 \sum_{k=1}^{n-1} W^{(k)}(t_1 - t_2, \ldots, t_1 - t_k) \\ &\quad \times W^{(n-k)}(t_1 - t_{k+1}, \ldots, t_1 - t_n) + \cdots + (W^{(1)})^n \bigg]. \end{aligned}$$

Let $t_c$ be the correlation time of the process. Then the functions $W^{(n)}(t_1, \ldots, t_n)$ must decrease rapidly as $|t_1 - t_k| \gg t_c$. Then, for $T \gg t_c$ we have the first term proportional to $T$, the second term proportional to $T^2$, etc.; so, the main contribution to the average is produced by the last term corresponding to the lowest-order moment,

$$\left\langle \left( \int A(t) dt \right)^n \right\rangle \simeq (-i W^{(1)})^n T^n = \langle A \rangle^n T^n. \tag{A6}$$

If the distribution is centered, $\langle A \rangle = 0$, and $n$ is even, the main contribution is produced by the term $\int dt_2 \langle A(t_1) A(t_2) \rangle_c^{n/2} T^{n/2}$.

In both cases the average is equal (up to $1/T$ accuracy) to that for a Gaussian distribution with the same average and pair correlation. This is the subject of the central limit theorem.

Just opposite is the situation with averages of the exponentials. Let us now consider $Q = \exp[-\int_0^T A(t)dt]$ and calculate its moments,

$$\langle Q^n \rangle = \langle e^{-n\int_0^T A(t)dt} \rangle = e^{W[\eta(t)=in\theta(t)\theta(T-t)]}. \quad \text{(A7)}$$

Substituting (A3), we see that all the terms in $W$ make contributions of the same order $T$,

$$\ln\langle Q^n \rangle \simeq \left[ \sum_k \frac{(in)^k}{k!} \int W^{(k)}(t_1-t_2,\dots,t_1-t_k)dt_2\cdots dt_k \right] T.$$

Taking into account (A4), we find that the moments of all orders contribute equally to $\langle Q^n \rangle$, and, unlike (A5), it cannot be calculated by neglecting the higher-order moments (i.e., by replacing the process with a Gaussian with the same average and pair correlation function).

Defining the cumulant function by

$$w(\eta) = \sum_k \frac{(\eta)^k}{k!} \int W^{(k)}(\tau_2,\dots,\tau_k)d\tau_2\cdots d\tau_k,$$

we get

$$\langle Q^n \rangle = e^{w(in)T}.$$

Though (11) is only valid for the functions $\eta(t)$ varying slowly at the correlation time $t_c$, these relations are universal for $T \gg t_c$.


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