Exact solution of the isotropic majority-vote model on complete graphs

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The isotropic majority-vote (MV) model, which, apart from the one-dimensional case, is thought to be nonequilibrium and violating the detailed balance condition. We show that this is not true when the model is defined on a complete graph. In the stationary regime, the MV model on a fully connected graph fulfills the detailed balance and is equivalent to the modified Ehrenfest urn model. Using the master equation approach, we derive the exact expression for the probability distribution of finding the system in a given spin configuration. We show that it only depends on the absolute value of magnetization. Our theoretical predictions are validated by numerical simulations.

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I. INTRODUCTION

The isotropic majority-vote (MV) model for opinion dynamics is a well-known nonequilibrium spin model that has been studied by many researchers (see, e.g., [1–12]). One of the reasons why physicists became interested in the model is its critical behavior. The model was originally introduced in Ref. [1], where, for the two-dimensional (2D) square lattice, it was shown to have a continuous phase transition with the same static critical exponents as the 2D Ising model. This result was an important confirmation of an earlier hypothesis according to which nonequilibrium models with up-down symmetry and spin-flip dynamics fall within the universality class of the equilibrium Ising model [13,14]. Since then, there have been many numerical studies aimed at final approval or rejection of that hypothesis for different models. In particular, a number of Monte Carlo simulations of the MV model on regular lattices and random networks have been carried out, providing contradictory conclusions (cf., e.g., [4] and [7]) and leaving unsolved the problem of its universality classes, although recent, very accurate numerical studies confirm this hypothesis at least for a wide class of Archimedean lattices [12]. Nevertheless, it is still controversial whether the upper critical dimension of the MV model on d-dimensional hypercubic lattices is 4 (as in the Ising model) or 6 (as suggested in [4]). Moreover, mean-field critical exponents for the MV model have not yet been exactly determined, although their numerical values seem to agree with the results known for the Ising model.

In this paper we present an exact solution of the isotropic majority-vote model on the complete graph of N nodes. We provide the probability distribution $P(\Omega)$ of finding the system in a certain microstate $\Omega = (\sigma_1,\sigma_2,\ldots,\sigma_N)$, where $\sigma_i = \pm 1$ denotes the spin variable associated with the site i and N is the total number of sites. We show that the probability depends only on the absolute value of magnetization: $P(\Omega) \propto \sqrt{(1-q)/q}^{|M(\Omega)|}$, where $M(\Omega) = \sum_{i=1}^{N} \sigma_i$ and q is the standard noise parameter of the majority-vote model [see Eq. (3) for its formal definition]. Given the result, we also show that the MV model on the complete graph fulfills the detailed-balance condition: just as the one-dimensional MV model [15] and unlike the model on the two-dimensional square lattice [1].

II. A BRIEF STATE OF THE ART ON THE MAJORITY-VOTE MODEL

The isotropic majority-vote model is a spin model in which to each site i a spin variable $\sigma_i = \pm 1$ is assigned. In the rest of this paper, the global configuration of the system is defined as

$$\Omega = (\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots\sigma_N), \tag{1}$$

where N is the total number of sites. The dynamics of the model is such that, at any time step, only one site has its spin modified. Let us assume that the considered spin is σ_i. The transition rate from a configuration Ω to another configuration

$$\Omega_i = (\sigma_1,\sigma_2,\ldots,-\sigma_i,\ldots\sigma_N), \tag{2}$$

which only differs from Ω by the sign of the ith spin variable, is given by

$$w_i(\Omega) = \tfrac{1}{2}[1-(1-2q)\sigma_i S_i], \tag{3}$$

where S_i takes one of three values

$$S_i = S(m_i) = \begin{cases} +1 & \text{for } m_i > 0 \\ 0 & \text{for } m_i = 0 \\ -1 & \text{for } m_i < 0 \end{cases} \tag{4}$$

depending on the local magnetization

$$m_i = \sum_{\langle i,j\rangle} \sigma_j \tag{5}$$

of the nearest neighborhood of σ_i.

The role of the noise parameter q can be easily deduced from Eq. (3). In the case, when, in the initial configuration Ω, the sign of the variable σ_i is consistent with the sign of its neighborhood, i.e., $\sigma_i S_i = +1$, the transition rate, which is proportional to the probability that σ_i changes the sign to the opposite, is equal to $w_i(\Omega) = q$. Otherwise, when $\sigma_i S_i = -1$ one has $w_i(\Omega) = 1-q$. Thus, the transition rate from a spin configuration Ω to Ω_i is $1-q$ if the flipping follows the majority rule among the nearest neighbors of i and q if it does not. The case when $S_i = 0$ corresponds to $w_i(\Omega) = \frac{1}{2}$, which means that the chosen site takes either sign with equal probability. From the above discussion it is clear that the parameter q in the MV model can be interpreted as an effective temperature. However, the meaning of q is much richer. In particular, when $0 \leqslant q < \frac{1}{2}$ the majority rule mimics a kind of ferromagnetic coupling in the system, while $\frac{1}{2} < q \leqslant 1$ corresponds to antiferromagnetic coupling. Finally, when $q = \frac{1}{2}$ the model behaves like a typical paramagnet.

The time evolution of the majority-vote model is governed by the master equation

$$\frac{d}{dt}P(\Omega,t) = \sum_i [w_i(\Omega_i)P(\Omega_i,t) - w_i(\Omega)P(\Omega,t)], \quad (6)$$

where $P(\Omega,t)$ is the time-dependent microstate distribution, i.e., the probability of occurrence of configuration Ω at time t, and $w_i(\Omega)$ is the transition rate from Ω to Ω_i, which is given by Eq. (3). In the stationary regime, when

$$\frac{d}{dt}P(\Omega,t) = 0, \quad (7)$$

i.e., $P(\Omega,t) \equiv P(\Omega)$, Eq. (6) simplifies to the balance equation

$$\sum_i [w_i(\Omega_i)P(\Omega_i) - w_i(\Omega)P(\Omega)] = 0, \quad (8)$$

which, in general and differently than it is in the Ising model, cannot be further simplified to the detailed balance condition

$$w_i(\Omega_i)P(\Omega_i) - w_i(\Omega)P(\Omega) \neq 0. \quad (9)$$

With regard to the majority-vote model, the only known exception to Eq. (9) is the one-dimensional chain of spins, which is equivalent to the 1D Glauber model, whose dynamics may be interpreted as a dynamics for the 1D Ising model (see [15], Chap. 11). In other words, in the stationary regime, the one-dimensional MV model is equivalent to the 1D Ising model. Therefore, its stationary distribution $P(\Omega)$ is the Boltzmann-Gibbs distribution. In higher dimensions, one believes that the MV model does not show microscopic reversibility that underlies the detailed balance, although, in a similar way to what happens in the microscopically reversible Ising model, the MV model exhibits continuous phase transitions. (The lack of microscopic reversibility in the square-lattice MV model is simply shown in Ref. [15], Chap. 12.6.) For example, in the stationary regime, for small values of the parameter q, the square-lattice MV model presents a ferromagnetic phase characterized by the presence of a majority of spins with the same sign. Above the critical value $q_c = 0.075 \pm 0.01$ [1] of the noise parameter, the model presents a paramagnetic state with an equal, on average, number of spins with distinct signs. Furthermore, it is known that the square-lattice MV model falls into the same universality class as the 2D Ising model. Recent numerical simulations indicate that these findings may also be true in higher dimensions, for $d \geqslant 3$ [7]. On the other hand, one still lacks strict theoretical results concerning, in particular, the mean-field critical behavior of the model.

In the next section, starting with the balance equation (8), we find recurrence relations for the probability $P(\Omega)$ of finding the MV model on a complete graph in a given spin configuration Ω. Then we derive exact formulas for $P(\Omega)$ and $P(M)$, with the latter being the probability that the system has a magnetization equal to M. We also show that the MV model on complete graphs fulfills the detailed balance condition. Our theoretical predictions are confirmed by numerical simulations.

III. MAJORITY-VOTE MODEL ON THE COMPLETE GRAPH

A. Probability of a configuration in the stationary regime

Let us note that, in the case of a complete graph, the local magnetization (5) of each node is

$$m_i = \sum_{j\neq i} \sigma_j = M - \sigma_i, \quad (10)$$

where

$$M = \sum_j \sigma_j \quad (11)$$

is the total magnetization of the system in a spin configuration Ω. Later in the text, if not explicitly stated otherwise, quantities such as magnetization M, the number of positive spins N_+, and the spin variable σ_i always refer to the configuration Ω. Accordingly, if we want to emphasize that these variables refer to another configuration, e.g., Ω_i, we write them in the following way: $M(\Omega_i)$, $N_+(\Omega_i)$, and $\sigma_i(\Omega_i)$.

Given Eq. (10), transition rates in the balance equation (8) can be written as

$$w_i(\Omega) = \tfrac{1}{2}[1 - (1-2q)\sigma_i S(M-\sigma_i)], \quad (12)$$

$$w_i(\Omega_i) = \tfrac{1}{2}[1 + (1-2q)\sigma_i S(M-\sigma_i)], \quad (13)$$

where the function $S(m_i)$ is that defined in Eq. (4) and $m_i(\Omega) = m_i(\Omega_i) = M - \sigma_i$. These rates only depend on the magnetization M and on the sign of the spin variable σ_i. Therefore, in a compete graph, since all the spins with the same sign are equivalent, the balance equation [cf. with Eq. (8)]

$$P(\Omega)\sum_i w_i(\Omega) = \sum_i P(\Omega_i)w_i(\Omega_i) \quad (14)$$

gets the following form:

$$N_+[1-(1-2q)S(M-1)]P(\Omega) + N_-[1+(1-2q)S(M+1)]P(\Omega) = N_+[1+(1-2q)S(M-1)]P(\Omega_-) + N_-[1-(1-2q)S(M+1)]P(\Omega_+), \quad (15)$$

where Ω_- and Ω_+ stand for the configurations Ω_i such that $\sigma_i(\Omega_-) = -\sigma_i(\Omega) = -1$ and $\sigma_i(\Omega_+) = -\sigma_i(\Omega) = +1$, respectively.

To proceed with the analysis of Eq. (15), one must assume that the system size N is even or odd. At the beginning, we assume that N is even. Therefore, the magnetization

$$M = 2N_+ - N \quad (16)$$

is also even (positive or negative) or zero and the balance equation (15) splits into three cases depending on the sign of M. Thus, we have

$$N_+(1-q)P(\Omega)+N_-qP(\Omega)=N_+qP(\Omega_-)+N_-(1-q)P(\Omega_+)\quad \text{for } M\leqslant -2, \tag{17}$$

$$N_+(1-q)P(\Omega)+N_-(1-q)P(\Omega)=N_+qP(\Omega_-)+N_-qP(\Omega_+)\quad \text{for } M=0, \tag{18}$$

$$N_+qP(\Omega)+N_-(1-q)P(\Omega)=N_+(1-q)P(\Omega_-)+N_-qP(\Omega_+)\quad \text{for } M\geqslant +2. \tag{19}$$

These equations can be significantly simplified if one assumes that the probability of a configuration $P(\Omega)$ only depends on the number of positive spins $N_+(\Omega)$, i.e.,

$$P(\Omega)\equiv f(N_+(\Omega))=f(N_+), \tag{20}$$

and correspondingly

$$\begin{aligned} P(\Omega_+)&\equiv f(N_+(\Omega_+))=f(N_++1),\\ P(\Omega_-)&\equiv f(N_+(\Omega_-))=f(N_+-1). \end{aligned} \tag{21}$$

The assumptions provided by Eqs. (20) and (21) are reasonable due to the symmetry of the system, in which all the spins with the same sign are equivalent. Also, they naturally arise from the balance equations (17)–(19), in which transition rates between different configurations only depend on N_+. According to these assumptions, one gets the following recurrence relations for the dummy function $f(N_+)$:

$$N_+(1-q)f(N_+)+N_-qf(N_+)=N_+qf(N_+-1)+N_-(1-q)f(N_++1)\quad \text{for } N_+<\frac{N}{2}, \tag{22}$$

$$N_+(1-q)f(N_+)+N_-(1-q)f(N_+)=N_+qf(N_+-1)+N_-qf(N_++1)\quad \text{for } N_+=\frac{N}{2}, \tag{23}$$

$$N_+qf(N_+)+N_-(1-q)f(N_+)=N_+(1-q)f(N_+-1)+N_-qf(N_++1)\quad \text{for } N_+>\frac{N}{2}. \tag{24}$$

At first glance the above relations seem quite complicated, but in fact they have a fairly simple structure. In particular, it is easy to see that Eq. (22), which is valid for $N_+ = 0,1,2,\dots,\frac{N}{2}-1$, can be written in the form

$$N_+F(N_+-1)=N_-F(N_+), \tag{25}$$

where

$$F(N_+)=(1-q)f(N_++1)-qf(N_+). \tag{26}$$

When examining Eq. (25) for $N_+=0$ one gets $F(0)=0$. Then, using $F(0)=0$ in the same equation for $N_+=1$, one gets $F(1)=0$. In a similar way, one can show that for each value of $N_+<\frac{N}{2}$, one has $F(N_+)=0$, i.e.,

$$f(N_+)=\frac{1-q}{q}f(N_++1)\quad \text{for } N_+=0,1,\dots,\frac{N}{2}-1. \tag{27}$$

Hence, for the subsequent values of N_+ one gets

$$f\left(\frac{N}{2}-1\right)=f\left(\frac{N}{2}\right)\left(\frac{1-q}{q}\right), \tag{28}$$

$$f\left(\frac{N}{2}-2\right)=f\left(\frac{N}{2}\right)\left(\frac{1-q}{q}\right)^2, \tag{29}$$

$$\vdots$$

and finally, for $N_+=\frac{N}{2}+\frac{M}{2}$, where $M<0$ [see Eq. (16)], the dummy function $f(N_+)$ is given by

$$f(N_+)=f\left(\frac{N}{2}\right)\left(\frac{1-q}{q}\right)^{-M/2}. \tag{30}$$

In a similar way, from Eq. (24) one can show that [cf. Eq. (27)]

$$f(N_+)=\frac{1-q}{q}f(N_+-1)\quad \text{for } N_+=\frac{N}{2}+1,\dots,N \tag{31}$$

and hence for $N_+=\frac{N}{2}+\frac{M}{2}$, where $M>0$, one gets [cf. with Eq. (30)]

$$f(N_+)=f\left(\frac{N}{2}\right)\left(\frac{1-q}{q}\right)^{M/2}. \tag{32}$$

Summarizing, from Eqs. (30) and (32) it directly follows that, in the stationary regime of the majority-vote model on a complete graph with an even number of nodes, the probability of occurrence of a configuration Ω is given by [see Eq. (20)]

$$P(\Omega)=P_0(q)\sqrt{\frac{1-q}{q}}^{\,|M(\Omega)|}, \tag{33}$$

where

$$P_0(q)=f\left(\frac{N}{2}\right) \tag{34}$$

is the probability of a microstate with zero magnetization, which can be calculated from the normalization condition

$$\begin{aligned}\sum_{\Omega}P(\Omega)&=\sum_{N_+=0}^{N}\binom{N}{N_+}f(N_+)\\ &=P_0(q)\sum_{N_+=0}^{N}\binom{N}{N_+}\sqrt{\frac{1-q}{q}}^{\,|2N_+-N|}=1.\end{aligned} \tag{35}$$

Let us also note that, in Eq. (35), the expression under the second sum stands for the probability of the system to have exactly N_+ positive spins or to have magnetization $M = 2N_+ - N$, i.e.,

$$P(M) = P_0(q)\binom{N}{\frac{N+M}{2}}\sqrt{\frac{1-q}{q}}^{|M|}. \quad (36)$$

Finally, proceeding in a similar way as shown in Eqs. (17)–(36), one can also find exact expressions for $P(\Omega)$ and $P(M)$ in systems with an odd number of nodes N and consequently, with an odd magnetization M. The mentioned expressions have the forms

$$P(\Omega) = P_{+1}(q)\sqrt{\frac{1-q}{q}}^{|M(\Omega)|-1} \quad (37)$$

and

$$P(M) = P_{+1}(q)\binom{N}{\frac{N+M}{2}}\sqrt{\frac{1-q}{q}}^{|M|-1}, \quad (38)$$

where $P_{+1}(q)$ is the probability of a microstate with magnetization $M = +1$ and it can be shown that $P_{+1}(q) = P_{-1}(q)$.

At this point, it is important to note that Eq. (36), describing the probability of the studied system to have exactly N_+ positive spins, is the same as Eq. (9.9) in [15]. The mentioned Eq. (9.9) characterizes the stationary state of the generalized Ehrenfest urn model and suggest that the two models are equivalent and can be mapped onto each other. In the Ehrenfest model, one has N particles that can occupy two states: A and B (+1 and −1, respectively). At successive time intervals one chooses a particle at random and checks in which state it is. If it is in a state with a larger number of particles, it changes its state with probability p. Otherwise, the particle's state changes with probability $1 - p$. Indeed, a thorough investigation of Eq. (3) for the complete graph confirms the equivalence for $q \equiv 1 - p$, where q is the MV parameter. This observation is interesting in itself because it is not obvious at first glance.

B. Comparison between theoretical predictions and numerical simulation results

The above theoretical predictions can be verified by numerical simulations. In particular, in Fig. 1, theoretical and numerical probability distributions of magnetization $P(M)$ for different values of the noise parameter q are shown to perfectly agree with each other. In this figure, for $q < \frac{1}{2}$, one can see that $P(M)$ is symmetric and bimodal. The behavior indicates that below the critical value of the noise parameter $q_c = \frac{1}{2}$, the considered MV model is in a ferromagnetic phase. [The value of $q_c = \frac{1}{2}$ has also been recently obtained from mean-field analysis as the limiting case of the critical noise in classical random graphs; see Eq. (12) in [9].] The ferromagnetic ordering for $q < q_c$ is characterized by a nonzero absolute value of the average magnetization per spin (see Fig. 2).

For $q > \frac{1}{2}$, the distribution $P(M)$ is unimodal with the most likely value of the magnetization equal to zero (see Fig. 1). In this range of the noise parameter, i.e., above the critical value of $q_c = \frac{1}{2}$, the probability that magnetization of the system is equal to zero $P(M = 0)$ is a monotonically increasing function of q (see Fig. 3). This probability reaches its maximum value of unity for $q = 1$, when the system is an ideal antiferromagnet. [See the discussion of the parameter q below Eq. (5) in Sec. II.]

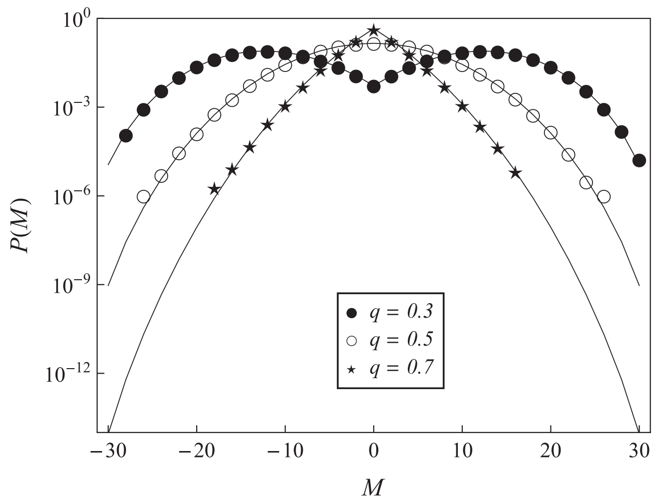

FIG. 1. Probability $P(M)$ that the magnetization of the majority-vote model on a complete graph of size $N = 30$ is equal to M. The scattered points represent the results of numerical simulations averaged over $10^4 \times N$ independent realizations of the model. Different symbols correspond to different values of the noise parameter q, given in the legend. The solid curves stand for the theoretical prediction according to Eq. (36).

C. Detailed balance condition

Knowing the exact expression for the probability distribution $P(\Omega)$, one can show that the MV model on a complete

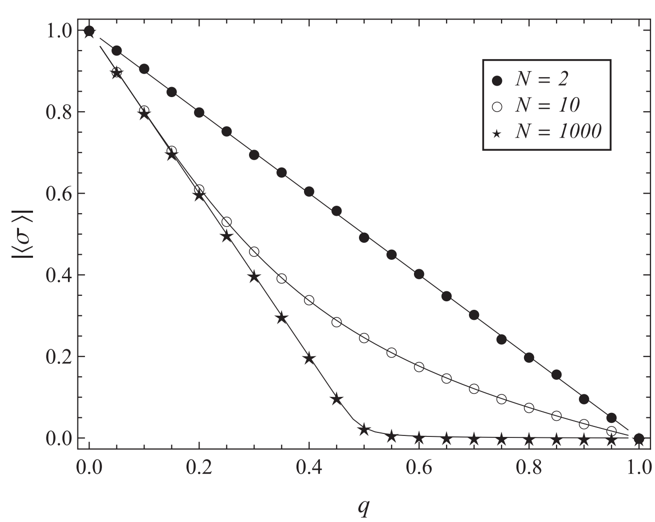

FIG. 2. Absolute value of the average magnetization per spin $|\langle\sigma\rangle|$ vs the noise parameter q in the MV model on complete graphs of various sizes N. The scattered points represent the results of numerical simulations for different values of N (see the legend). The solid curves result from the theoretical prediction $|\langle\sigma\rangle| = \frac{\langle|M|\rangle}{N}$, where $\langle|M|\rangle = \sum_M |M|P(M)$, with $P(M)$ given by Eq. (36).

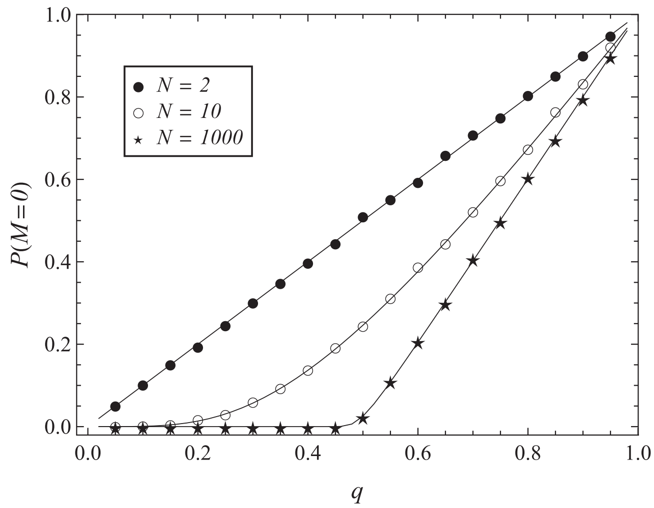

FIG. 3. Probability that magnetization of the system is equal to zero $P(M=0)$ vs the noise parameter q for different system sizes N. As in the previous figures, the scattered points represent numerical simulation results, while the solid curves are the theoretical predictions according to Eq. (36), i.e., $P(M=0)=\binom{N}{N/2}P_0(q)$.

graph fulfills the detailed balance condition [cf. Eq. (9)]

$$\frac{P(\Omega)}{P(\Omega_i)}=\frac{w_i(\Omega_i)}{w_i(\Omega)}. \tag{39}$$

Below, as in Sec. III A, we consider in detail only the case of an even system size N. The case of an odd N may be analyzed in a similar way. Thus, from Eq. (33) one gets that

$$\frac{P(\Omega)}{P(\Omega_i)}=\sqrt{\frac{1-q}{q}}^{\Delta|M|}, \tag{40}$$

where

$$\Delta|M|=|M(\Omega)|-|M(\Omega_i)|. \tag{41}$$

It is easy to see that, since

$$|M(\Omega_i)|=|M(\Omega)-2\sigma_i|=\begin{cases}|M|+2 & \text{for } \sigma_i M\leqslant 0\\ |M|-2 & \text{for } \sigma_i M\geqslant 2,\end{cases} \tag{42}$$

then the difference of the absolute values of magnetization in successive spin configurations is given by

$$\Delta|M|=\begin{cases}-2 & \text{for } \sigma_i M\leqslant 0\\ +2 & \text{for } \sigma_i M\geqslant 2.\end{cases} \tag{43}$$

Therefore, the left-hand side of the detailed balance condition (40) becomes

$$\frac{P(\Omega)}{P(\Omega_i)}=\begin{cases}q/(1-q) & \text{for } \sigma_i M\leqslant 0\\ (1-q)/q & \text{for } \sigma_i M\geqslant 2.\end{cases} \tag{44}$$

In a similar way, one can show that the right-hand side of Eq. (39), which is a quotient of the rate transitions $w_i(\Omega)$ and $w_i(\Omega_i)$, also depends only on $\sigma_i M$. To see this, let us note that, in Eqs. (12) and (13), the product $\sigma_i S(M-\sigma_i)$ may have only two values:

$$\begin{aligned}\sigma_i S(M-\sigma_i)&=S(\sigma_i(M-\sigma_i))\\ &=S(\sigma_i M-1)\\ &=\begin{cases}-1 & \text{for } \sigma_i M\leqslant 0\\ +1 & \text{for } \sigma_i M\geqslant 2.\end{cases}\end{aligned} \tag{45}$$

Accordingly, the right-hand side of Eq. (39) can be written as

$$\frac{w_i(\Omega_i)}{w_i(\Omega)}=\begin{cases}q/(1-q) & \text{for } \sigma_i M\leqslant 0\\ (1-q)/q & \text{for } \sigma_i M\geqslant 2.\end{cases} \tag{46}$$

The correspondence between Eqs. (44) and (46) allows one to state that the detailed balance condition holds true in the majority-vote model on complete graphs. This means that the considered system is ergodic and, in the stationary regime, there exists its equilibrium representation in the sense of the canonical ensemble,

$$P(\Omega)\propto e^{-\mathcal{H}(\Omega)}, \tag{47}$$

with the Hamiltonian given by [cf. Eqs. (33) and (37)]

$$\mathcal{H}(\Omega)=\ln\sqrt{\frac{q}{1-q}}|M(\Omega)|. \tag{48}$$

IV. SUMMARY

The presented work is theoretical in nature. We have studied the isotropic majority-vote model, which, apart from the one-dimensional case, is thought to be nonequilibrium. We found that if this model is defined on a complete graph, then, in the stationary regime, it is equivalent to the modified Ehrenfest urn model and has an equilibrium representation in the sense of the canonical ensemble. We showed that the probability distribution $P(\Omega)$ of finding the system in a certain microstate $\Omega=(\sigma_1,\sigma_2,\ldots,\sigma_N)$, where $\sigma_i=\pm 1$, depends only on the absolute value of magnetization: $P(\Omega)\propto\sqrt{(1-q)/q}^{|M(\Omega)|}$, where $M(\Omega)=\sum_{i=1}^N\sigma_i$ and q is the noise parameter of the model. The result was obtained from the master equation for the microstate distribution $P(\Omega)$ and it agrees with the equivalent result for the Ehrenfest model that was previously obtained from the rate equation for the magnetization distribution $P(M)$. Our theoretical predictions perfectly agree with the results of numerical simulations performed for systems of various sizes, $N\geqslant 2$. Analytical results, which were described in this work, are the first step to determine exact values of the mean-field critical exponents of the MV model.

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