Properties and relative measure for quantifying quantum synchronization

Wenlin Li,[1] Wenzhao Zhang,[2] Chong Li,[1] and Heshan Song[1,*]
[1] *School of Physics, Dalian University of Technology, Dalian 116024, China*
[2] *Beijing Computational Science Research Center, Beijing 100193, China*
(Received 9 April 2017; published 14 July 2017)

Although quantum synchronization phenomena and corresponding measures have been widely discussed recently, it is still an open question how to characterize directly the influence of nonlocal correlation, which is the key distinction for identifying classical and quantum synchronizations. In this paper, we present basic postulates for quantifying quantum synchronization based on the related theory in Mari's work [Phys. Rev. Lett. **111**, 103605 (2013)], and we give a general formula of a quantum synchronization measure with clear physical interpretations. By introducing Pearson's parameter, we show that the obvious characteristics of our measure are the relativity and monotonicity. As an example, the measure is applied to describe synchronization among quantum optomechanical systems under a Markovian bath. We also show the potential by quantifying generalized synchronization and discrete variable synchronization with this measure.

DOI: 10.1103/PhysRevE.96.012211

I. INTRODUCTION

Synchronization phenomena in the quantum regime have recently been attracting a great deal of interest, and they have been investigated in different branches of quantum physics, such as quantum phase transitions [1,2], quantum clocks [3,4], and quantum information processing (QIP) [5–8]. Basically, quantum synchronization can be regarded as an extension of classical synchronization, which was first observed by Huygens in the 17th century [9]. Numerous mature quantum systems have served as platforms to demonstrate quantum synchronization. As representatives, cavity electrodynamics systems with two- or three-level emitters [10–12] and quantum oscillators [13–15] are widely used to illustrate quantum synchronization with discrete variables and continuous variables (CV), respectively. A number of theoretical works have proposed effective schemes to synchronize quantum systems with inequality dynamic parameters, e.g., the couplings of two systems with appropriate intensity [6,16,17], or the exertion of an additional control field based on quantum control theory [18]. Subsequently, the influence of a quantum environment, even a non-Markovian environment, on quantum synchronization is also discussed in open quantum systems [6,19–21]. The possibilities of probing the spectral density of a dissipative environment and inducing synchronization with noise were proposed in Refs. [22,23], respectively. In the QIP domain, some straightforward applications of quantum synchronization are applying synchronous systems as channels to transfer or share quantum states (signals) [7,24,25], even in a quantum array or a quantum network [6,8,20,26,27].

With the increasingly deepening and broadening developments of quantum synchronization theory and its applications, a fundamental problem becomes ever more noticeable, that is, how to describe quantitatively the degree of synchronization between two quantum systems so that cross-comparison among different synchronization phenomena can be done. In the classical regime, quantification of synchronization has developed into a fairly mature technology [28–31]; however, hardly any mature technologies can be used to measure quantum synchronization directly because quantum correlation plays an important role in quantum synchronization [6,32]. Synchronization and quantum correlation have been studied for a long time in their respective communities; however, the relation between the two started to be explored only recently. Therefore, it is still challenging to define a suitable measure for quantum synchronization with correlation terms. Additional insights corresponding to this problem can be divided into two categories: some of the theoretical descriptions about synchronization only focus on the expectation value of a quantum system so that the synchronizations can be measured via classical measures [20,32–34]. Recently, local quantum fluctuation was also considered in this kind of measure [32]. Notably, quantum correlation has not been introduced into this kind of measure. Other forms of measures are motivated by applying a quantum correlation measure, such as quantum discord, and a mutual information measure based on entropy theory, in the synchronization domain [15,16,35]. However, it seems that there does not exist a simple homology relationship between synchronization and quantum correlation.

Until now, a desirable combination of synchronization and quantum correlation has been Mari's measure ($\mathcal{S}_c$), because synchronization error and a nonlocal term are both included in its definition [6]. Basically, this measure satisfies the properties that people intuitively think quantum synchronization should have. However, some defects remain that restrict further applications. Most notably, this measure is designed based on absolute error but not relative error, and it is defined as an inverse function of error [36]. Due to these two characteristics, the measure value is not linearly dependent on the magnitude of the error. In the interval [0.4,1] the measure is quite sensitive, and a small increment of error will result in a significant decrease of the measure value. On the contrary, the measure is insensitive in the interval with a low value, and the disparity between the values corresponding, respectively, to synchronous and asynchronous systems is small.

The aim of this work is to address this problem, that is, we will develop a consistent and quantitative theory of synchronization for quantum system evolution on the premise of inheriting the advantages of Mari's measure. For a clearer

*hssong@dlut.edu.cn

explanation, we focus on a physical interpretation of quantum synchronization from the standpoint of quantum measurement, and we summarize some basic frameworks and postulates with which the synchronization measure should comply. The four postulates are boundedness, criterion, monotonicity, and relativity, and we will discuss them in detail in the following sections. In accordance with these basic principles, we will give a general form of the quantum synchronization measure in which Mari's measure can also be contained, and we will show two calculable measures as examples based on this general formula. Meanwhile, as an intuitive verification, we also calculate our synchronization measures in an optomechanical mechanical system [37,38].

In the quantum domain, generalized synchronizations have been investigated recently in various contexts, such as phase synchronization, antisynchronization, constant error, and time delay synchronization [6,14,18,39–41]. It is gratifying that our measure could also be extended well to these cases. Finally, we try to apply this quantitative description in the case of a discrete variable.

II. POSTULATES OF QUANTUM SYNCHRONIZATION MEASURE

Let us begin our discussion with a brief review of quantum synchronization theory in the Heisenberg picture. For two CV quantum systems whose dynamics are described by quadrature operators $\hat{q}_j$ and $\hat{p}_j$ $(j = 1,2)$, an intuitive idea to study quantum synchronization is to define two error operators: $\hat{q}_- = (\hat{q}_1 - \hat{q}_2)/\sqrt{2}$ and $\hat{p}_- = (\hat{p}_1 - \hat{p}_2)/\sqrt{2}$, which is equivalent to extending the classical concept "error" into the quantum domain by canonical quantization. A synchronization measure (or a criterion) should be a mapping between the error operator and the real number field $\mathcal{R}$. In Ref. [6], Mari *et al.* designed a measure:

$$\mathcal{S}_c(t) := \langle \hat{q}_-(t)^2 + \hat{p}_-(t)^2 \rangle^{-1}, \quad (1)$$

where $\langle \cdots \rangle$ denotes taking the operator expectation value with respect to the quantum state. Under this definition, $\mathcal{S}_c \in \mathcal{R}$ is obvious. By rewriting the error operator as a sum of the c-number expectation value and its corresponding fluctuation operator, that is, $\hat{o}_j = \langle \hat{o}_j \rangle + \delta o_j$ $(o \in \{q,p\},\ j = 1,2)$ [42,43], Eq. (1) can be divided into two parts, which are, respectively, as follows: first-order criterion,

$$\lim_{t\to\infty} |q_1 - q_2| = 0, \\ \lim_{t\to\infty} |p_1 - p_2| = 0, \quad (2)$$

and second-order criterion,

$$\mathcal{S}_c'(t) := \langle \delta q_-(t)^2 + \delta p_-(t)^2 \rangle^{-1}. \quad (3)$$

Here, o_j denotes $\langle \hat{o}_j \rangle$ for convenience. First- and second-order synchronizations have quite different physical meanings. Specifically, first-order synchronization reflects the dynamic characteristics of the system's expectation value, and the synchronization effect depends mainly on the dynamic parameters and external control of the system. Since nonlocal correlation and quantum noise are not included in the expectation value equation, the synchronization at this level should be regarded as semiclassical, and its performance is in good agreement with classical synchronization theories [44,45]. The second-order measure, by contrast, describes the pure quantum natures of the synchronized systems. Note that the systematic synchronization error due to slightly different average trajectories is automatically excluded by a first-order criterion. As a consequence, the only source of disturbance bounding this synchronization measure will be quantum (or thermal) fluctuation or nonlocal correlation. Therefore, the unique characteristics of quantum synchronization should be the relation with quantum correlation. For example, the effect of quantum synchronization will decrease significantly with increased heat reservoir temperature [6], even though the classical synchronization theory has proved that synchronization has a strong robustness to the extra noise [46].

The difference between classical and quantum synchronizations can be further understood from the perspective of quantum measurement. For an observer, the synchronization phenomenon of two systems can be regarded as a consistent evolution in a single measurement process. Semiclassical systems obeying determined c-number dynamic equations will give exactly the same measurement results if we repeat the same observation many times. Therefore, criterion (2) is adequate to describe classical synchronization. For quantum systems, the quantum measurement hypothesis yields results that are random distributions, and they are constrained by the Heisenberg relation. This means that there may be significant error between the two systems for a single observation, even if the two systems have the same expectation value or quantum state. From this perspective, two identical thermal states (or maximum mixed states) are obviously unsynchronized. It also implies that some independent measures (e.g., fidelity) that have no cross (nonlocal) term cannot be used as quantum synchronization measures.

According to the above properties, which we think quantum synchronization should obey, we introduce a basic framework for the quantification of quantum synchronization. For CV quantum systems, an effective synchronization measure $\mathcal{S}$ should be a function of system variables satisfying the following four postulates:

(S1) *Boundedness*: $|\mathcal{S}| \leq 1$, and $|\mathcal{S}| = 1$ if and only if system errors satisfy the standard quantum limit.

(S2) *Criterion*: Classical synchronization criterion (2) should be a necessary condition for $|\mathcal{S}| = 1$.

(S3) *Monotonicity*: For the same system, $|\mathcal{S}(n_{\text{th}})| < |\mathcal{S}(n'_{\text{th}})|$ if $n_{\text{th}} > n'_{\text{th}}$, i.e., decreasing under the increased thermal environment phonon number n_{th}.

(S4) *Relativity*: $|\mathcal{S}(o_1,o_2,o_-)| < |\mathcal{S}(o'_1,o'_2,o_-)|$ if $o_i < o'_i$, i.e., errors with the same size will lead to a weaker sensitivity to synchronization if the dynamic variables of the systems have a larger value.

The notation $\mathcal{S}(x)$ here means that the other variables remain the same except for x. We emphasize here that the above properties are phenomenological to a certain extent, and they are not rigorous properties such as the ones used to quantify quantum entanglement and coherence [47,48]. Nevertheless, they are still significant constraints for common CV systems with sufficient physical meaning. Here, $\mathcal{S} \geq 0$ is not a mandatory requirement because $\mathcal{S} < 0$ corresponds to antisynchronization in some measures. The monotonicity originates from our belief that two thermal states are completely unsynchronous, and it is a relatively important standard to determine whether a synchronization measure is appropriate in the quantum regime. Relativity is required for a crosswise comparison between two different synchronization phenomena. Practical quantum systems, especially quantum open systems, can easily have different energies. A measure will be unfair to the larger energy system if it is defined based on the absolute size of the error.

Based on the above four properties, we provide some judgments about existing quantum synchronization measures that have been summarized recently in a review article [32]. We think that some local measures are not excellent quantum synchronization measures because the nonlocal terms are not considered, even though quantum fluctuations are included in them, and they may not satisfy the monotonicity. For monotonous measures based on mutual information and correlation [16], they are also not good measures since the classical synchronization criterion is not a necessary condition for them. To the best of our knowledge, Eq. (1) is the closest measure to our postulates, and only relativity is not satisfied.

III. RELATIVE MEASURE OF QUANTUM SYNCHRONIZATION

With the four postulates proposed in the preceding section, a natural followup question is how to design a measure satisfying them simultaneously. A basic idea in this work is that the quantum synchronization measure should be an equal-weighted average of the synchronization measure results for two dynamic trajectories after a quantum measurement. This idea is derived from the measurement interpretation of quantum synchronization in the previous section. A mature classical synchronization measure is enough to describe the similarity between two trajectories from a single quantum measurement, and the quantum properties of the synchronization are hidden in the measurement probability. Therefore, we give a general formula of the quantum synchronization measure:

$$\mathcal{L}_c(t) := \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \mathcal{F}\big(q_1^i, p_1^i, q_2^i, p_2^i\big). \tag{4}$$

Here, the superscript i denotes the observation result for the ith measurement. $N \to \infty$ implies enough times for measuring the quantum system. $\mathcal{F}$ is a function to reflect the difference of two system trajectories. Note that the relation $\mathcal{L}_c(t) = \mathcal{S}_c(t)^{-1}$ is obvious if the function $\mathcal{F}$ is selected as

$$\mathcal{F} = \left(\frac{q_1^i - q_2^i}{\sqrt{2}}\right)^2 + \left(\frac{p_1^i - p_2^i}{\sqrt{2}}\right)^2. \tag{5}$$

From this perspective, our measure can be regarded as a more generalized measure, and it will degenerate to Mari's measure with some appropriate designs.

For a normalized measure, here we introduce the well-known Pearson's parameter [49]

$$\mathcal{C}_{f,g}(t,\Delta t) = \frac{\overline{\delta f\,\delta g}}{\sqrt{\overline{\delta f^2}\;\overline{\delta g^2}}} \tag{6}$$

to provide a basis for the synchronization measure. In this expression, $\delta o = o - \bar{o}$ and $\bar{o} = \Delta t^{-1}\int_t^{t+\Delta t} o(t')dt'$, where $o \in \{g, f\}$. In general, a perfect synchronization corresponds to $\bar{\mathcal{C}} = 1$ and an antisynchronization will be $\bar{\mathcal{C}} = -1$. With the requirement of the relativity, we also give a relative error measure

$$\mathcal{E}_{f,g} = 1 - \frac{|f-g|}{|f|+|g|}. \tag{7}$$

For quantum synchronization, it requires us to consider quadrature variables simultaneously, which leads us to define the measures as $\mathcal{F}_c = (\mathcal{C}_{q_1^i,q_2^i} + \mathcal{C}_{p_1^i,p_2^i})/2$ or $\mathcal{F}'_c = (\mathcal{E}_{q_1^i,q_2^i} + \mathcal{E}_{p_1^i,p_2^i})/2$. Hence, our measures can be finally defined as

$$\mathcal{L}_c(t) := \lim_{N\to\infty} \frac{1}{2N}\sum_{i=1}^{N} \big(\mathcal{C}_{q_1^i,q_2^i} + \mathcal{C}_{p_1^i,p_2^i}\big) \tag{8}$$

and

$$\mathcal{L}'_c(t) := \lim_{N\to\infty} \frac{1}{2N}\sum_{i=1}^{N} \big(\mathcal{E}_{q_1^i,q_2^i} + \mathcal{E}_{p_1^i,p_2^i}\big). \tag{9}$$

Equations (8) and (9) are calculable measures if one can solve the probability distribution of the quantum system at the moment t, and we will show examples in the next section to illustrate this property.

In addition to having good agreement with the four postulates, another advantage of our measure is its scalability. For example, Eq. (4) can be extended easily to measure phase synchronization and other generalized synchronizations by setting

$$\begin{aligned}
\mathcal{F}_p &= \mathcal{C}_{\phi_1^i,\phi_2^i} \quad \text{or} \quad \mathcal{F}_p = \mathcal{E}_{\phi_1^i,\phi_2^i},\\
\mathcal{F}_g &= \big(\mathcal{C}_{g(q_1^i),g(q_2^i)} + \mathcal{C}_{g(p_1^i),g(p_2^i)}\big)/2,\\
&\text{or}\\
\mathcal{F}_g &= \big(\mathcal{E}_{g(q_1^i),g(q_2^i)} + \mathcal{E}_{g(p_1^i),g(p_2^i)}\big)/2.
\end{aligned} \tag{10}$$

It is very hard, however, to measure explicitly those synchronizations in previous works.

IV. EXAMPLE IN OPTOMECHANICAL SYSTEMS

Now we give some examples to explain further the effectiveness and computability of our measure in the CV quantum synchronization regime. In recent years, optomechanical devices provided the perfect platform for studying quantum synchronization where our measure can be applied directly [6,8,22,34,38]. In this section, we adopt two methods to analyze the dynamics of optomechanical systems (in Sec. IV A), and we show the CV quantum synchronization of such systems with our measure (in Sec. IV B).

A. Dynamics of optomechanical systems

A representative Fabry-Pérot resonator consists of two highly reflective mirrors, one of which is a moving mirror with eigenfrequency ω_m [see Fig. 1(a)]. By setting $\hbar = 1$, the Hamiltonian of such a system can be expressed as $H = H_0 + H_I + H_d$, where $H_0 = \omega_c\hat{a}^\dagger\hat{a} + \omega_m\hat{b}^\dagger\hat{b}$ are the free terms of the optical mode and the oscillator mode,

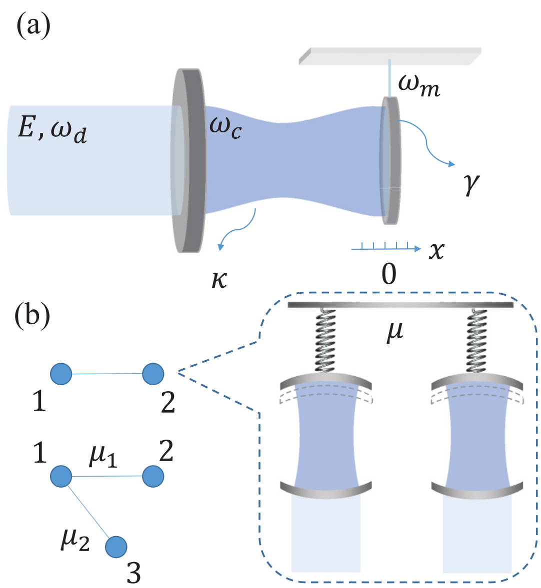

FIG. 1. (a) Schematic diagram of an optomechanical system: a mechanical resonator is coupled with a Fabry-Pérot cavity through the nonlinear radiation pressure force of a quantized optical mode. (b) Two kinds of coupling structures. Here, each node represents an optomechanical system, and the lines denote phonon tunnelings.

$H_I = -g\hat{a}^\dagger\hat{a}(\hat{b}^\dagger + \hat{b})$ [37,38] describes the well-known radiation pressure interaction, and $H_d = iE(\hat{a}^\dagger e^{-i\omega_d t} - \hat{a}e^{i\omega_d t})$ is the driving term. Here we consider two approximately identical mechanical resonators interacting mutually through phonon tunneling with intensity μ [see Fig. 1(b)]. Then the total Hamiltonian of this coupled system can be expressed as [50]

$$H = \sum_{j=1,2}[-\Delta_j\hat{a}_j^\dagger\hat{a}_j + \omega_{mj}\hat{b}_j^\dagger\hat{b}_j - g\hat{a}_j^\dagger\hat{a}_j(\hat{b}_j^\dagger + \hat{b}_j) \\ + iE(\hat{a}_j^\dagger - \hat{a}_j)] - \mu(\hat{b}_1\hat{b}_2^\dagger + \hat{b}_1^\dagger\hat{b}_2) \tag{11}$$

after a frame rotating. In this expression, for $j = 1,2$, a_j and b_j are the optical and mechanical annihilation operators, and $\Delta_j = \omega_{dj} - \omega_{cj}$ is the detuning between the frequency of cavity and driving. E is the laser intensity driving the optical cavities, and g is the single-photon coupling coefficient. The dynamics of such a system can be solved by considering dissipative effects in the Heisenberg picture, and we can write the following quantum Langevin equations [51,52]:

$$\begin{aligned}\dot{\hat{a}}_j &= [-\kappa + i\Delta_j + ig(\hat{b}_j^\dagger + \hat{b}_j)]\hat{a}_j + E + \sqrt{2\kappa}\hat{a}_j^{\text{in}},\\ \dot{\hat{b}}_j &= [-\gamma - i\omega_{mj}]\hat{b}_j + ig\hat{a}_j^\dagger\hat{a}_j + i\mu\hat{b}_{3-j} + \sqrt{2\gamma}\hat{b}_j^{\text{in}},\end{aligned} \tag{12}$$

where κ and γ are, respectively, the optical and mechanical damping rates. a_j^{in} and b_j^{in} are the input bath operators, and they are assumed to be white Gaussian fields obeying standard correlation relations $\langle\hat{a}_j^{\text{in}}(t)^\dagger\hat{a}_{j'}^{\text{in}}(t') + \hat{a}_{j'}^{\text{in}}(t')\hat{a}_j^{\text{in}}(t)^\dagger\rangle = \delta_{jj'}\delta(t - t')$ and $\langle\hat{b}_j^{\text{in}}(t)^\dagger\hat{b}_{j'}^{\text{in}}(t') + \hat{b}_{j'}^{\text{in}}(t')\hat{b}_j^{\text{in}}(t)^\dagger\rangle = (2\bar{n}_b + 1)\delta_{jj'}\delta(t - t')$, where $\bar{n}_b = [\exp(\hbar\omega_{mj}/k_bT) - 1]^{-1}$ is the mean occupation number of the mechanical bath and it gauges the temperature T of the system [53]. Note that Eq. (12) can also be used to discuss a single optomechanical system by setting $\mu = 0$.

To simulate the quantum trajectory of the ith observation, we translate the operator equations (12) into two c-number differential equations with stochastic noise terms and initial conditions [22], that is,

$$\begin{aligned}\dot{\alpha}_j^i &= \left[-\kappa + i\Delta_j + 2ig\,\text{Re}\left(\beta_j^i\right)\right]\alpha_j^i + E + \sqrt{2\kappa}\alpha_j^{\text{in}},\\ \dot{\beta}_j^i &= [-\gamma - i\omega_{mj}]\beta_j^i + ig\left|\alpha_j^i\right|^2 + i\mu\beta_{3-j}^i + \sqrt{2\gamma}\beta_j^i,\end{aligned} \tag{13}$$

and the initial Gaussian state ρ_0 of the system is simulated by the Gaussian random complex number obeying $\mathcal{N}(o_j,\delta o_j)$, where

$$o = \text{Tr}(\hat{o}\rho_0) \tag{14}$$

and

$$\delta o = \sqrt{\text{Tr}(\hat{o}^2\rho_0) - [\text{Tr}(\hat{o}\rho_0)]^2}. \tag{15}$$

Moreover, the noise operators are also simulated by the Gaussian distribution $\mathcal{N}(0,1)$ without time correlation. Note that the c number does not have the commutation relation, and the correlation relations become [22]

$$\begin{aligned}\left\langle\alpha_j^{\text{in}*}\alpha_{j'}^{\text{in}} + \alpha_{j'}^{\text{in}}\alpha_j^{\text{in}*}\right\rangle &= 2\left\langle\alpha_j^{\text{in}*}\alpha_{j'}^{\text{in}}\right\rangle = \delta_{jj'},\\ \left\langle\beta_j^{\text{in}*}\beta_{j'}^{\text{in}} + \beta_{j'}^{\text{in}}\beta_j^{\text{in}*}\right\rangle &= 2\left\langle\beta_j^{\text{in}*}\beta_{j'}^{\text{in}}\right\rangle = (2\bar{n}_b + 1)\delta_{jj'}.\end{aligned} \tag{16}$$

By using the transformation

$$\begin{aligned}x_j^i &= \left(\alpha_j^{i*} + \alpha_j^i\right)/\sqrt{2}, \quad y_j^i = i\left(\alpha_j^{i*} - \alpha_j^i\right)/\sqrt{2},\\ q_j^i &= \left(\beta_j^{i*} + \beta_j^i\right)/\sqrt{2}, \quad p_j^i = i\left(\beta_j^{i*} - \beta_j^i\right)/\sqrt{2},\end{aligned} \tag{17}$$

our measure can be calculated conveniently by substituting (17) into Eq. (8) or (9).

The ensemble-averaged quantities and the quantum fluctuation can also be obtained by simulating the stochastic Langevin equations a large number of times, i.e., $a_j = \sum\alpha_j^i/N$ and $\langle\delta a_j^2\rangle = \sum{\alpha_j^i}^2/N - (\sum\alpha_j^i/N)^2$, and the correlation terms are reconfigurable by using the definition $\langle\delta a_1\delta a_2\rangle = \sum\alpha_1^i\alpha_2^i/N - (\sum\alpha_1^i/N)(\sum\alpha_2^i/N)$. The mechanical modes $b_{1,2}$ can also be deduced in the same manner. It is well known that a Gaussian state can be characterized completely by its corresponding covariance matrix, and therefore we can obtain all the quantum properties at time t by adopting the stochastic Langevin equations (13).

Before further discussion about quantum synchronization, one may wonder if the stochastic Langevin equations are effective and accurate to simulate the CV quantum system. To clarify this query, we have utilized the mean-field approximation to resolve the quantum Langevin equations (12) in the strong driving and weak-coupling regime ($g/\kappa \ll 1$) [54,55], and we showed the comparisons of the results based on two methods. By dividing the operators in the quantum Langevin equations in the forms of $\hat{a}_j = a_j + \delta a_j$ and $\hat{b}_j = b_j + \delta b_j$, we can write directly the dynamic equations, including the expectation value and fluctuation, as

$$\begin{aligned}\dot{a}_j &= [-\kappa + i\Delta_j + ig(b_j^* + b_j)]a_j + E,\\ \dot{b}_j &= [-\gamma - i\omega_{mj}]b_j + ig|a_j|^2 + i\mu b_{3-j},\end{aligned} \tag{18}$$

and

$$\begin{aligned}\dot{\delta a_j} &= [-\kappa + i\Delta_j]\delta a_j \\ &\quad +2ig\,\mathrm{Re}(b_j)\delta a_j + iga_j(\delta b_j^\dagger + \delta b_j) + \sqrt{2\kappa}a_j^{\rm in}, \\ \dot{\delta b_j} &= [-\gamma - i\omega_{mj}]\delta b_j \\ &\quad +ig(a_j\delta a_j^\dagger + a_j^*\delta a_j) + i\mu\delta b_{3-j} + \sqrt{2\gamma}b_j^{\rm in} \end{aligned} \tag{19}$$

after neglecting the nonlinear fluctuation terms. The system evolutions can be solved by simulating Eqs. (18) and (19) in the correct order.

Adopting a similar transformation corresponding to Eq. (17), the fluctuation equation can be given in a more compact form [43]:

$$\frac{d}{dt}\hat{u} = S\hat{u} + \hat{\xi}, \tag{20}$$

where S is the coefficient matrix (see Appendix B for details) and $\hat{u} = (\delta x_1,\delta y_1,\delta x_2,\delta y_2,\delta q_1,\delta p_1,\delta q_2,\delta p_2)^\top$. $\hat{\xi} = (\delta x_1^{\rm in},\delta y_1^{\rm in},\delta x_2^{\rm in},\delta y_2^{\rm in},\delta q_1^{\rm in},\delta p_1^{\rm in},\delta q_2^{\rm in},\delta p_2^{\rm in})^\top$ is the input noise vector. Under this representation, the evolution of the covariance matrix should obey

$$\frac{d}{dt}C = SC + CS^\top + N, \tag{21}$$

where $c_{ij}(t) = c_{ji}(t) = \langle u_i(t)u_j(t) + u_j(t)u_i(t)\rangle/2$ and $N = \mathrm{diag}[\kappa,\kappa,\kappa,\kappa,\gamma(2\bar{n}_b+1),\gamma(2\bar{n}_b+1),\gamma(2\bar{n}_b+1),\gamma(2\bar{n}_b+1)]$ with the bath phonon number $\bar{n}_b$. So Mari's measure can also be calculated by the coefficient matrix C.

In Fig. 2, we present some comparisons of results calculating by the mean-field approximation and the stochastic dynamic method, respectively. In (a) and (b), we plot the expectation values and the corresponding fluctuations of the coordinate evolutions of oscillator 1. One can observe intuitively that the two simulation methods can achieve consistent evolutions. The subfigure on the right-hand side of (a) shows the local magnification of the curve based on stochastic dynamics. Here, each blue (dashed) line denotes a random quantum trajectory that can be considered as a measurement result of the CV quantum system, and the black line is the ensemble average of all random trajectories. One may wonder why this consistency can be satisfied only when we calculate local variables. To clarify this point further, we also plot the nonlocal measure $\mathcal{S}_c$, and Fig. 2(c) shows that the two methods indeed provide us with the same results.

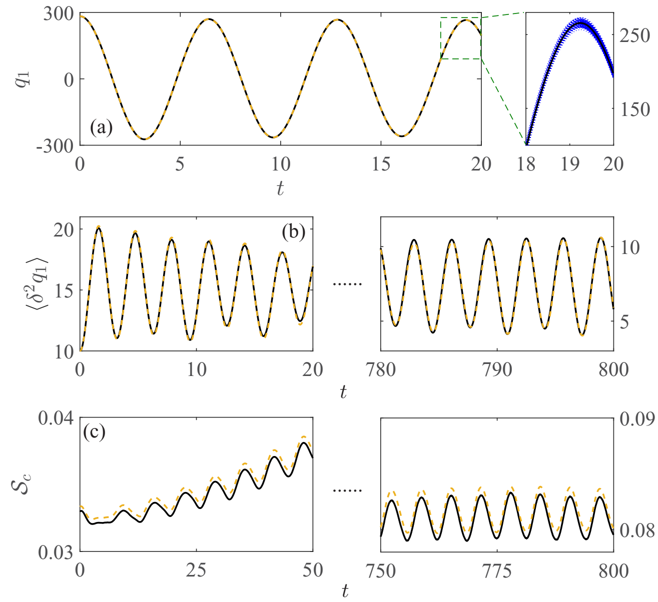

FIG. 2. (a), (b), and (c) Evolutions of q_1, $\langle\delta^2 q_1\rangle$, and $\mathcal{S}_c$ based on the mean-field approximation (yellow dashed line) and the stochastic dynamic method (black solid line), respectively. The blue dashed lines on the right side of (a) present the 100 stochastic trajectories. Here, we set $\omega_{m1} = 1$ as a unit, and other parameters are $\omega_{m2} = 1$, $\Delta_j = 1$, $g = 0.003$, $E = 10$, $\mu = 0.02$, $\bar{n}_b = 0$, $\kappa = 0.15$, and $\gamma = 0.005$. In this simulation, the stochastic dynamics (black solid lines) are obtained based on 10 000 calculations of the stochastic Langevin equations.

B. Synchronization of optomechanical systems

Now we use an extreme example to explain that local measures, such as fidelity, are not good measures for quantum synchronization. Consider two optomechanical systems coupled as in Fig. 1(b). The states of two oscillators will always be identical under the same dynamic parameters $(\omega_m,\Delta,g,\kappa,\gamma,E)$ and initial states even though there does not exist any interaction between two subsystems ($\mu = 0$). The fidelity of two oscillator states should always be 100% in this case, indicating that two oscillators have the same expectations, even the identical size of the fluctuation $(\langle\delta^2 o_1\rangle = \langle\delta^2 o_2\rangle)$. Under the definitions of classical or local quantum synchronization measures, the two oscillators here have already become perfect synchronization, and it seems that the synchronization degree does not depend on the magnitudes of fluctuations since their difference is always zero. However, because there is no correlation, the respective measurement results of two oscillators will be independent and random, meaning that the oscillators may still be unsynchronized if the magnitudes of fluctuations are too large, and the only way to improve synchronization is to increase correlation. In other words, the synchronization in the quantum regime cannot be measured by the fidelity or the difference of the quantum fluctuations. On the contrary, a good measure should consider nonlocal terms $(\langle\delta o_1\delta o_2\rangle)$ and the sum of the quantum fluctuations $(\langle\delta^2 o_1\rangle + \langle\delta^2 o_2\rangle)$, which is in agreement with Mari's theory. According to this criterion, it is quite natural to understand the thermal monotonicity in mathematics because the increased bath temperature can amplify the fluctuations and destroy the nonlocal correlation simultaneously. It is also notable that quantum synchronization does not require the system to be entangled, although nonlocal terms are necessary for the quantum synchronization measure.

For a quantitative explanation, we plot some curves to compare Gaussian fidelity [56], the local measure $(\mathcal{C}_{\langle q_1^2\rangle,\langle q_2^2\rangle})$, Mari's measure, and our measure in Figs. 3(a) and 3(b). The results confirm that the local measure could not get any response to the varied interaction intensity μ and bath phonon number $\bar{n}_b$, implying that it will lose effectiveness in the quantum synchronization regime. From this perspective, the local measure should be a necessary and insufficient condition for quantum synchronization. In contrast, both Mari's measure and our measure show reasonable responses to nonlocal

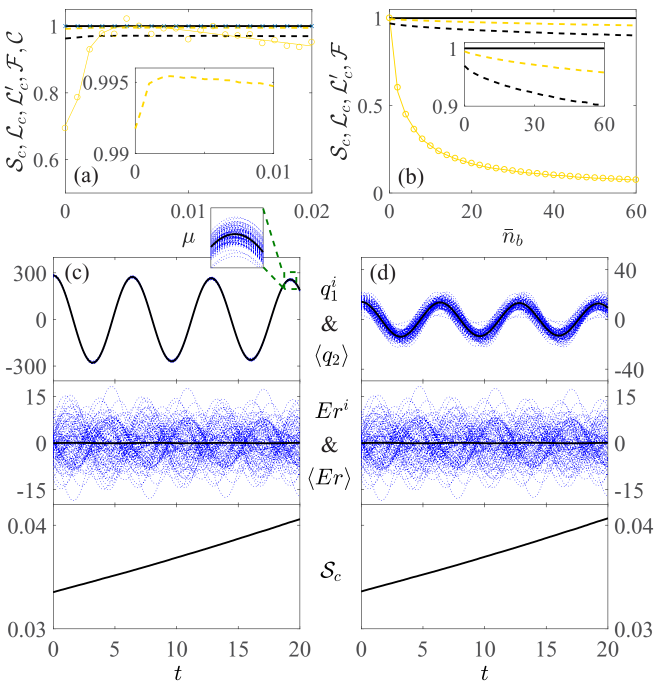

FIG. 3. (a) and (b) Comparisons of Gaussian fidelity, local measure [blue marks in (a)], Mari's measure, and our measure with varied interaction intensity μ and bath phonon number $\bar{n}_b$. Parts (a) and (b) are obtained, respectively, by 5000 and 3000 calculations of the stochastic Langevin equations. Here, the black (dark) solid lines denote Gaussian fidelity, and black (dark) and yellow (pale) dashed lines are our measures $\mathcal{L}_c$ and $\mathcal{L}'_c$. The yellow (pale) circles and lines are normalized Mari's measure $\mathcal{S}_c/\max\{\mathcal{S}_c\}$ calculated by stochastic Langevin equations and the mean-field method, respectively. The local measure and fidelity are calculated by the mean-field method. (c) and (d) Synchronizations corresponding to the same value of Mari's measure. The blue dashed lines are the 100 stochastic trajectories of q_1 and $\mathrm{Er} = q_1 - q_2$, and the black lines denote $\langle q_2 \rangle$ and $\langle \mathrm{Er} \rangle$. Here we set $\Delta t = 0.1$ for $\mathcal{L}_c$ and $g = 0$ in (c,d), and the other parameters are the same as those in Fig. 2.

interaction and a thermal environment [57]. In particular, in Fig. 3(b) the thermal monotonicity is well suited for all three measures, certifying that it can indeed measure inside the quantum effect in quantum synchronization. One can also observe from Fig. 3(a) that $\mathcal{S}_c$, $\mathcal{L}_c$, and $\mathcal{L}'_c$ exhibit consistent evolution with varied μ. Note that it is not a monotonic evolution, and all three measures point out that the most perfect synchronization effect corresponds to $\mu \sim 0.005$. Therefore, we think the three measures are in agreement with each other and are all effective when the crosswise comparison is not involved.

Now we discuss relativity, which is an essential characteristic of our measure compared with Mari's measure. According to our relativity measure, a larger amplitude can weaken the disturbance of errors on the synchronization, which is different from Mari's definition, that is, the same size errors always damage the synchronization equivalently. To illustrate this, we consider another extreme example: only two oscillators with the same dynamic parameters (ω_m, γ) and initial states. This model becomes equivalent to our system by setting

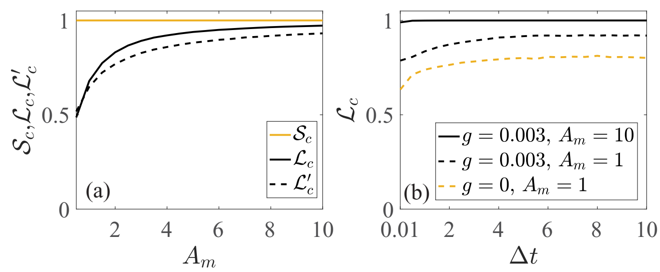

FIG. 4. (a) Comparisons of Mari's measure ($\mathcal{S}_c/\max\{\mathcal{S}_c\}$) and our measure with varied classical amplitudes. Here we set $\langle \hat{b}_j \rangle(0) = 20A_m$. (b) Comparisons of our measure $\mathcal{L}_c$ with a different time window Δt. The parameters in (a) and (b) are the same as those in Figs. 3(c) and 2, respectively.

$g = 0$, and it becomes a pure linear dynamic system with Hamiltonian $H = \sum_{j=1,2} \omega_m \hat{b}_j^\dagger \hat{b}_j - \mu(\hat{b}_1 \hat{b}_2^\dagger + \hat{b}_1^\dagger \hat{b}_2)$. Such a linear Hamiltonian causes the dynamic equations of each order quantity to be closed [58]. In other words, the evolutions of quantum fluctuation and the covariance matrix are not dependent on the first-order expectation in this case. Therefore, this model will correspond to an unchanged Mari measure $\mathcal{S}_c$ with varied amplitudes of expectation.

In Figs. 3(c) and 3(d), we plot two different synchronization processes, and one can observe visually that the damage degree of synchronization by quantum fluctuation is quite different when the system corresponds, respectively, to larger or smaller amplitudes in the expectation value level even though $\mathcal{S}_c$ is equal. Because Mari's measures give an equal value corresponding to the above two processes, we believe that it is inaccurate, meaning that Mari's measure and other absolute measures have a limited ability to discuss quantum synchronization crosswise. For our measures, however, either $\mathcal{L}_c$ or $\mathcal{L}'_c$ exhibits significant monotonicity with varied amplitude. As is shown in Fig. 4(a), the values of two measures will decrease with the increased amplitudes even if the size of the error is invariable. In other words, this result confirms that $\mathcal{L}_c$ and $\mathcal{L}'_c$ satisfy the relativity, and therefore we have proved that all four postulates are satisfied by our measures up to now. Figure 4(b) shows that time-averaged $\mathcal{L}_c$ is little affected by the time window Δt. This is an advantage of Peason's parameter, and one can set Δt arbitrarily when this measure is adopted.

In addition to these special examples, we finally give a dynamic evolution process in which three different optomechanical systems (with different oscillator frequencies and initial states) interact each other as shown in Fig. 1(b). We have already proved in previous work that three oscillators will evolve from unsynchronized to synchronized states if they satisfy the dissipative condition [8]. This process can show us the performance of our measure in unsynchronized-synchronized crossover. Corresponding to the coupling structure in Fig. 1(b), the dissipative condition reads

$$\begin{aligned} \omega_1 - \omega_s &= \mu_1 + \mu_2, \\ \omega_2 - \omega_s &= \mu_1, \\ \omega_3 - \omega_s &= \mu_2, \end{aligned} \tag{22}$$

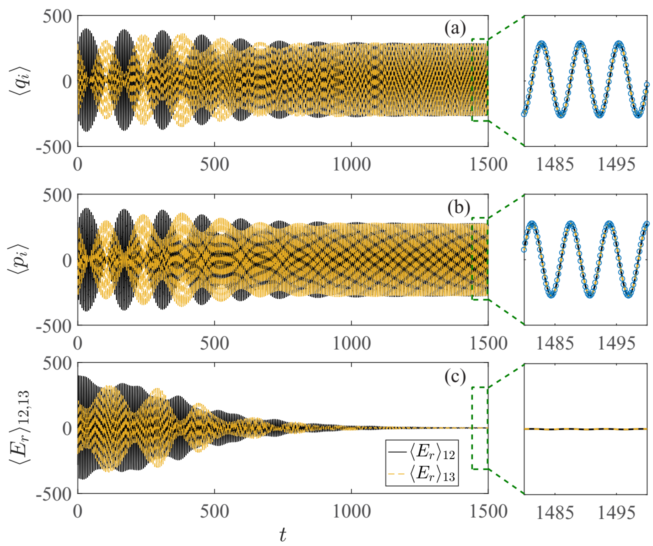

FIG. 5. Coordinates (a), momenta (b), and errors (c) of three coupled optomechanical systems. In (a) and (b), the black (dark, solid) lines correspond to system 1, and the yellow (pale, dashed) lines denote system 2. The subfigures on the right side are the corresponding local magnifications in which the blue circles correspond to system 3. Here we set $g = 0.005$, $\omega_s = 1$, $\omega_1 = 1.03$, $\omega_2 = 1.02$, and $\omega_3 = 1.01$, which leads to $\mu_1 = 0.02$ and $\mu_2 = 0.01$. The other parameters are the same as those in Fig. 2. In this simulation, the stochastic dynamics are obtained by 1000 calculations of the stochastic Langevin equations.

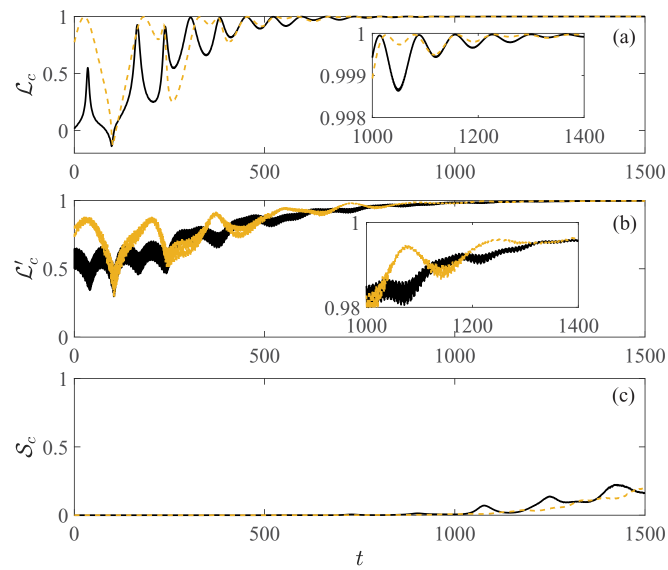

FIG. 6. Comparisons of Mari's measure $\mathcal{S}_c$ (a) and our measures $\mathcal{L}_c$ (b) and $\mathcal{L}'_c$ (c). Here the black (dark, solid) lines denote the measure between systems 1 and 2, and the yellow (pale, dashed) lines present the measure between systems 1 and 3. The parameters are the same as those in Fig. 5. In (a), the time window of the measure is set as $\Delta t = 10$.

where ω_s is a stated quantum standard frequency. To avoid a small coordinate or momentum to make the denominator in Eq. (7) tend to zero, here we modify $\mathcal{L}'_c$ by redefining $\mathcal{F}_c = (\mathcal{E}^r_{b^i_1,b^i_2}/2 + \mathcal{E}^i_{b^i_1,b^i_2}/2)/2$, where

$$\begin{aligned} \mathcal{E}^r_{f,g} &= 1 - \frac{|\mathrm{Re}(f) - \mathrm{Re}(g)|}{|f| + |g|}, \\ \mathcal{E}^i_{f,g} &= 1 - \frac{|\mathrm{Im}(f) - \mathrm{Im}(g)|}{|f| + |g|}. \end{aligned} \tag{23}$$

Then the evolution and measure can be calculated according to this model.

It can be found from Figs. 5(a) and 5(b) that three different systems will achieve coincident coordinate and momentum with enough evolution time, and the classical errors among them tend to zero after $t > 1300$, as shown in Fig. 5(c). This dynamic process can be measured by using the methods mentioned above, and the evolutions of $\mathcal{S}_c$, $\mathcal{L}_c$, and $\mathcal{L}'_c$ are shown in Fig. 6. The agreement between the measure and the synchronization effect can be seen intuitively by contrasting Figs. 5 and 6. Although all three measures show a similar tendency to evolve from a lower value to a higher value, the details of their performances are essentially different. Mari's measure is restricted in such an interval with a low value [0,0.3] even though the other two measures tend to 1. Moreover, $\mathcal{S}_c$ remains almost unchanged and tends to 0 before $t = 1000$ even if the synchronization degree has an obvious improvement in this time interval. This defect is mainly caused by the reciprocal operation in the definition of $\mathcal{S}_c$. Correspondingly, $\mathcal{S}_c$ will be quite sensitive, and a small increment of error will lead to a significant decrease of its value in the interval $\mathcal{S}_c \in [0.3,1]$. In fact, the possibility to reach the maximum bound of $\mathcal{S}_c$ in an actual physical system has not been reported, so that the value of Mari's measure is changed only in a small range. It finally causes Mari's measure to be inadequate in the unsynchronized-synchronized crossover process since the values corresponding to synchronous and asynchronous systems are closed. On the contrary, Figs. 6(a) and 6(b) show that both $\mathcal{L}_c$ and $\mathcal{L}'_c$ increase significantly in the crossover interval $t \in [0,1000]$, and their values corresponding to the synchronized and unsynchronized systems will change in a larger range. Therefore, we think the change of our measures is uniform, and it is more effective for applying $\mathcal{L}_c$ and $\mathcal{L}'_c$ to describe the unsynchronized-synchronized crossover.

V. EXTENSION IN THE DISCRETE VARIABLE CASE

Now we discuss the potential for applying our measure in the discrete variable case. In the research area of quantum synchronization, a discrete variable system is always a controversial issue. On the one hand, the theory of quantum properties in a discrete variable system is more mature, so the relationship between synchronization and quantum correlation can be studied more rigorously [15,16,59,60]. On the other hand, there is no good agreement for some concepts in discrete variables because synchronization is a generalization from classical mechanics to quantum mechanics, which means that it is still a defect for measuring discrete variable synchronization.

Let us reconsider the general formula (4) of our measure, which requires an evaluation for each measurement result of dynamical variables. The dynamical variables $\hat{q} = (\hat{a}^\dagger + \hat{a})/\sqrt{2}$ and $\hat{p} = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$ can be extended in Fock space with finite dimension [15], and the evaluation function $\mathcal{F}$ can be designed as

$$\mathcal{F}_q = 1 - \frac{2|q_1 - q_2|}{|q_{-\max}|}, \quad \mathcal{F}_p = 1 - \frac{2|p_1 - p_2|}{|p_{-\max}|}, \tag{24}$$

where, for the discrete variable system, $q(p)_i$ are a series of discrete eigenvalues and $q(p)_{-\max}$ has their largest absolute value.

For the two-level system $q(p)$ becomes $\sigma_{x(y)}/\sqrt{2}$, and the relation between the measurement result and the synchronization measure should be

$$\mathcal{F}_o = \begin{cases} 1, & o_1 = o_2 = c \quad \text{or} \quad o_1 = o_2 = -c, \\ -1, & o_1 = c, \quad o_2 = -c, \\ -1, & o_1 = -c, \quad o_2 = c, \end{cases} \tag{25}$$

where $c = 1/\sqrt{2}$ and $o \in \{q,p\}$. Its weighted average can be calculated by summing $\mathcal{F}_o$ times their corresponding probability, which is exactly $\langle\sigma_{x1}\sigma_{x2}\rangle$ or $\langle\sigma_{y1}\sigma_{y2}\rangle$. Hence our measure for the two-level system is finally expressed as

$$\mathcal{L}_c = \frac{\langle\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}\rangle}{2} = 2\,\mathrm{Re}\langle\sigma_1^+\sigma_2^-\rangle, \tag{26}$$

and it is exactly the same as the measure used widely in previous works [10–12,32]. Therefore, we think that our formula is indeed an effective and general conclusion.

An advantage of our measure in a discrete variable system is that the boundedness of the measure can be well guaranteed under its definition. In contrast, if we extend directly Mari's measure to the discrete variable system,

$$\mathcal{S}_c = \langle\sigma_{x-}^2 + \sigma_{y-}^2\rangle^{-1}, \tag{27}$$

the boundedness $\mathcal{S}_c \leqslant 1$ may not hold because the commutation of σ_{x-}^2 and σ_{y-}^2 is no longer a constant.

VI. CONCLUSION

In summary, we have studied quantitatively quantum synchronization from the perspective of quantum measurement. The quantum synchronization theory proposed in Ref. [6] pointed out that nonlocal correlation plays an important role in quantum synchronization. By adopting this definition, four postulates for quantifying quantum synchronization are proposed in this work. These postulates, especially the monotonicity, can help us to distinguish whether the synchronization is in a quantum or semiclassical level. From the perspective of measurement, we have introduced a measure of CV quantum synchronization that can satisfy all four of our postulates. To clarify the advantages of our measure, some extreme dynamical processes are analyzed to illustrate that some local measures and absolute measures will lose their accuracy to some extent. As an example, we discuss CV quantum synchronization between optomechanical systems, and we provide a detailed calculation method of our measure. Finally, we discuss the potential of applying our measure in a discrete variable quantum system.

In addition to the above characteristics, we also want to emphasize here that our measure only needs the measurement result statistics of mechanical quantities, but it does not require the reconstruction of the density matrix or the complete covariance matrix. Moreover, our measure is not restricted in linear evolution and a Gaussian state. Therefore, we think that our measure is more suitable for quantum synchronization experiments in QIP. To sum up, we believe that our theory can be applied to reveal the interplay between quantum correlation and synchronization, and it can be adopted as a useful resource measure for quantum communication and quantum control.

ACKNOWLEDGMENTS

All of the authors thank J. Cheng, J. Zhang, and Y. Zhang for useful discussion. This research was supported by the National Natural Science Foundation of China (Grants No. 11574041 and No. 11175033).

APPENDIX A: STOCHASTIC SYSTEM DYNAMICS

In this appendix, we explain the numerical method for solving stochastic Langevin equations. For a single optomechanical system, Eq. (13) in the main text can be expressed in a more compact form:

$$\begin{aligned} \dot{\alpha}^i &= f_\alpha(\alpha^i,\beta^i) + \sqrt{2\kappa}\alpha^{\text{in}}, \\ \dot{\beta}^i &= f_\beta(\alpha^i,\beta^i) + \sqrt{2\gamma}\beta_j^i, \end{aligned} \tag{A1}$$

and they can be further expanded as

$$\begin{aligned} \alpha^i(t+h) &= \alpha^i(t) + f_\alpha(\alpha^i(t),\beta^i(t))h + dW_\alpha, \\ \beta^i(t+h) &= \beta^i(t) + f_\beta(\alpha^i(t),\beta^i(t))h + dW_\beta \end{aligned} \tag{A2}$$

by using the time-difference method. Here, W is a Brownian random process, and it can be simulated by $G_\alpha N$ and $G_\beta N$, where $G_\alpha = \sqrt{\kappa h}$ and $G_\beta = \sqrt{\gamma(2\bar{n}_b + 1)h}$. N is a Gaussian random complex number that can be generated by $N = (z_1 + iz_2)/\sqrt{2}$, where $z_1, z_2 \in \mathcal{R}$ and $z_1, z_2 \sim \mathcal{N}(0,1)$ are obtained by substituting the modified correlation function (16) into Eq. (A2). The fourth-order Runge-Kutta method corresponding to differential equations (A2) provides the following general formulas:

$$\begin{aligned} \alpha^i(t+h) &= \alpha^i(t) + h\sum_{i=1}^4 v_j K_{\alpha i} + G_\alpha N, \\ \beta^i(t+h) &= \beta^i(t) + h\sum_{i=1}^4 v_j K_{\beta i} + G_\beta N, \end{aligned} \tag{A3}$$

where the coefficients are $v_1 = v_4 = 1/6$ and $v_2 = v_3 = 1/3$. Here, for $l \in \{\alpha,\beta\}$,

$$\begin{aligned} K_{l1} &= f_l(\alpha(t),\beta(t)), \\ K_{l2} &= f_l\left(\alpha(t) + \frac{hK_{\alpha1}}{2} + \frac{G_\alpha N}{2}, \beta(t) + \frac{hK_{\beta1}}{2} + \frac{G_\beta N}{2}\right), \\ K_{l3} &= f_l\left(\alpha(t) + \frac{hK_{\alpha2}}{2} + \frac{G_\alpha N}{2}, \beta(t) + \frac{hK_{\beta2}}{2} + \frac{G_\beta N}{2}\right), \\ K_{l4} &= f_l(\alpha(t) + hK_{\alpha3} + G_\alpha N, \beta(t) + hK_{\beta3} + G_\beta N). \end{aligned}$$

Note that the noise term is simulated in the first-order level, and therefore the above simulation method is accurate only if G_l is constant.

APPENDIX B: COEFFICIENT MATRIX OF SYSTEM DYNAMICS

The coefficient matrix S in the main text is

$$S = \begin{pmatrix} -\kappa & -\Delta_1 - 2g\,\mathrm{Re}(b_1) & 0 & 0 & -2g\,\mathrm{Im}(a_1) & 0 & 0 & 0 \\ \Delta_1 + 2g\,\mathrm{Re}(b_1) & -\kappa & 0 & 0 & 2g\,\mathrm{Re}(a_1) & 0 & 0 & 0 \\ 0 & 0 & -\kappa & -\Delta_2 - 2g\,\mathrm{Re}(b_2) & 0 & 0 & -2g\,\mathrm{Im}(a_2) & 0 \\ 0 & 0 & \Delta_2 + 2g\,\mathrm{Re}(b_2) & -\kappa & 0 & 0 & 2g\,\mathrm{Re}(a_2) & 0 \\ 0 & 0 & 0 & 0 & -\gamma & \omega_{m1} & 0 & -\mu \\ 2g\,\mathrm{Re}(a_1) & 2g\,\mathrm{Im}(a_1) & 0 & 0 & -\omega_{m1} & -\gamma & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu & -\gamma & \omega_{m2} \\ 0 & 0 & 2g\,\mathrm{Re}(a_2) & 2g\,\mathrm{Im}(a_2) & \mu & 0 & -\omega_{m2} & -\gamma \end{pmatrix}. \tag{B1}$$

[1] J. Jin, D. Rossini, R. Fazio, M. Leib, and M. J. Hartmann, Phys. Rev. Lett. **110**, 163605 (2013).
[2] J. Jin, A. Biella, O. Viyuela, L. Mazza, J. Keeling, R. Fazio, and D. Rossini, Phys. Rev. X **6**, 031011 (2016).
[3] R. Kohlhaas, A. Bertoldi, E. Cantin, A. Aspect, A. Landragin, and P. Bouyer, Phys. Rev. X **5**, 021011 (2015).
[4] J. Wang, Z. Tian, J. Jing, and H. Fan, Phys. Rev. D **93**, 065008 (2016).
[5] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
[6] A. Mari, A. Farace, N. Didier, V. Giovannetti, and R. Fazio, Phys. Rev. Lett. **111**, 103605 (2013).
[7] C. L. Zhu, N. Yang, Y. X. Liu, F. Nori, and J. Zhang, Phys. Rev. A **92**, 042327 (2015).
[8] W. L. Li, C. Li, and H. S. Song, Phys. Rev. E **95**, 022204 (2017).
[9] C. Huygens, *Oeuvres Complètes de Christiaan Huygens* (Nijhoff, The Hague, 1893), Vol. 15, p. 243.
[10] M. Xu, D. A. Tieri, E. C. Fine, J. K. Thompson, and M. J. Holland, Phys. Rev. Lett. **113**, 154101 (2014).
[11] M. Xu and M. J. Holland, Phys. Rev. Lett. **114**, 103601 (2015).
[12] A. Roth and K. Hammerer, Phys. Rev. A **94**, 043841 (2016).
[13] T. E. Lee and H. R. Sadeghpour, Phys. Rev. Lett. **111**, 234101 (2013).
[14] S. Walter, A. Nunnenkamp, and C. Bruder, Phys. Rev. Lett. **112**, 094102 (2014).
[15] T. E. Lee, C. K. Chan, and S. Wang, Phys. Rev. E **89**, 022913 (2014).
[16] V. Ameri, M. Eghbali-Arani, A. Mari, A. Farace, F. Kheirandish, V. Giovannetti, and R. Fazio, Phys. Rev. A **91**, 012301 (2015).
[17] S. Walter, A. Nunnenkamp, and C. Bruder, Ann. Phys. **527**, 131 (2015).
[18] W. L. Li, C. Li, and H. S. Song, Phys. Rev. E **93**, 062221 (2016).
[19] U. Parlitz and A. Schlemmer, and S. Luther, Phys. Rev. E **83**, 057201 (2011).
[20] G. Manzano, F. Galve, G. L. Giorgi, E. Hernández-Garcá, and R. Zambrini, Sci. Rep. **3**, 1439 (2013).
[21] G. L. Giorgi, F. Plastina, G. Francica, and R. Zambrini, Phys. Rev. A **88**, 042115 (2013).
[22] T. Weiss, A. Kronwald, and F. Marquardt, New J. Phys. **18**, 013043 (2016).
[23] G. L. Giorgi, F. Galve, and R. Zambrini, Phys. Rev. A **94**, 052121 (2016).
[24] G. Heinrich, M. Ludwig, J. Qian, B. Kubala, and F. Marquardt, Phys. Rev. Lett. **107**, 043603 (2011).
[25] J. Zhang, Y. X. Liu, Ş. K. Özdemir, R. B. Wu, F. Gao, X. B. Wang, L. Yang, and F. Nori, Sci. Rep. **3**, 2211 (2013).
[26] M. Ludwig and F. Marquardt, Phys. Rev. Lett. **111**, 073603 (2013).
[27] J. Nokkala, F. Galve, R. Zambrini, S. Maniscalco, and J. Piilo, Sci. Rep. **6**, 26861 (2016).
[28] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, Phys. Rep. **366**, 1 (2002).
[29] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[30] L. Lü, C. Li, G. Li, A. Sun, Z. Yan, T. Rong, and Y. Gao, Commun. Nonlin. Sci. Numer. Sim. **47**, 267 (2017).
[31] F. Gomez, T. Lorimer, and R. Stoop, Phys. Rev. Lett. **116**, 108101 (2016).
[32] F. Galve, G. L. Giorgi, and R. Zambrini, *Quantum Correlations and Synchronization Measures*, edited by F. Fanchini, P. D. Soares, and G. Adesso, Lectures on General Quantum Correlations and their Applications, Quantum Science and Technology (Springer, 2017), pp. 393–420.
[33] M. Zhang, S. Shah, J. Cardenas, and M. Lipson, Phys. Rev. Lett. **115**, 163902 (2015).
[34] L. Ying, Y. C. Lai, and C. Grebogi, Phys. Rev. A **90**, 053810 (2014).
[35] B. Zhu, J. Schachenmayer, M. Xu, F. Herrera, J. G. Restrepo, M. J. Holland, and A. M. Rey, New J. Phys. **17**, 083063 (2015).
[36] We make an agreement here that the measure depending on the absolute error is called an absolute measure, and correspondingly the measure depending on the relative error is called a relative measure.
[37] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).
[38] F. Marquardt and S. M. Girvin, Physics **2**, 40 (2009).
[39] M. H. Matheny, M. Grau, L. G. Villanueva, R. B. Karabalin, M. C. Cross, and M. L. Roukes, Phys. Rev. Lett. **112**, 014101 (2014).
[40] K. Shlomi, D. Yuvaraj, I. Baskin, O. Suchoi, R. Winik, and E. Buks, Phys. Rev. E **91**, 032910 (2015).
[41] Q. Chen, Y. C. Lai, J. Chae, and Y. Do, Phys. Rev. B **87**, 144304 (2013).
[42] A. Mari and J. Eisert, Phys. Rev. Lett. **103**, 213603 (2009).
[43] A. Farace and V. Giovannetti, Phys. Rev. A **86**, 013820 (2012).

[44] L. Lü, C. Li, L. Chen, and G. Zhao, Nonlin. Dyn. **86**, 655 (2016).
[45] L. Lü, L. Chen, C. Han, L. Ge, and L. Gao, Physica A **467**, 74 (2017).
[46] A. Pikovsky, M. Rosenblum, and J. Kurth, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2003).
[47] S. X. Wu and C. S. Yu, Phys. Rev. A **85**, 032332 (2012).
[48] X. D. Yu, D. J. Zhang, C. L. Liu, and D. M. Tong, Phys. Rev. A **93**, 060303(R) (2016).
[49] G. L. Giorgi, F. Galve, G. Manzano, P. Colet, and R. Zambrini, Phys. Rev. A **85**, 052101 (2012).
[50] C. K. Law, Phys. Rev. A **51**, 2537 (1995).
[51] W. Z. Zhang, J. Cheng, W. D. Li, and L. Zhou, Phys. Rev. A **93**, 063853 (2016).
[52] J. Cheng, W. Z. Zhang, L. Zhou, and W. Zhang, Sci. Rep. **6**, 23678 (2016).
[53] C. W. Gardiner and P. Zoller *Quantum Noise* (Springer, Berlin, 2000).
[54] G. Wang, L. Huang, Y. C. Lai, and C. Grebogi, Phys. Rev. Lett. **112**, 110406 (2014).
[55] L. Bakemeier, A. Alvermann, and H. Fehske, Phys. Rev. Lett. **114**, 013601 (2015).
[56] A. Isar, Phys. Part. Nuclei Lett. **6**, 567 (2009).
[57] Fidelity contains all local quantum properties of two quantum states, therefore we use fidelity to represent all the local measures in the following discussions. Correspondingly, $\mathcal{C}_{\langle q_1^2\rangle,\langle q_2^2\rangle}$ will no longer be discussed because $\mathcal{F}=1$ inevitably leads to $\mathcal{C}_{\langle q_1^2\rangle,\langle q_2^2\rangle}=1$.
[58] Y. C. Liu, Y. F. Xiao, X. Luan, and C. W. Wong, Phys. Rev. Lett. **110**, 153606 (2013).
[59] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
[60] S. S. Mirkhalaf and A. Smerzi, Phys. Rev. A **95**, 022302 (2017).