Finite-size scaling in the system of coupled oscillators with heterogeneity in coupling strength

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(Received 28 February 2017; published 14 July 2017)

We consider a mean-field model of coupled phase oscillators with random heterogeneity in the coupling strength. The system that we investigate here is a minimal model that contains randomness in diverse values of the coupling strength, and it is found to return to the original Kuramoto model [Y. Kuramoto, Prog. Theor. Phys. Suppl. **79**, 223 (1984)] when the coupling heterogeneity disappears. According to one recent paper [H. Hong, H. Chaté, L.-H. Tang, and H. Park, Phys. Rev. E **92**, 022122 (2015)], when the natural frequency of the oscillator in the system is "deterministically" chosen, with no randomness in it, the system is found to exhibit the finite-size scaling exponent $\bar{\nu} = 5/4$. Also, the critical exponent for the dynamic fluctuation of the order parameter is found to be given by $\gamma = 1/4$, which is different from the critical exponents for the Kuramoto model with the natural frequencies randomly chosen. Originally, the unusual finite-size scaling behavior of the Kuramoto model was reported by Hong *et al.* [H. Hong, H. Chaté, H. Park, and L.-H. Tang, Phys. Rev. Lett. **99**, 184101 (2007)], where the scaling behavior is found to be characterized by the unusual exponent $\bar{\nu} = 5/2$. On the other hand, if the randomness in the natural frequency is removed, it is found that the finite-size scaling behavior is characterized by a different exponent, $\bar{\nu} = 5/4$ [H. Hong, H. Chaté, L.-H. Tang, and H. Park, Phys. Rev. E **92**, 022122 (2015)]. Those findings brought about our curiosity and led us to explore the effects of the randomness on the finite-size scaling behavior. In this paper, we pay particular attention to investigating the finite-size scaling and dynamic fluctuation when the randomness in the coupling strength is considered.

DOI: 10.1103/PhysRevE.96.012213

I. INTRODUCTION

Collective synchronization phenomena have been widely explored via various models of coupled phase oscillators. In particular, the Kuramoto model and its many variant models have been mostly considered to understand the mechanism of the synchronization behavior observed in diverse systems in nature [1–14].

The models mostly contain the positive coupling (interaction) between the oscillators, where the positive coupling can be regarded as the reasonable one, based on the fact that the attractive interaction is realistic and natural in most physical systems. However, in some biological systems, not only the positive interaction between the components in the system but also the negative one can exist [15]. Considering that, there have been some works on collective behavior in the systems of coupled oscillators with mixed coupling of both positive and negative interactions [11–14].

In addition to the positivity and negativity of the interaction, the heterogeneity of the coupling strength is also worth being considered, since the coupling strength in most real systems may not be identical; rather, it may be diverse in values. Based on these facts, in this paper, we consider the heterogeneity of the coupling strength and explore how the diverse values of the coupling strength affect the phase-synchronization behavior. In particular, we focus on the effects of the "random" heterogeneity of the coupling strength.

There have been previous works on the effects of randomness on collective synchronization [2,3,8–14,16]. In particular, the unusual finite-size scaling behavior of the Kuramoto model was reported in Ref. [3], where the finite-size scaling exponent $\bar{\nu} = 5/2$ was derived. On the other hand, if the randomness is removed from the system, the finite-size scaling behavior is found to be changed, showing another exponent, $\bar{\nu} = 5/4$ [2]. The dynamic fluctuation has been also investigated, and the exponent for the dynamic fluctuation is found to be also changed by removing the randomness. Those findings raised curiosity and led us to explore the effects of the randomness on the finite-size scaling and dynamic fluctuation behavior in the system, which is the main issue of this paper.

On the one hand, we might consider various types of randomness: for example, randomness in the coupling strength (which we study in this paper), randomness in natural frequency, randomness in the delay of the coupling function, and randomness in connection weights. Among those, the randomness in connection weights might be regarded as some kind of random coupling strength between the oscillators. In fact, about that issue there have been some previous studies [10,11], where the authors in the studies considered the random constraints J_{ij} distributed according to a Gaussian distribution function and explored collective behavior of the system. In particular, the authors of the paper reported a new type of ordered state (in some sense glassy) that is characterized by quasientrainment and algebraic relaxation. Interestingly, instead of the usual synchronization-desynchronization transition some sort of "glassy" transition has been reported, which is the effect of the randomness in the connection or coupling weights. However, the works have been mostly done by the numerics due to the difficulties in treating the issue analytically; accordingly, further studies on the glassy behavior are still required.

There exists another study that has worked on the randomness in the delay of the coupling function [16], where the authors have considered heterogeneous delays in the coupling and explored how the heterogeneous delays affect the cluster synchronization. The authors found that the parity of heterogeneous delays plays a crucial role in determining the mechanism of cluster formation, inducing a rich cluster pattern.

Also, there have some studies on the effects of the randomness in the connection or link between the oscillators [8], where the authors have paid attention to exploring the effects by the connectivity or link disorder on the synchronization transition. In particular, the case of complex networks including scale-free networks with degree distribution given by the power law $P(k) \sim k^{-\lambda}$ has been considered, and the nature of the synchronization transition has been investigated in detail. According to the work, the finite-size scaling behavior can be differently characterized depending on the degree exponent: the finite-size scaling exponent is found to be $\bar{\nu} = (\lambda - 1)/(\lambda - 3)$ for the region with $3 < \lambda < 4$ and $\bar{\nu} = (2\lambda - 5)/(\lambda - 3)$ for $4 < \lambda < 5$, respectively. For the region with large degree exponent ($\lambda > 5$), the finite-size scaling is found to be same as that of the original Kuramoto model, displaying $\bar{\nu} = 5/2$, which is consistent with the findings reported in Ref. [3]. This result can be acceptable as a reasonable one, considering the fact that the region with high degree exponent ($\lambda > 5$) corresponds to the mean-field regime.

In this paper, we consider the randomness in heterogeneous coupling strength and pay attention to exploring how the randomness affects the finite-size scaling behavior and dynamic fluctuation in the system.

This paper consists of six sections. Section II introduces the model we study in the current paper. In Sec. III, we explore the phase-synchronization behavior in the system. The self-consistency equation for the order parameter is derived, and the synchronization transition is investigated. The dynamic fluctuation of the order parameter is also examined in Sec. III. The analysis on the finite-size scaling is presented in Sec. IV. Section V is devoted to understanding the change of the finite-size scaling exponent by investigating the sample-dependent correction in the entrained oscillators. Section VI gives a brief summary.

II. MODEL

We begin with the model system governed by

$$\dot{\phi}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N} J_j \sin(\phi_j - \phi_i), \quad i = 1,\ldots,N, \qquad (1)$$

where ϕ_i represents the ith oscillator's state described by an angle in $[0,2\pi)$. The ω_i on the right-hand side of Eq. (1) is the natural frequency of the ith oscillator, chosen from a distribution function $g(\omega)$. For convenience, we take it as the Lorentzian one given by $g(\omega) = \frac{\Delta}{\pi}\frac{1}{\omega^2+\Delta^2}$ with zero mean and width Δ. The J_j in Eq. (1) denotes the coupling strength of the jth oscillator, chosen from another random distribution function $h(J)$. We take the Gaussian distribution function given by

$$h(J) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(J-\mu)^2/2\sigma^2} \qquad (2)$$

for convenience, where σ and μ denote the standard deviation and the mean of $h(J)$, respectively. Note that ω_i in Eq. (1) is "deterministically" chosen from $g(\omega)$; thus, it does not contain any randomness in it. Instead, the coupling strength J_i is "randomly" chosen from $h(J)$; accordingly, it contains randomness in it. We are here curious about the effects of the randomness in J_i in the system with deterministically chosen ω_i, which is the reason why we take ω_i and J_i like that.

In the model described as Eq. (1), each oscillator is assigned its own coupling strength as well as its own natural frequency. In particular, the coupling strength J_j can be either positive ($J_j > 0$) or negative ($J_j < 0$), where the positive and negative values can be regarded as the "attractive" and "repulsive" interactions between the oscillators, respectively. We note that the interaction is not *mutually equal* one between the oscillators. In other words, the coupling strength from the ith oscillator to the jth one can be different from that of the jth oscillator to the ith one. In practice, in some biological systems, such as neural networks with excitatory and inhibitory neurons [15], such types of interaction can be more realistic.

The system with positive and negative coupling strength was previously studied in Refs. [14,17], where the double-δ distribution function such as $h(J) = p\delta(J-1) + (1-p)\delta(J+1)$ has been mostly considered, where p is the probability that the oscillators have positive coupling strength. In the previous studies, only two values of the coupling strength have been taken for simplicity, and various interesting features of the system have been reported. We here consider a variety of random values for the coupling strength, by randomly choosing the coupling strength from the Gaussian distribution given by Eq. (2), and explore how the randomness in the heterogeneous coupling strength affects the phase-synchronization transition and the finite-size scaling behavior.

III. PHASE-SYNCHRONIZATION TRANSITION

In this section, we investigate the phase-synchronization behavior, with particular attention to the possibility of the synchronization transition and the critical behavior near the transition. Collective synchronization has been conveniently described by the complex order parameter Z defined as [1]

$$Z \equiv Re^{i\Theta} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}, \qquad (3)$$

where R measures the magnitude of the phase coherence (synchronization), and Θ denotes the average phase (angle) of the synchronized oscillators. On the one side, when the system has random heterogeneity in the coupling strength, such as in the present model given by Eq. (1), we can consider another order parameter given by [17,18]

$$W \equiv Se^{i\Phi} = \frac{1}{N}\sum_{j=1}^{N} J_j e^{i\phi_j}. \qquad (4)$$

We note that the coupling parameter J_j is multiplied in front of $e^{i\phi_j}$, as shown in Eq. (4). In that sense, the order parameter W can be regarded as either an "effective" coupling strength or a sort of "weighted" mean field. In fact, it is advantageous to take the new order parameter W: Theoretical analysis can be possible by introducing W into the system governed by Eq. (1). Moreover, since R $(= |Z|)$ is found to be proportional to S $(= |W|)$ near the transition, we can estimate the behavior of R via the analysis of S.

For the theoretical analysis on Eq. (1), we first insert the order parameter W $(\equiv Se^{i\Phi})$ into the model. Equation (1) is then rewritten as

$$\dot{\phi}_i = \omega_i - S\sin(\phi_i - \Phi), \tag{5}$$

where $i = 1,\ldots,N$. The system is now expected to be divided into two subpopulations, depending on the condition: One is the population of the oscillators with $|\omega_i| \leqslant S$, and the other one is that of the oscillators with $|\omega_i| > S$. In other words, one population consists of the "locked" oscillators having zero phase velocity ($\dot{\phi}_i = 0$), and the other one consists of the "unlocked" oscillators having nonzero phase velocity ($\dot{\phi}_i \neq 0$). The locked oscillators have the stationary phase given by $\phi_i^{(s)} = \Phi + \sin^{-1}(\omega_i/S)$, mostly contributing to the synchronization behavior in the system. On the other hand, the unlocked oscillators exhibit drifting behavior, having their own natural frequency ω_i, which means that the phase of the drifting oscillator is given by $\phi_i \approx \phi_i(0) + \omega_i t$. We note that the above description is similar to that for the original Kuramoto model. However, for the case of the Kuramoto model, the order parameter S should be replaced with the order parameter R multiplied by the identical coupling strength.

Based on the idea of the division of the oscillators into two subpopulations of the locked and unlocked oscillators, we find that the self-consistency equation for the order parameter S is given by [18]

$$\begin{aligned} Se^{i\Phi} &= \int_{-\infty}^{\infty}\int_{|\omega|\leqslant S} Je^{i\Phi}e^{i\sin^{-1}(\omega/S)}h(J)g(\omega)\,d\omega\,dJ \\ &= \langle J\rangle \int_{-S}^{S}\sqrt{1-(\omega/S)^2}g(\omega)\,d\omega, \end{aligned} \tag{6}$$

where $\langle J\rangle = \int Jh(J)\,dJ$ is the mean value of the distribution function $h(J)$. For the Gaussian distribution shown in Eq. (2), the mean is given by $\langle J\rangle = \mu$. Expanding $g(\omega)$ near $\omega = 0$, Eq. (6) reads

$$S = \frac{\pi}{2}g(0)\mu S + \frac{\pi}{16}g''(0)\mu S^3 + O(S^5). \tag{7}$$

Note that the first derivative, $g'(0)$, vanishes due to the symmetric property of $g(\omega)$ around $\omega = 0$; thus, the term is invisible in Eq. (7). As shown in Eq. (7), when the mean is zero ($\mu = 0$), the order parameter S always vanishes; therefore, the order parameter R also goes to zero. This implies that the system with the coupling strength chosen from the symmetric distribution having zero mean does not exhibit phase synchronization. On the other hand, when the mean is a nonzero positive value ($\mu > 0$) the system is expected to show phase synchronization.

In Eq. (7), to have the nonzero solution with $S \neq 0$, it should be

$$S^2 = \frac{1 - \frac{\pi}{2}g(0)\mu}{\frac{\pi}{16}g''(0)\mu} > 0, \tag{8}$$

where $\mu > 0$. At μ_c, S goes to zero, which means that the transition occurs at

$$\mu_c = \frac{2}{\pi g(0)}, \tag{9}$$

which reads $\mu_c = 2\Delta$ for $g(\omega)$ chosen as the Lorentzian distribution given by $g(\omega) = \Delta/[\pi(\omega^2 + \Delta^2)]$.

We are curious about the effects induced by the randomness in the system. To see it, we first assume that the $\{\omega_i\}$ in Eq. (1) does not contain any randomness. Under this condition, we pay attention to exploring how the randomness affects the synchronization behavior of the system when the randomness comes into the system via the heterogeneity of the coupling strength $\{J_i\}$.

Unfortunately, the analysis of the self-consistency equation shown as Eqs. (6) and (7) does not discern between cases with or without randomness in the system, even though the transition point can be predicted. To discriminate the two cases, therefore, we now resort to the numerical analysis, performing the numerical simulations on Eq. (1).

First, to remove the randomness in the set of $\{\omega_i\}$, we *deterministically* assign ω_i on each site. Namely, the ω_i can be chosen according to the procedure given by [2]

$$\frac{i}{N} - \frac{1}{2N} = \int_{-\infty}^{\omega_i} g(\omega)\,d\omega. \tag{10}$$

For the Lorentzian distribution given by $g(\omega) = \frac{\Delta}{\pi}\frac{1}{\omega^2+\Delta^2}$, the above procedure generates the ω_i as follows:

$$\omega_i = \Delta\tan\left[\frac{i\pi}{N} - \frac{(N+1)\pi}{2N}\right], \quad i = 1,\ldots,N. \tag{11}$$

This comes from

$$\begin{aligned} \frac{i-1/2}{N} &= \frac{\Delta}{\pi}\int_{-\infty}^{\omega_i}\frac{1}{\omega^2+\Delta^2}\,d\omega, \\ &= \frac{1}{\pi}\tan^{-1}\left(\frac{\omega_i}{\Delta}\right) + \frac{1}{2}. \end{aligned} \tag{12}$$

We note that the system with $\{\omega_i\}$ given by Eq. (11) does not contain any randomness in it. Instead, the randomness is included in the coupling strength J_i that is randomly chosen according to the random Gaussian distribution function $h(J)$ given as Eq. (2).

To see how the randomness in the heterogeneous coupling strength affects the phase-synchronization behavior, we now numerically explore it. We perform the numerical integrations on Eq. (1), using the fourth-order Runge-Kutta method with $dt = 0.01$. For the total $N_t = 1\times10^6$ time steps, Eq. (1) was integrated, where the first $N_t/2$ steps were discarded for the equilibrium state, after which all quantities of interest were measured and averaged over time. We measure the order parameters R and S for various values of the mean μ, for a given value of the width Δ. Figure 1 shows the behavior of R and S as a function of μ, where $\Delta = 0.5$, varying the system size N. The theoretical prediction on the transition ($\mu_c = 2\Delta$) reads $\mu_c = 1$ for $\Delta = 0.5$, which is shown together in Fig. 1. We observe that both R and S show decreasing behavior for $\mu < \mu_c$ when the system size N increases. On the other hand, the order parameters for $\mu > \mu_c$ do not show size-dependent behavior. This suggests the possibility of the synchronization transition at $\mu = \mu_c$, which is consistent with the theoretical prediction in Eq. (9).

To see the existence of the synchronization transition at μ_c further, we now measure another quantity such as Binder's

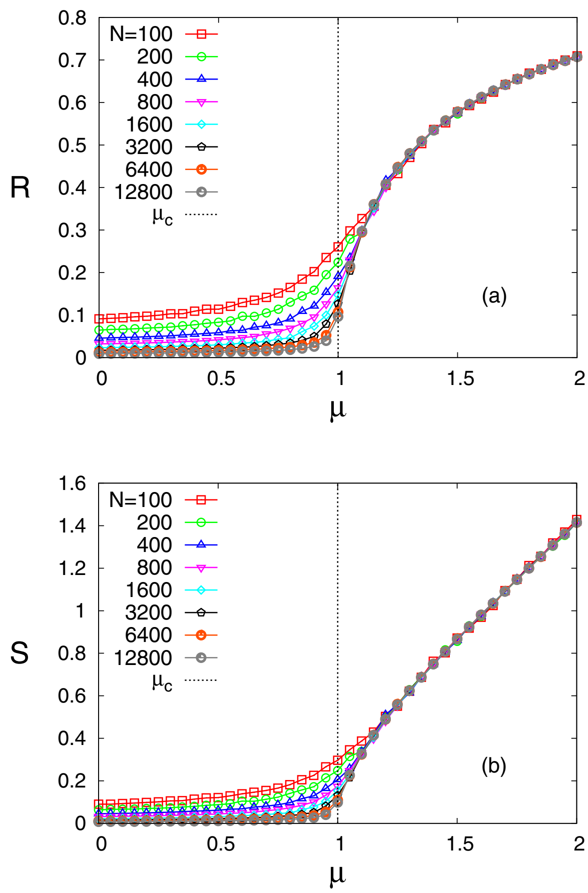

FIG. 1. (a) Behavior of the order parameter R as a function of the mean μ, varying the system size N from $N = 100$ to $N = 12\,800$. (b) Behavior of the order parameter S plotted as a function of μ, for various system size N. The width of the natural-frequency distribution function is set to $\Delta = 0.5$. The theoretical prediction about the transition point, $\mu_c = 2\Delta$, is also shown as a dotted line with the label μ_c. The data for both R and S have been averaged over one hundred samples with different sets $\{J_j\}$ and different initial conditions $\{\phi_i(0)\}$, where the magnitude of the error is the symbol size.

fourth-order cumulant B_R, defined as [19]

$$B_R \equiv \left[1 - \frac{\langle R^4 \rangle}{3\langle R^2 \rangle^2}\right], \tag{13}$$

where $R = |Z|$, and $\langle \cdots \rangle$ and $[\cdots]$ represent the time average and the sample average, respectively. Here, the sample average means the average over various configurations with different sets of the coupling strength, $\{J_j\}$, and different initial conditions $\{\phi_i(0)\}$. The Binder cumulant has been widely used as a useful indicator for predicting the transition point in many physical systems. When the transition exists in the system, the Binder cumulant in the subcritical region (e.g., $\mu < \mu_c$) approaches $1/3$ as the system size N increases. On the other hand, in the supercritical region ($\mu > \mu_c$) it goes to $2/3$. At the transition ($\mu = \mu_c$), the Binder cumulant should have a unique crossing point for different sizes N. This characteristic behavior is shown in Fig. 2, which supports the existence of the synchronization transition at $\mu = \mu_c$.

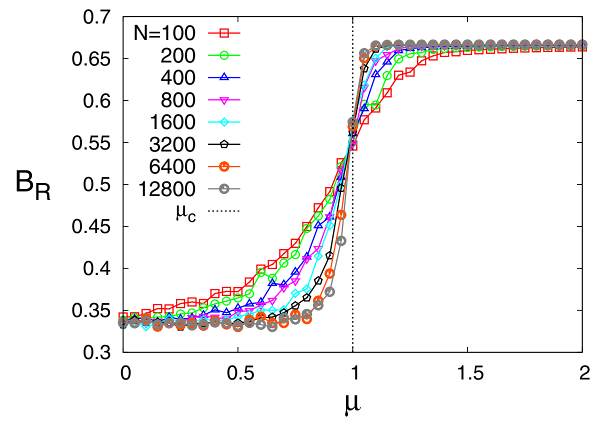

FIG. 2. Binder's fourth-order cumulant for the order parameter R as a function of the mean value μ for various system size N, for a given value of $\Delta = 0.5$. The theoretical prediction for the transition point, $\mu_c = 1$, is also shown by the dotted line with the label μ_c. The data have been averaged over one hundred samples with different sets $\{J_j\}$ and different initial conditions $\{\phi_i(0)\}$, where the magnitude of the error is the symbol size.

We also measured the dynamic fluctuation of the order parameter, where the dynamic fluctuation is defined as

$$\chi_R = N[\langle R^2 \rangle - \langle R \rangle^2] \tag{14}$$

with $R = |Z|$. Figure 3 shows the behavior of χ_R as a function of μ, for a given value of $\Delta = 0.5$, which also supports the presence of the synchronization transition at $\mu_c = 1$. The χ_R is found to increase near the transition, as the system size N increases, which implies that the dynamic fluctuation diverges in the thermodynamic limit ($N \to \infty$), suggesting the possible phase transition at μ_c.

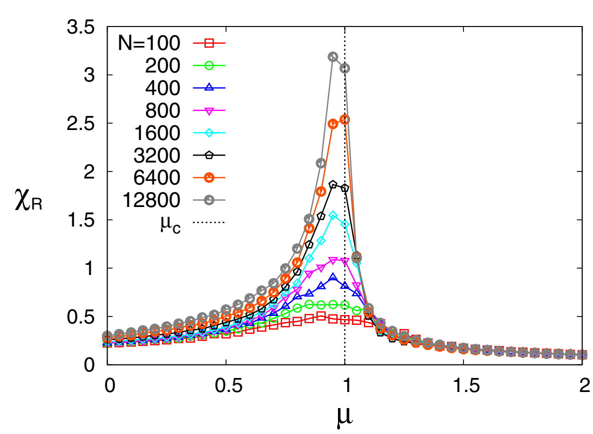

FIG. 3. Dynamic fluctuation χ_R as a function of the mean value μ for various system size N, where $\Delta = 0.5$. The theoretical prediction of the transition point, $\mu_c = 1$, is also displayed as the dotted line with the label μ_c. The data have been averaged over one hundred samples with different sets $\{J_j\}$ and different initial conditions $\{\phi_i(0)\}$, where the magnitude of the error is the symbol size.

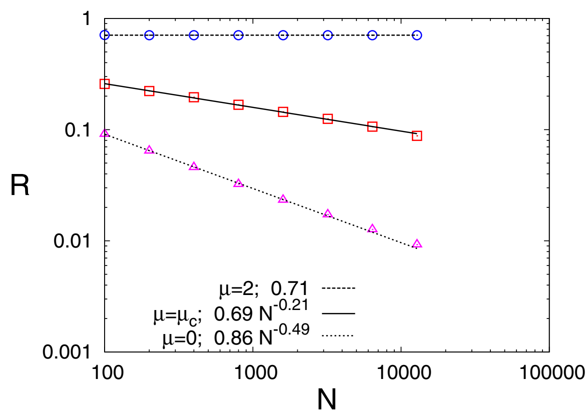

FIG. 4. Order parameter R versus the system size N is shown in log-log scale, for three different values: $\mu = 0(<\mu)$, $\mu = 1(=\mu_c)$, and $\mu = 2(>\mu_c)$. At the transition ($\mu = \mu_c$), R is found to behave as $R \sim N^{-0.21}$.

IV. FINITE-SIZE SCALING

In this section, we investigate the critical behavior of R at the transition μ_c. To do it, we now examine the finite-size scaling behavior of R depending on the value of μ. Figure 4 shows the size-dependent behavior of the order parameter R for three different regions with $\mu < \mu_c$, $\mu > \mu_c$, and $\mu = \mu_c$, respectively. We find that the order parameter R behaves as $R \sim N^{-1/2}$ when the mean is small enough ($\mu < \mu_c$), which implies the presence of the incoherent phase ($R = 0$) in the thermodynamic limit ($N \to \infty$). On the other hand, when the mean is large enough ($\mu > \mu_c$), the order parameter shows a finite value ($R > 0$), irrespective of the various sizes N, which implies the existence of the synchronized state. This finding is consistent with the mean-field analysis, showing that the transition occurs at μ_c, from the incoherent state ($R = 0$) to the coherent one ($R > 0$).

According to the finite-size scaling theory [2], the order parameter is expected to show a critical decaying behavior characterized by $R \sim N^{-\beta/\bar{\nu}}$ at the transition $\mu = \mu_c$. Here, β is the critical exponent for the order parameter, and $\bar{\nu}$ is the finite-size scaling exponent, respectively, which is shown later. Analyzing the self-consistency equation given by Eqs. (6) and (7), we find $S \sim (\mu - \mu_c)^{1/2}$, where $\mu_c = 2\Delta$. This means that the order parameter exponent is given by $\beta = 1/2$. We numerically investigated the size-dependent behavior of R at the transition ($\mu = \mu_c$). Figure 4 shows that the order parameter R at the transition behaves as $R \sim N^{-0.21}$, which implies $\beta/\bar{\nu} = 0.21$. With $\beta = 1/2$, this suggests $\bar{\nu} \approx 5/2$.

Note that, for the case of the Kuramoto model with deterministically chosen natural frequencies, it has been reported that the model shows $\bar{\nu} = 5/4$ [2], which is different from $\bar{\nu} \approx 5/2$ for the present system in this paper. In the current system, we have chosen ω_i deterministically, following the procedure given by Eq. (11); thus, there is no randomness in the set of the natural frequencies. However, the coupling strength J_i in the current system contains the randomness in it: We have chosen it randomly according to the Gaussian distribution function $h(J)$.

To understand $\bar{\nu} = 5/2$ for the current system, we now revisit the finite-size scaling theory of the order parameter further. According to the conventional finite-size scaling theory [2], we expect that the order parameter R at finite size N can be written as

$$R(\mu,N) = N^{-\beta/\bar{\nu}} f(\epsilon N^{1/\bar{\nu}}), \tag{15}$$

where $\epsilon = \mu - \mu_c$, and $f(x)$ is a scaling function having the asymptotic properties

$$f(x) = \begin{cases} \text{const}, & x = 0 \\ x^{\beta}, & x \gg 1 \\ (-x)^{\beta-\bar{\nu}/2}, & x \ll -1. \end{cases} \tag{16}$$

The critical decay of the order parameter R is thus expected to be characterized as $R \sim N^{-\beta/\bar{\nu}}$ at the transition ($\epsilon = 0$). Accordingly, the size dependence of R at the transition ($\mu = \mu_c$), shown in Fig. 4, implies $\beta/\bar{\nu} = 0.21$. Since we found that the order parameter exponent is given by $\beta = 1/2$ from the analysis of the self-consistency equation given by Eq. (7), we deduce $\bar{\nu} \approx 5/2$. This result means that the finite-size scaling exponent $\bar{\nu}$ is the same as that for the Kuramoto model with randomly (not deterministically) chosen natural frequencies [2]. In other words, the system has the same finite-size scaling exponent ($\bar{\nu} = 5/2$) when the randomness of the coupling strength newly comes into the system with no randomness in $\{\omega_i\}$.

Meanwhile, in the case of the dynamic fluctuation given by Eq. (14), its critical behavior is characterized as [2]

$$\chi_R = \begin{cases} (-\epsilon)^{-\gamma}, & \epsilon < 0 \\ \epsilon^{-\gamma}, & \epsilon > 0 \end{cases} \tag{17}$$

in the thermodynamic limit ($N \to \infty$). We here assume that the scaling of χ_R can be controlled by one single exponent, γ, which is valid for most homogeneous systems. According to the finite-size scaling theory [2], the critical increasing behavior of χ_R at the transition ($\mu = \mu_c$) can be characterized as $\chi_R \sim N^{\gamma/\bar{\nu}}$. We numerically measured χ_R at the transition for various system size N (see Fig. 5). We find that χ_R behaves as $\chi_R \sim N^{0.41}$, where the power value 0.41 means $\gamma/\bar{\nu} \approx 0.41$, according to the finite-size scaling theory. With the value $\bar{\nu} = 5/2$, the power 0.41 gives us the value of γ: $\gamma \approx 1$.

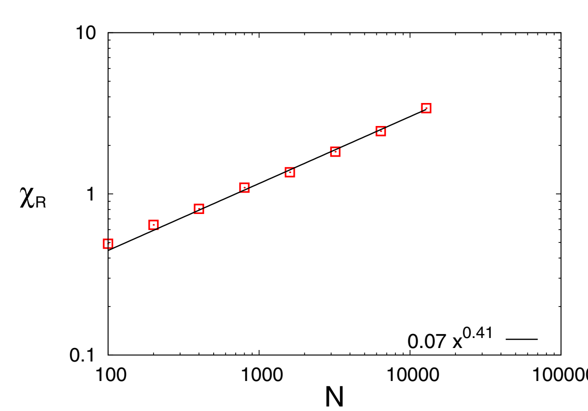

FIG. 5. Dynamic fluctuation χ_R is shown as a function of the system size N at the transition ($\mu = \mu_c$) in log-log scale. The χ_R is found to behave as $\chi_R \sim N^{0.41}$.

Summarizing, the critical exponents we found for the present system can be given by

$$\beta = 1/2, \quad \bar{\nu} \approx 5/2, \quad \gamma \approx 1, \tag{18}$$

which shows that the hyperscaling relation does not hold:

$$\bar{\gamma} \neq \bar{\nu} - 2\beta. \tag{19}$$

The breakdown of the hyperscaling relation is the same as that for the original Kuramoto model with randomly chosen natural frequencies. This finding means that adding some new randomness in the coupling strength makes the system comes back to the same universal class as the Kuramoto model with randomly chosen natural frequencies, which is the main result of this paper.

V. SAMPLE-DEPENDENT CORRECTION IN ENTRAINED OSCILLATORS

In this section, we try to understand the origin of $\bar{\nu} \approx 5/2$ even for the absence of any randomness in $\{\omega_i\}$.

We note that the self-consistency equation given by Eq. (7) is valid for the thermodynamic limit ($N \to \infty$). For a finite-size system, on the other hand, it can be written as

$$S = \frac{\pi}{2} g(0)\mu S + \frac{\pi}{16} g''(0)\mu S^3 + \Gamma_N, \tag{20}$$

where Γ_N is the term induced by the finite-size correction. Note that Γ_N is proportional to the number of entrained oscillators [2]:

$$\Gamma_N \propto \sqrt{N_s}/N, \tag{21}$$

where N_s is the number of entrained oscillators with $|\omega_i| < S$. We note that N_s is linearly proportional to $S \times N$, which reads $\Gamma_N \sim \sqrt{S/N}$. At $\mu = \mu_c$, Eq. (20) then leads to

$$S^3 \sim \sqrt{S/N}. \tag{22}$$

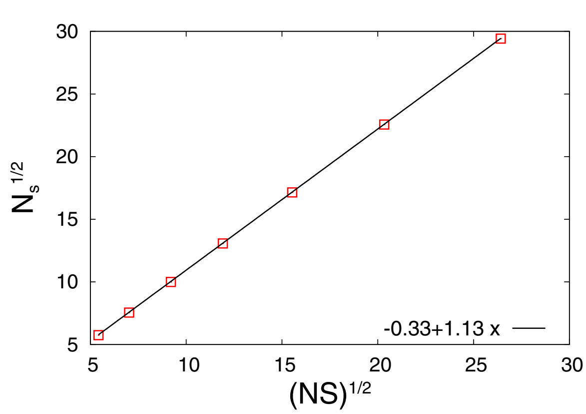

FIG. 6. Behavior of $\sqrt{N_s}$ as a function of $\sqrt{NS}$ at the transition ($\mu = \mu_c$), displaying the linearly proportional relation.

If we assume that $S \sim N^{-a}$ with an arbitrary power a, Eq. (22) then reads

$$N^{-3a} \sim N^{-(a+1)/2}, \tag{23}$$

which yields $a = 1/5$. This means $S \sim N^{-1/5}$ at the transition, where 1/5 corresponds to the critical exponents $\beta/\bar{\nu} = 1/5$. Substituting $\beta = 1/2$, we then find $\bar{\nu} = 5/2$, which is the same as the Kuramoto model with randomly distributed natural frequencies [2].

We have numerically measured N_s as a function of $N \times S$. Figure 6 shows $N_s^{1/2} \propto (N \times S)^{1/2}$, which is consistent with the prediction.

VI. SUMMARY

We have considered a minimal model of globally coupled phase oscillators with random heterogeneity in the coupling strength, and investigated how the randomness affects the phase synchronization behavior. In particular, we have paid attention to the synchronization transition and the finite-size scaling behavior at the transition. We found that the system exhibits the phase transition from the incoherent state to the synchronized one at a finite mean value of the distribution of the coupling strength. In particular, we found that the system exhibits $\beta = 1/2$, $\bar{\nu} \approx 5/2$, and $\gamma \approx 1$, which implies that the present system returns to the same universal class as that for the Kuramoto model with randomly chosen natural frequencies.

Based on the findings in the present paper, it seems that the randomness in coupling strength and the randomness in natural frequency play (effectively) the same role in the finite-size scaling of the synchronization behavior. Here, it is noteworthy that the two randomnesses can be regarded as a sort of "quenched disorder" that does not depend on time. In other words, once it is chosen initially from a certain random distribution function, its value persists for the whole dynamics of the synchronization, which is different from the usual "thermal noise" that does depend on time. In some sense, the quenched disorder can be considered stronger than the thermal noise, which allows us to expect that the quenched disorder yields stronger finite-size scaling in the synchronization behavior.

The different characteristics of the noise are found to induce different finite-size scaling behavior when the thermal noise comes into the system, where the finite-size scaling behavior is found to be characterized by the exponent, the same as that for many other conventional mean-field systems, i.e., $\bar{\nu} = 2$ [9]. According to the findings in many works, it seems that the quenched randomness in the natural frequency, randomness in coupling strength, and randomness in link or connectivity induce the same finite-size scaling exponent, $\bar{\nu} = 5/2$ [2,3,8], which is another interesting issue that requires further study.

ACKNOWLEDGMENTS

We thank Prof. H. Park for useful discussions. This research was supported by "Research Base Construction Fund Support Program" funded by Chonbuk National University in 2016 and NRF Grant No. 2015R1D1A3A01016345.

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