Pseudo-Hermitian anti-Hermitian ensemble of Gaussian matrices

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It is shown that the ensemble of pseudo-Hermitian Gaussian matrices recently introduced gives rise in a certain limit to an ensemble of anti-Hermitian matrices whose eigenvalues have properties directly related to those of the chiral ensemble of random matrices.

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I. INTRODUCTION

One of the most impressive successes of random matrix theory (RMT) was the fitting of the local statistical fluctuations of the nontrivial zeros of the Riemann ζ function by statistics extracted from the eigenvalues of the complex Hermitian matrices of the unitary class of the Gaussian ensemble (GUE). Those zeros are the matter of the famous Riemann conjecture that states they are the only complex ones and have a real part equal to one-half.

Despite this success there is an incongruity in the fact that the GUE eigenvalues are real while the zeros are complex conjugate. It would be a more conformable agreement with eigenvalues of some class of anti-Hermitian matrices which are known to have pure imaginary eigenvalues. However, eigenvalues of anti-Hermitian matrices though pure imaginary are not necessarily conjugate. Imaginary and conjugate are the eigenvalues of antisymmetric matrices. Moreover, Mehta has shown that oddly their properties have similarities with the eigenvalues of the complex matrices of the unitary class of random matrix theory [1]. But matrices with real elements are associated with time reversal symmetry which is not expected to be a relevant symmetry in the case of the zeros.

In fact, another important hypothesis regarding the zeros is that they might be eigenvalues of some physical Hamiltonian. This has been proposed by Hilbert in the 1920s and if this Hamiltonian exists it is not expected to be invariant under the reversal of the time.

More recently, Berry and Keating [2] further elaborated upon this idea and came up with the suggestion that this Hamiltonian would have the unidimensional form

$$H = xp, \tag{1}$$

where (x,p) are classical conjugate variables. This Hamiltonian is not invariant under time reversal as can easily be seen. But, if simultaneously with the reversal of the time also the sign of the coordinate is changed then the Hamiltonian remains the same. This means that it is a $\mathcal{PT}$-symmetric operator, that is, it belongs to a class of operators invariant under the parity (P) and the time reversal (T) transformations.

The interest in PT-symmetric systems started when the spectrum of the complex non-Hermitian Hamiltonian

$$H = p^2 - (ix)^\nu \tag{2}$$

was analyzed [3,4], and it was found that, as a function of the parameter ν, the eigenvalues undergo a transition from the real axis to the complex plane in conjugate pairs. This Hamiltonian, Eq. (2), contains a Hermitian part (the first term) and a PT-symmetric one, the second term.

This type of operator may be connected to another class of operators known as pseudo-Hermitian operators [5]. An operator, say A, satisfying this symmetry would be connected to its adjoint by a similarity transformation

$$A^\dagger = \eta A \eta^{-1}, \tag{3}$$

where the η is Hermitian. Accordingly, their eigenvalues would be real or come as complex conjugate pairs. It also has been found that quantum relations still work and can be extended to these physical systems if η becomes a metric to define the inner product as

$$(.,\eta.). \tag{4}$$

This kind of symmetry has been a matter of studies in quantum mechanics in the last decades and more recently a $\mathcal{PT}$-symmetric operator with these properties has been proposed in particular also to investigate the zeros of the Riemann ζ function [6]. In a different approach, the zeros have been interpreted as missing spectral lines [7]. Motivated by this context, it is our purpose to introduce a random matrix model which may be connected to both $\mathcal{PT}$-symmetric quantum mechanics and, through its eigenvalues, to the statistics of the zeros of the Riemmann ζ function.

The study of non-Hermitian random matrices can be traced back at least as far as to the seminal works of Ginibre [8], which explored the properties of general random matrix models for real, complex, and quaternion elements. Additionally, complex networks were studied as early as the 1970s and 1980s using non-Hermitian matrices [9–11] and that line of research has been actively pursued since then (see, e.g., Refs. [12,13] and references therein). In physics, non-Hermitian random matrices appeared in such contexts as dissipative quantum maps [14], quantum chromodynamics [15], the statistics of quantum chaotic scattering processes [16], and localization problems in condensed matter physics [17–19]. In a broader context, further properties of non-Hermitian random matrices were the focus of continued study in itself, and several properties of non-Hermitian matrices were studied in the literature such as the eigenvalue density on the complex plane of almost-Hermitian matrices [20,21], the statistics of non-Gaussian non-Hermitian random matrices and their relation to free random variables [22,23], and many other statistical properties [24–32]. A concise but insightful review may be found in Chap. 18 of Ref. [33].

However, since the early studies of PT-symmetric systems there was an interest in investigating random matrix ensembles to model the $\mathcal{PT}$-symmetric kind of non-Hermitian Hamiltonians. This interest comes naturally as time reversal symmetry plays an important role in RMT [1]. The ensembles initially proposed were restricted to the case of 2×2 matrices [34,35], with further extensions shortly thereafter going as far as treat explicitly the 3×3 case [36], and random walks and cyclic matrices were approached in the $N \times N$ case a few years later [37].

More recently, general $N \times N$ models were proposed in at least three different approaches. The first introduced the use of split-complex and split-quaternion ensembles of random matrices [38], using the properties of those numbers to model the $\mathcal{PT}$-symmetric aspects of the system. The second consisted of considering matrices of tridiagonal form like the ones of the so-called β ensemble [39,40]. In this ensemble the Dyson index β can assume any real positive value in contrast with Gaussian case in which $\beta = 1,2,4$.

But here we focus on the last of those approaches, a recently introduced ensemble of pseudo-Hermitian Gaussian matrices [41]. This approach shares some traits with that of [38] such as the use of $N \times N$ Gaussian matrices. However, instead of introducing $\mathcal{PT}$-symmetric properties through the properties of the split numbers, projection operators are introduced for this purpose. Starting from a Hermitian matrix with standard real, complex, or quaternion Gaussian elements, pseudo-Hermiticity is introduced through the use of projection operators to define Hermitian and anti-Hermitian blocks. It has been found that, introducing an interaction between the blocks as a function of a real positive parameter controlling the intensity of that interaction, the eigenvalues of this special class of Gaussian matrices leave the real axis and, as the parameter increases, fill an ellipsis and finally approach the imaginary axis as conjugate pairs. Our main purpose in the present paper is to investigate properties of these complex eigenvalues.

II. PSEUDO-HERMITIAN GAUSSIAN ENSEMBLE

In the construction of the pseudo-Hermitian ensemble of Ref. [41], the starting point is the standard RMT ensemble defined by the distribution [1]

$$P(H) = Z_N^{-1} \exp\left[-\frac{\beta}{2}\mathrm{tr}(H^\dagger H)\right], \tag{5}$$

where H is a matrix whose elements can be written as

$$H_{ij} = H_{ij}^0 + iH_{ij}^1 + jH_{ij}^2 + kH_{ij}^3 \tag{6}$$

with $i^2 = j^2 = k^2 = ijk = -1$. The number of nonzero elements in Eq. (6) denoted by β can be equal to 1, 2, or 4 [1]. Therefore, the elements are Gaussian distributed and can be real, complex, or quaternion. In Ref. [41], these matrices were used to define a family of matrices which satisfy the pseudo-Hermitian condition (3). One feature of that model is that the metric η is found to be an involution, that is $\eta^2 = 1$. Here we are interested in taking advantage of this fact to restate the formalism of that model. To do this let us consider a matrix which may be written as

$$A(r) = \mathcal{H} + r\mathcal{S} \tag{7}$$

such that $A(r)$ verifies Eq. (3) and where $\mathcal{H}$ is a Hermitian matrix and $\mathcal{S}$ is an anti-Hermitian matrix, with r a positive parameter. The operator η will then commute with the Hermitian part $\mathcal{H}$,

$$[\mathcal{H},\eta] = 0, \tag{8}$$

and anticommute with the anti-Hermitian part $\mathcal{S}$,

$$\{\mathcal{S},\eta\} = 0. \tag{9}$$

Moreover, as an involution, the metric η has eigenvalues ± 1, which leads to a split of the space into subspaces labeled by these values. Introducing the operators P and Q that project into these subspaces, the metric can be expressed as

$$\eta = P - Q. \tag{10}$$

We are now in position to use the Gaussian matrices of Eq. (6) used to construct two realizations of the pseudo-Hermitian matrix A. The first of these realizations is just

$$A = PHP + QHQ + r(PHQ - QHP), \tag{11}$$

where

$$P_i = |i\rangle\langle i| \tag{12}$$

and

$$P = \sum_{i=1}^{M} P_i \quad \text{and} \quad Q = \sum_{j=M+1}^{N} P_j. \tag{13}$$

The matrix defined in (11) verifies both (8) and (9). This can be seen by noting that, for (11), $\mathcal{H} = PHP + QHQ$ and $\mathcal{S} = PHQ - QHP$ and therefore

$$\begin{aligned}
&(P - Q)\mathcal{H} - \mathcal{H}(P - Q)\\
&\quad = PHP - QHQ - (PHP - QHQ) = 0,\\
&(P - Q)\mathcal{S} + \mathcal{S}(P - Q)\\
&\quad = PHQ + QHP + (-QHP - PHQ) = 0.
\end{aligned} \tag{14}$$

The second realization is motivated by the fact that given a PT-symmetric potential $V(x)$, that is one such that $[V(-x)]^* = V(x)$, transforming it into a matrix using a complete set of eigenfunctions with a definite parity produces, with the usual inner product over the real line, elements

$$V_{ij} = \langle \Psi_i(x)|V(x)|\Psi_j(x)\rangle. \tag{15}$$

Taking the complex conjugate and changing x to $-x$ then as a consequence of the parity of the eigenfunctions we obtain

$$V_{ij}^* = (-1)^{i+j} V_{ji}, \tag{16}$$

which shows that the elements follow a structure in which the subdiagonals alternate sign.

In order to obtain $N \times N$ matrices whose structures follow (16), the set of N projectors $\{P_k\}_{k=1,2,\ldots,N}$ as defined in (12) is used to define [41]

$$\begin{aligned}
A = &\sum_{k=1}^{N} P_k H P_k + \sum_{j>i} r^{s_{ij}} P_i H P_j\\
&+ \sum_{j<i} r^{s_{ij}} \cos[(j-i)\pi] P_i H P_j,
\end{aligned} \tag{17}$$

where $s_{ij} = 1/2 - 1/2\cos[(i-j)\pi]$ and r is a real positive parameter. In this case, the metric can still be written as

$\eta = P - Q$ with

$$P = \sum_{i=1}^{[N+1/2]} P_{2i-1} \text{ and } Q = \sum_{j=1}^{[N/2]} P_{2j}, \tag{18}$$

where [.] means integer part. The first, P, is the projector on the subspace created by the odd elements of the base, whereas the second, Q, does the same for the even elements. This method creates a chessboardlike structure, where the Hermitian and anti-Hermitian parts occupy alternating nature of the subdiagonal elements of the matrix.

The Hermitian and anti-Hermitian parts of (17) may be written as, respectively,

$$\mathcal{H} = \sum_{\text{mod }(|i-j|,2)=0}^{N} P_k H P_j,$$
$$\mathcal{S} = \sum_{\text{mod }(|i-j|,2)=1} [P_i H P_j - P_j H P_i], \tag{19}$$

where mod (x,y) denotes the remainder of the division between integers x and y and $\delta_{i,j}$ is the Kronecker δ. This means that the Hermitian part consists of all the products $P_i H P_j$ where both i and j have the same integer parity and the anti-Hermitian part consists of all the analogous products where i and j have different parities.

The commutation and anticommutation relations, (8) and (9), are also valid in this case. This can be seen fairly straightforwardly by first noting that

$$P\mathcal{H} = \sum_{i,j \text{ odd}} P_i H P_j = \mathcal{H}P,$$
$$Q\mathcal{H} = \sum_{i,j \text{ even}} P_i H P_j = \mathcal{H}Q, \tag{20}$$

and that

$$P\mathcal{S} = \sum_{\substack{i \text{ odd} \\ j \text{ even}}} [P_i H P_j - P_j H P_i] = \mathcal{S}Q,$$
$$Q\mathcal{S} = \sum_{\substack{i \text{ even} \\ j \text{ odd}}} [P_i H P_j - P_j H P_i] = \mathcal{S}P. \tag{21}$$

Equation (20) implies immediately that $(P - Q)\mathcal{H} = \mathcal{H}(P - Q)$ which in turn implies (8), whereas (21) similarly implies that $(P - Q)\mathcal{S} = \mathcal{S}(Q - P)$, which in turn implies (9).

In [41] it is shown that in both cases discussed above, the joint density distribution of the matrix elements of A is given by

$$P(A) = \zeta_N(r) \exp\left\{-\frac{\beta}{2}\text{tr}\left[\left(1 - \frac{1}{r^2}\right)\frac{AA + A^\dagger A^\dagger}{4} + \left(1 + \frac{1}{r^2}\right)\frac{AA^\dagger + A^\dagger A}{4}\right]\right\}. \tag{22}$$

Moreover, it is shown that this form suggests the ansatz that the resulting distribution of eigenvalues are likely to follow an elliptic law [11,42,43] with axes

$$a = \sqrt{\frac{N}{1+r^2}}, \tag{23}$$

$$b = r^2\sqrt{\frac{N}{1+r^2}}. \tag{24}$$

The behavior of these ellipses is shown in Fig. 1, where the semimajor axis is real for $r = 0.75$ in Fig. 1(a), becomes equal to the semiminor axis for $r = 1.00$ in Fig. 1(b), and finally becomes imaginary for $r = 1.25$ in Fig. 1(c).

Therefore, this distribution describes a transition from a situation in which the eigenvalues lie in the real axis, for $r = 0$, to one in which the eigenvalues move into the complex plane, and approaches the imaginary axis when r becomes large. In Fig. 2 we present the behavior of the eigenvalues as the r parameter increases. As the eigenvalues with no imaginary part evaporate into the complex plane, as shown in Fig. 2(a), their imaginary part, scaled by $1/r$, condensates into the imaginary axis, as shown in Fig. 2(b).

Putting $r = 0$, we have the eigenvalue equation

$$A\Psi = \mathcal{H}\Psi = \lambda\Psi \tag{25}$$

and the equation

$$\mathcal{H}\eta\Psi = \lambda\eta\Psi \tag{26}$$

obtained with the commutator relation (8). Therefore, in this case the eigenvalues of η are good quantum numbers.

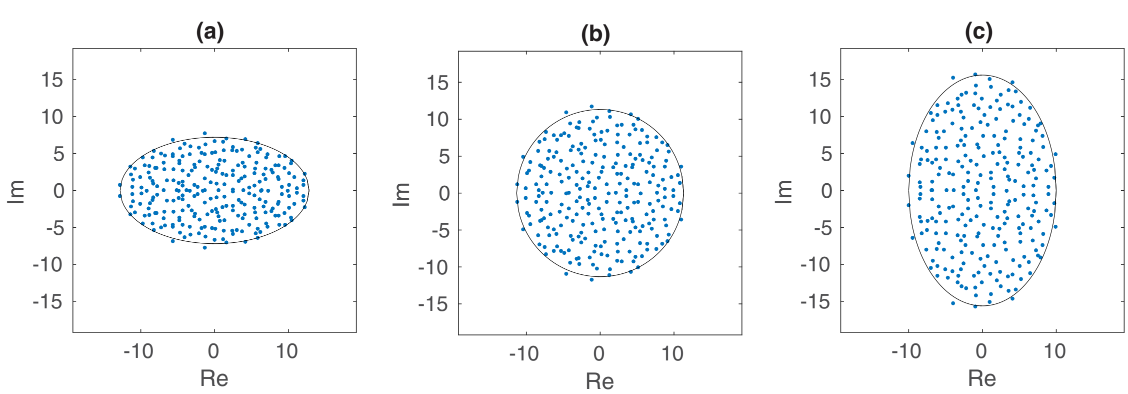

FIG. 1. Effect on the eigenvalues of varying parameter r for a single matrix of size $n = 256$ for (a) $r = 0.75$, (b) $r = 1.00$, and (c) $r = 1.25$.

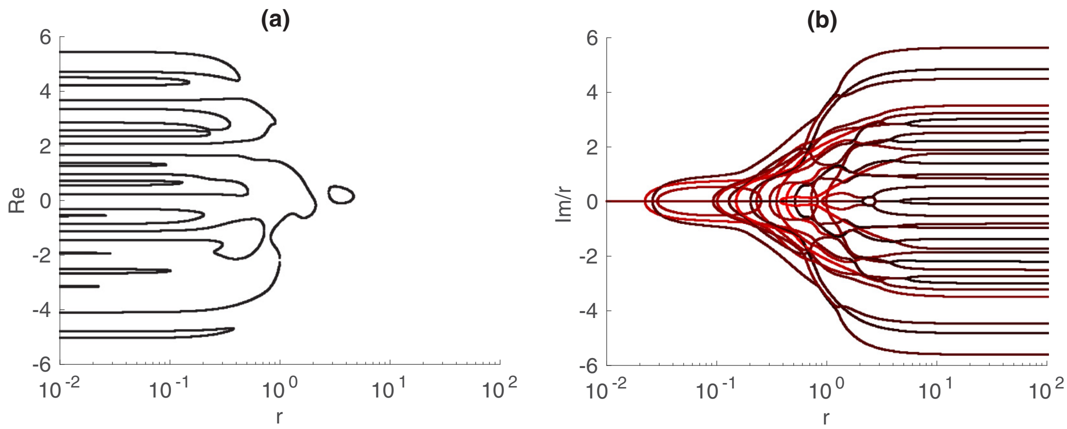

FIG. 2. Real eigenvalues (a) and complex eigenvalues, scaled by $1/r$ (b), of the matrix as a function of the parameter r. Color denotes the absolute vale of the real part of the eigenvalues, with red denoting the largest relative intensity of the whole.

By adding and subtracting these equations we obtain

$$P\mathcal{H}P\Psi = \lambda P\Psi \tag{27}$$

and

$$Q\mathcal{H}Q\Psi = \lambda Q\Psi \tag{28}$$

in which the relations

$$P = \frac{1+\eta}{2} \tag{29}$$

and

$$Q = \frac{1-\eta}{2} \tag{30}$$

have been used. Therefore, we have two decoupled matrices whose eigenvalues are those of the Gaussian matrices. The dimensions of the blocks are $M \times M$ and $(N-M)\times(N-M)$ in the first realization, Eq. (11), in which the decoupled matrices have a block diagonal structure. In the second one, Eq. (17), denoting as n the integer part of $N/2$, the two decoupled matrices are both $n \times n$ for N even and for N odd the $P\mathcal{H}P$ block has dimension $(n+1)\times(n+1)$, and their structure is such that one of them has only the elements of $\mathcal{H}$ that have a given parity, that is either even or odd.

III. ANTI-HERMITIAN LIMIT

The anti-Hermitian part of the pseudo-Hermitian matrix A [44] is obtained by taking the limit

$$\mathcal{S} = \lim_{r\to\infty} \frac{A}{r}, \tag{31}$$

whose joint density distribution of its matrix elements is immediately found to be given by

$$P(\mathcal{S}) = \frac{1}{W_N} \exp\left(-\frac{\beta}{2}\text{tr}\,\mathcal{S}\mathcal{S}^\dagger\right). \tag{32}$$

We remark that this should not be mistaken as being an anti-pseudo-Hermitian matrix, which is defined in the literature [44] as an operator which verifies (3) for an antilinear, anti-Hermitian η.

There is a noteworthy property of matrices which are both pseudo- and anti-Hermitian regarding their trace. Let S be an $N \times N$ diagonalizable pseudo-Hermitian matrix verifying

$$S^\dagger = \mu S \mu^{-1} = -S \tag{33}$$

and with diagonalizations

$$SU = U\Xi,$$

$$SV = V\bar{\Xi}.$$

Using the pseudo-Hermitian relation (33) we obtain the connection between the two above diagonalizations,

$$V^{-1} = U^\dagger \mu,$$

and applying this to the trace of $S^\dagger S$, we obtain through the cyclic property of the trace and the anti-Hermiticity of S,

$$\text{tr}(S^\dagger S) = \text{tr}(S^\dagger S V U^\dagger \mu) = \text{tr}(U^\dagger \mu S^\dagger S V) = -\text{tr}(U^\dagger \mu S V \bar{\Xi})$$
$$= -\text{tr}(U^\dagger \mu V \bar{\Xi}\bar{\Xi}) = \text{tr}(\Xi\bar{\Xi}).$$

Therefore, for any pseudo- and anti-Hermitian diagonalizable matrix S, we have that

$$\text{tr}(S^\dagger S) = \text{tr}(\bar{\Xi}\Xi) = \text{tr}(\Xi\bar{\Xi}) = \sum_{k=1}^{N} |\xi_k|^2, \tag{34}$$

where Ξ is the diagonal matrix containing the eigenvalues $\{\xi_k\}_{k=1,2,\ldots,N}$ of S.

The anticommutation of $\mathcal{S}$ with the metric η defines a discrete symmetry characteristic of the presence of chirality, that is of a P symmetry [45]. So, apart from the fact that we are dealing with conjugate pairs of imaginary eigenvalues, we should expect to recover properties of the so-called chiral ensemble. To see this, let us consider the eigenvalue equation

$$\mathcal{S}\Psi = \lambda\Psi \tag{35}$$

together with the equation

$$\mathcal{S}\eta\Psi = -\lambda\eta\Psi \tag{36}$$

obtained using the anticommutator relation (9). Adding and subtracting them and using relations (29) and (30) we have the coupled equations

$$Q\mathcal{S}P\Psi = \lambda Q\Psi \tag{37}$$

and

$$PSQ\Psi = \lambda P\Psi. \tag{38}$$

Easily, these equations decouple as

$$(QSP)^\dagger(QSP)P\Psi = -\lambda^2 P\Psi \tag{39}$$

and

$$(PSQ)^\dagger(PSQ)Q\Psi = -\lambda^2 Q\Psi. \tag{40}$$

It is convenient to introduce matrices $\mathcal{S}^\pm$ whose elements are the elements of the matrices $Q\mathcal{S}P$ and $P\mathcal{S}Q$ respectively. In the realization defined by Eq. (11), $\mathcal{S}^+$ has dimension $(N-M)\times M$ whose elements are

$$\mathcal{S}^+_{ij} = (Q\mathcal{S}P)_{ij} = -H^*_{M+i,j} \tag{41}$$

and $\mathcal{S}^-$ has dimension $M\times(N-M)$ and elements

$$\mathcal{S}^-_{ij} = (P\mathcal{S}Q)_{ij} = H_{i,M+j}. \tag{42}$$

The above matrices generate the Wishart matrices $(\mathcal{S}^\pm)^\dagger\mathcal{S}^\pm$ with distribution [40]

$$P(\mathcal{S}^\pm) = C_M^{-1}\exp\left[-\frac{\beta}{2}\mathrm{tr}(\mathcal{S}^\pm)^\dagger\mathcal{S}^\pm\right], \tag{43}$$

which share the same set of eigenvalues with the distribution of a Laguerre ensemble

$$P_\beta(y_1,\dots,y_M) = \frac{1}{C_M}\prod_{k=1,}^{M} y_k^{\beta(N-2M+1)/2-1}e^{-\beta y_k/2}\prod_{j>i}|y_j-y_i|^\beta. \tag{44}$$

Turning now to the realization defined by Eq. (17) with $M=[N/2]$ and $\mathrm{sgn}(x)=x/|x|$, we have the matrices

$$\mathcal{S}^+_{ij} = (Q\mathcal{S}P)_{2i,2j-1} = \mathrm{sgn}(2i-2j+1)H_{2i,2j-1} \tag{45}$$

and

$$\mathcal{S}^-_{ij} = (P\mathcal{S}Q)_{2i-i,2j} = \mathrm{sgn}(2i-2j-1)H_{2i-1,2j}. \tag{46}$$

Their dimensions are $M\times M$ if N is even and $M\times(M+1)$ and $(M+1)\times M$ respectively if N is odd. This means that these matrices may still be reduced to the Wishart case and (43) still holds. Therefore, making the substitution $y_k=\lambda_k^2$, the joint distribution of the eigenvalues of the anti-pseudo-Hermitian ensemble, is obtained as

$$P_\beta(\lambda_1,\dots,\lambda_M) = \frac{1}{Z_M}\prod_{\gamma=1}^{M}\exp\left(-\frac{\beta|\lambda_\gamma|^2}{2}\right)|\lambda_\gamma|^{\beta(N-2M+1)-1} \times\prod_{\xi>\gamma}|\lambda_\xi^2-\lambda_\gamma^2|^\beta. \tag{47}$$

In (47), the term $N-2M$ in the exponent is the number of zeros and M is the number of complex conjugate pairs. This corresponds to the number of differences between nonzero eigenvalues and the zeros in the former case, and the term $|\lambda|^{\beta-1}$ corresponds to the number of differences between eigenvalues and their complex conjugates. This suggests that the factorization of the Jacobian in the Hermitian case extends to the pseudo-Hermitian case in terms of these differences; cf. the Appendix.

IV. SPECTRAL STATISTICS

In the specific case of $\beta=2$ and even matrix size, this corresponds to the GUE, which follows the same statistics as the zeros of the Riemann ζ function [46]. We remark that if the terms $|\lambda_k|^{\beta-1}$ are considered as the differences between each eigenvalue and its conjugate, then the total number of differences is equal to the number of independent elements of the matrix S.

To write the spectral properties of the anti-Hermitian ensemble it is more convenient to consider Eq. (44) and following Ref. [47] the properties for square matrices and $\beta=2$ will be given for a given $m\geqslant 1$ by the Kernel

$$K_m(x,y) = \sum_{k=0}^{m-1}\phi_k(x)\phi_k(y), \tag{48}$$

where the functions are the Laguerre functions

$$\phi_k(y) = \exp\left(-\frac{y}{2}\right)L_k(y), \tag{49}$$

where $\{L_k(y)\}_{k=1,2,\dots}$ are the orthogonal Laguerre polynomials. We may then use the Christoffel-Darboux [48] to obtain

$$K_m(x,y) = a_m\frac{\phi_m(x)\phi_{m-1}(y)-\phi_m(y)\phi_{m-1}(x)}{x-y}. \tag{50}$$

The eigenvalue density is given by the diagonal part of the kernel $K_m(x,x)$ which in the large $m\to N\gg 1$ asymptotic limit becomes the Marchenko-Pastur distribution [40,49,50]

$$\rho(y) = \frac{N}{\pi}\sqrt{\frac{1-y}{y}} \tag{51}$$

for $\beta=1,2$. In terms of the eigenvalue variable of the anti-Hermitian ensemble the density is the Wigner semicircle,

$$\rho(\lambda) = \frac{N}{\pi}\sqrt{1-\lambda^2}. \tag{52}$$

The fluctuations properties are contained in the kernel. In order to do so, we use the asymptotic expansions for large N [48],

$$\phi_N(y) \approx \frac{1}{\pi^{1/2}N^{1/4}}\frac{1}{y^{1/4}}\cos\left(2\sqrt{Ny}-\frac{\pi}{4}\right),$$
$$\phi_{N-1}(y) \approx \frac{1}{\pi^{1/2}(N-1)^{1/4}}\frac{1}{y^{1/4}}\cos\left(2\sqrt{(N-1)y}-\frac{\pi}{4}\right). \tag{53}$$

In order to turn (53) into a form that can be used in (50), we use the Taylor expansions

$$f(u;\epsilon) = \cos\left(2\sqrt{N(1+\epsilon)u}-\frac{\pi}{4}\right) \approx \cos\left(2\sqrt{Nu}-\frac{\pi}{4}\right)-\sqrt{Nu}\epsilon\sin\left(2\sqrt{Nu}-\frac{\pi}{4}\right), \tag{54}$$

$$[N(1+\delta)]^{-1/4} \approx N^{-1/4}-\tfrac{1}{4}N^{-5/4}\delta \tag{55}$$

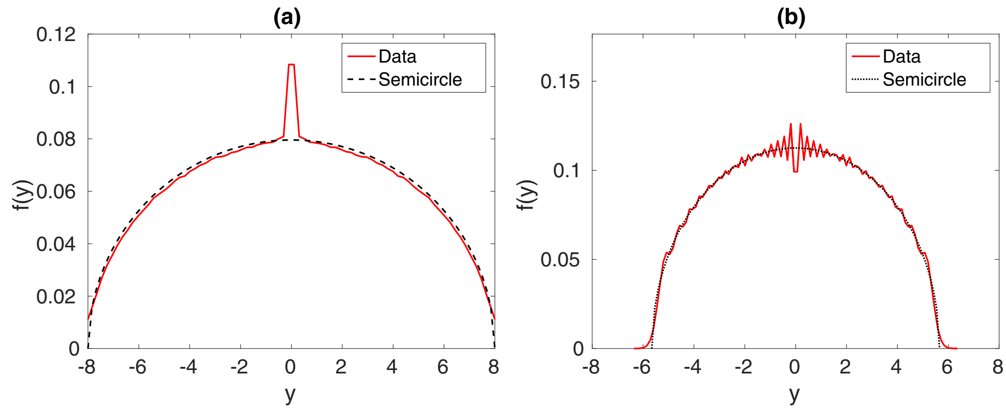

FIG. 3. Deviations from the semicircle distribution for 10^5 matrices of order $N = 32$ for (a) $\beta = 1$ and (b) $\beta = 2$.

and therefore, for $\epsilon = -\frac{1}{N}$,

$$\phi_N(y) \approx \frac{1}{\pi^{1/2}N^{1/4}}\frac{1}{y^{1/4}}\cos w_N(y),$$
$$\phi_{N-1}(y) \approx \frac{1}{\pi^{1/2}N^{1/4}}\left[\frac{\cos w_N(y)}{y^{1/4}} + \frac{y^{1/4}}{\sqrt{N}}\sin w_N(y)\right],$$
$$w_N(y) = 2\sqrt{Ny} - \frac{\pi}{4}, \tag{56}$$

where only terms up to order $1/N$ were kept. We may then obtain

$$K_N(x,y) \approx \frac{1}{\pi\sqrt{N}(xy)^{1/4}}\left\{\frac{\sin(2\sqrt{Nx} - 2\sqrt{Ny})}{2\sqrt{Nx} - 2\sqrt{Ny}} + \frac{\cos(2\sqrt{Nx} + 2\sqrt{Ny})}{2\sqrt{Nx} + 2\sqrt{Ny}}\right\}. \tag{57}$$

Considering now the substitution

$$(x,y) \to \left(\frac{\xi^2}{2\sqrt{N}},\frac{\gamma^2}{2\sqrt{N}}\right) \tag{58}$$

we then have

$$K_N(\xi,\gamma) \approx \frac{1}{\pi N^{3/2}}\sqrt{|\xi\gamma|}\left[\frac{\sin(\xi-\gamma)}{\xi-\gamma} + \frac{\cos(\xi+\gamma)}{\xi+\gamma}\right]. \tag{59}$$

The first term leads to Wigner-Dyson statistics while the second one produces oscillations near the origin [1]. This conclusion is confirmed by numerical simulations as shown in Fig. 3 for $\beta = 1$ in Fig. 3(a) and $\beta = 2$ in Fig. 3(b).

In the figures, this density is compared with numerical results for the antisymmetric and anti-Hermitian cases. The semicircle gives a good fit of the average density but there are discrepancies near the origin. Using the semicircle, the spectra are unfolded and in Fig. 4 the spacing distributions for the two cases are compared with the Wigner surmise [1] for the orthogonal and the unitary cases.

V. CONCLUSION

In the present paper we further investigate properties of the ensemble recently introduced in Ref. [41] whose Gaussian matrices satisfy the pseudo-Hermitian condition, namely Eq. (3). Decomposing the matrices into Hermitian and anti-Hermitian parts, it is found that the former commutes with the metric while the latter anticommutes with the same metric. As the anticommutation with the metric means parity invariance, the matrices contain a parity invariant term besides a term that belongs to the symmetry classes of the Gaussian ensemble. As a function of a real positive parameter that couples the two terms, the ensemble undergoes a transition from a Hermitian to a pseudo-Hermitian anti-Hermitian family of matrices. It was then shown that after this transition, this pseudo-anti-Hermitian family is directly related to the chiral ensemble of random matrices. It is also worthy of note that the eigenvalues come in pure imaginary conjugate pairs, and that the distribution of those imaginary parts, for the $\beta = 2$ case, follows the same distribution as the nontrivial zeros of the Riemann ζ function.

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APPENDIX: DERIVATION OF THE JACOBIAN FROM MATRIX ELEMENTS TO EIGENVALUES

In order to obtain the density in terms of eigenvalues for the general matrix A with pseudo-Hermiticity operator η, we must first obtain an equation for the diagonalization of A [1]. We begin by writing

$$AU = U\Lambda \tag{A1}$$

and the diagonalization in terms of the complex conjugate eigenvalues,

$$AV = V\bar{\Lambda}, \tag{A2}$$

which corresponds to switching the positions of the eigenvectors of the conjugate pairs in (A1). Taking the adjoint of (A1) and considering the pseudo-Hermiticity of A as in (3), however we have that

$$U^\dagger\eta A = \bar{\Lambda}U^\dagger\eta \tag{A3}$$

such that, multiplying Eq. (A2) by $U^\dagger\eta$ to the left and subtracting (A3) multiplied by V to the right, we get

$$U^\dagger\eta V\bar{\Lambda} - \bar{\Lambda}U^\dagger\eta V = 0 \tag{A4}$$

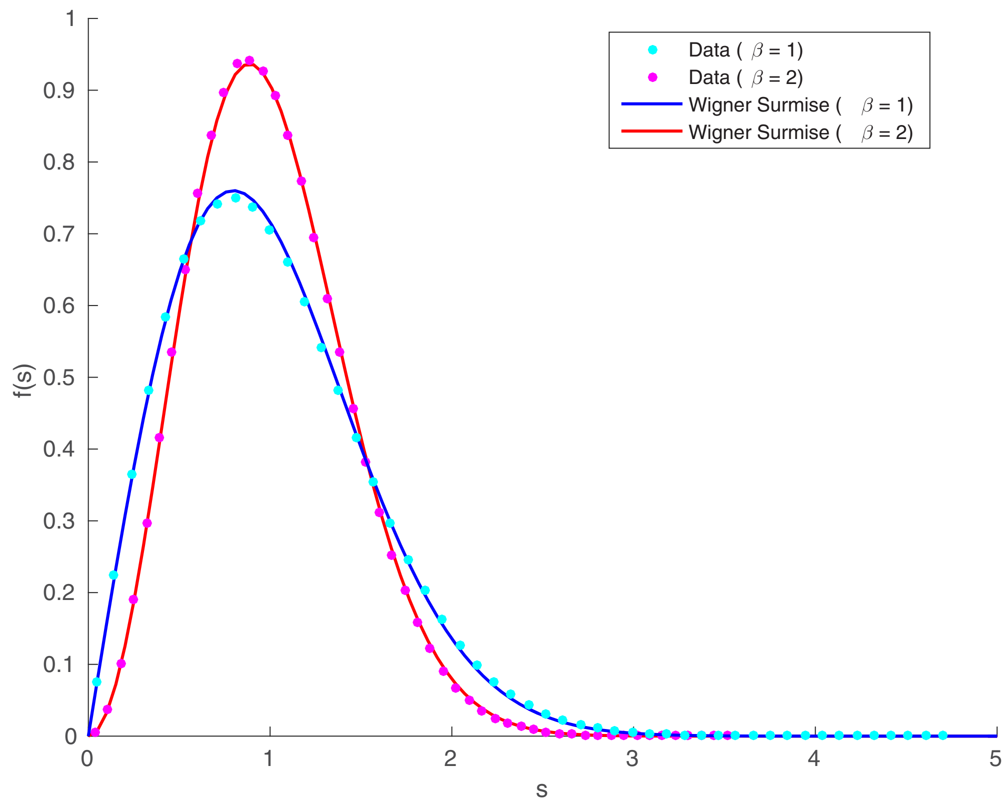

FIG. 4. Spacing distribution for 10^5 matrices of order $N = 32$. Circles denote the numerical data and the solid lines denote the Wigner surmise for $\beta = 1$ and $\beta = 2$.

and, therefore, $U^{\dagger}\eta V = \mathbf{1}$ and thus the Hermiticity of η implies that $V^{-1} = (\eta U)^{\dagger}$ or, alternatively,

$$U^{-1} = (\eta V)^{\dagger}. \tag{A5}$$

Now, to define the Jacobian of interest we must perform a change of variables that maps the elements of A into the N eigenvalues and another set of M variables which describe the remaining degrees of freedom. In other words,

$$P\left(\Gamma_j, p_\mu^{\{k\}}\right) = P(A)\det\left[\frac{\partial\left(a_{i,i}, a_{i,j}^{\{k\}}\right)}{\partial(\Gamma_j, p_\mu)}\right], \tag{A6}$$

where Γ_j are the eigenvalues, p_μ are the M additional variables needed by the constraints in the eigenfunctions, and the last term denotes the Jacobian of the transformation.

To obtain this Jacobian, we must consider first that

$$A = U\Lambda U^{-1}. \tag{A7}$$

Therefore

$$\frac{\partial}{\partial\Gamma_j}A_{\gamma,\xi} = \frac{\partial}{\partial\Gamma_j}[U\Lambda U^{-1}]_{\gamma,\xi} = \frac{\partial}{\partial\Gamma_j}\sum_{\delta,\epsilon}U_{\gamma,\delta}\Lambda_{\delta,\epsilon}U^{-1}_{\xi,\epsilon}$$

and since Λ is a diagonal matrix containing the eigenvalues of A, and noting that the remaining terms are of U and its inverse,

$$\frac{\partial A_{\gamma,\xi}}{\partial\Gamma_j} = \delta_{j,\gamma}\delta_{\gamma,\xi}. \tag{A8}$$

The inverse relation $U^{-1}U = \mathbf{1}$ gives us

$$\frac{\partial}{\partial p_\mu}U^{-1}U = \left(\frac{\partial}{\partial p_\mu}U^{-1}\right)U + U^{-1}\left(\frac{\partial}{\partial p_\mu}U\right) = \mathbf{0}$$
$$\to \left(\frac{\partial}{\partial p_\mu}U^{-1}\right)U = -U^{-1}\left(\frac{\partial}{\partial p_\mu}U\right) \equiv \Theta_\mu. \tag{A9}$$

This new matrix Θ_μ has a property of interest,

$$\Theta_\mu^{\dagger} = U^{\dagger}\left(\frac{\partial}{\partial p_\mu}U^{-1}\right)^{\dagger} = V^{-1}\left(\frac{\partial}{\partial p_\mu}V\right), \tag{A10}$$

which may be used in specific realizations of A to obtain further information about the change of variables.

Using the definition in the above Eq. (A9), multiplying (A1) from the left by U^{-1} and taking the derivative of the resulting equation by a parameter p_μ we obtain

$$U^{-1}\frac{\partial A}{\partial p_\mu}U = U^{-1}\left(\frac{\partial}{\partial p_\mu}U\right)\Lambda - \Lambda\left(\frac{\partial}{\partial p_\mu}U^{-1}\right)U = (\Theta_\mu\Lambda - \Lambda\Theta_\mu) \tag{A11}$$

or, in component terms,

$$\left(U^{-1}\frac{\partial A}{\partial p_\mu}U\right)_{\gamma,\xi} = (\Theta_\mu)_{\gamma,\xi}(\Gamma_\xi - \Gamma_\gamma). \tag{A12}$$

Therefore, instead of writing the Jacobian matrix directly, we follow Mehta's approach [1] and write the auxiliary matrix

$$\Omega = \begin{pmatrix} \left[U^{-1}\frac{\partial A}{\partial \Gamma_j}U\right]_{\xi,\xi} & \left[U^{-1}\frac{\partial A}{\partial \Gamma_j}U\right]_{\gamma,\xi} \\ \left[U^{-1}\frac{\partial A}{\partial p_\mu}U\right]_{\xi,\xi} & \left[U^{-1}\frac{\partial A}{\partial p_\mu}U\right]_{\gamma,\xi} \end{pmatrix} = \begin{pmatrix} \delta_{j,\gamma} & 0 \\ 0 & [\Theta_\mu]_{\gamma,\xi}(\Gamma_\xi - \Gamma_\gamma) \end{pmatrix}. \tag{A13}$$

The determinant of this matrix is then the Jacobian up to a function on the eigenvector parameters

$$\phi(p_\mu)\det\Omega = \det\left[\frac{\partial(a_{i,i},a_{i,j})}{\partial(\Gamma_j,p_\mu)}\right]. \tag{A14}$$

This is, therefore, a general expression that shows that the Jacobian factors as a product of differences, due to Ω, and a term dependent only in eigenvector terms, which may be integrated out of the probability distribution.

[1] M. L. Mehta, *Random Matrices*, 3rd ed., Pure and Applied Mathematics Vol. 142 (Academic, New York, 2004).
[2] M. V. Berry and J. P. Keating, in *H=xp and the Riemann Zeros*, edited by I. V. Lerner, J. P. Keating, and D. E. Khmelnitskii, Supersymmetry and Trace Formulae, Nato Science Series B, Vol. 370 (Springer, New York, 1999), p. 355.
[3] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians having PT Symmetry, Phys. Rev. Lett. **80**, 5243 (1998).
[4] C. M. Bender, S. Boettcher, and P. N. Meisinger, PT-symmetric quantum mechanics, J. Math. Phys. **40**, 2201 (1999).
[5] A. Mostafazadeh, Pseudo-hermitian representation of quantum mechanics, Int. J. Geom. Methods Mod. Phys. **07**, 1191 (2010).
[6] C. M. Bender, D. C. Brody, and M. P. Müller, Hamiltonian for the Zeros of the Riemann Zeta Function, Phys. Rev. Lett. **118**, 130201 (2017).
[7] A. Connes, Trace formula in noncommutative geometry and the zeros of the riemann zeta function, Selecta Math. **5**, 29 (1999).
[8] J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices, J. Math. Phys. **6**, 440 (1965).
[9] R. M. May, Will a large complex system be stable? Nature (London) **238**, 413 (1972).
[10] D. R. Nelson and N. M. Shnerb, Non-hermitian localization and population biology, Phys. Rev. E **58**, 1383 (1998).
[11] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, Spectrum of Large Random Asymmetric Matrices, Phys. Rev. Lett. **60**, 1895 (1988).
[12] R. Chaudhuri, A. Bernacchia, and X.-J. Wang, A diversity of localized timescales in network activity, eLife **3**, e01239 (2014).
[13] A. Amir, N. Hatano, and D. R. Nelson, Non-hermitian localization in biological networks, Phys. Rev. E **93**, 042310 (2016).
[14] R. Grobe, F. Haake, and H.-J. Sommers, Quantum Distinction of Regular and Chaotic Dissipative Motion, Phys. Rev. Lett. **61**, 1899 (1988).
[15] J. J. M. Verbaarschot and T. Wettig, Random matrix theory and chiral symmetry in QCD, Annu. Rev. Nucl. Part. Sci. **50**, 343 (2000).
[16] Y. V. Fyodorov and H.-J. Sommers, Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering: Random matrix approach for systems with broken time-reversal invariance, J. Math. Phys. **38**, 1918 (1997).
[17] N. Hatano and D. R. Nelson, Localization Transitions in Non-Hermitian Quantum Mechanics, Phys. Rev. Lett. **77**, 570 (1996).
[18] N. Hatano and D. R. Nelson, Non-hermitian delocalization and eigenfunctions, Phys. Rev. B **58**, 8384 (1998).
[19] F. L. Metz, I. Neri, and D. Bollé, Localization transition in symmetric random matrices, Phys. Rev. E **82**, 031135 (2010).
[20] Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers, Almost-hermitian random matrices: Eigenvalue density in the complex plane, Phys. Lett. **A226**, 46 (1997).
[21] Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers, Almost Hermitian Random Matrices: Crossover from Wigner-Dyson to Ginibre Eigenvalue Statistics, Phys. Rev. Lett. **79**, 557 (1997).
[22] J. Feinberg and A. Zee, Non-gaussian non-hermitian random matrix theory: Phase transition and addition formalism, Nucl. Phys. B **501**, 643 (1997).
[23] R. A. Janik, M. A. Nowak, G. Papp, J. Wambach, and I. Zahed, Non-hermitian random matrix models: Free random variable approach, Phys. Rev. E **55**, 4100 (1997).
[24] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Non-hermitian random matrix models, Nucl. Phys. B **501**, 603 (1997).
[25] J. Feinberg and A. Zee, Spectral curves of non-hermitian hamiltonians, Nucl. Phys. B **552**, 599 (1999).
[26] J. T. Chalker and B. Mehlig, Eigenvector Statistics in Non-Hermitian Random Matrix Ensembles, Phys. Rev. Lett. **81**, 3367 (1998).
[27] B. Mehlig and J. T. Chalker, Eigenvector correlations in non-hermitian random matrix ensembles, Ann. Phys. **7**, 427 (1998).
[28] B. Mehlig and J. T. Chalker, Statistical properties of eigenvectors in non-hermitian gaussian random matrix ensembles, J. Math. Phys. **41**, 3233 (2000).
[29] J. Feinberg, Non-hermitian random matrix theory: Summation of planar diagrams, the 'single-ring' theorem and the disc–annulus phase transition, J. Phys. A: Math. Gen. **39**, 10029 (2006).
[30] Y. N. Joglekar and W. A. Karr, Level density and level-spacing distributions of random, self-adjoint, non-hermitian matrices, Phys. Rev. E **83**, 031122 (2011).
[31] O. Bohigas, J. X. De Carvalho, and M. P. Pato, Structure of trajectories of complex-matrix eigenvalues in the hermitian–non-hermitian transition, Phys. Rev. E **86**, 031118 (2012).
[32] N. Hatano and J. Feinberg, Chebyshev-polynomial expansion of the localization length of hermitian and non-hermitian random chains, Phys. Rev. E **94**, 063305 (2016).
[33] G. Akemann, J. Baik, and P. Di Francesco, *The Oxford Handbook of Random Matrix Theory* (Oxford University Press, New York, 2011).
[34] Z. Ahmed and S. R. Jain, Pseudounitary symmetry and the gaussian pseudounitary ensemble of random matrices, Phys. Rev. E **67**, 045106 (2003).
[35] Z. Ahmed and S. R. Jain, Gaussian ensemble of 2 x 2 pseudo-hermitian random matrices, J. Phys. A: Math. Gen. **36**, 3349 (2003).

[36] Q. H. Wang, S. Z. Chia, and J. H. Zhang, $\mathcal{PT}$ symmetry as a generalization of hermiticity, J. Phys. A: Math. Theor. **43**, 295301 (2010).
[37] S. C. L. Srivastava and S. R. Jain, Pseudo-hermitian random matrix theory, Fortschr. Phys. **61**, 276 (2012).
[38] E.-M. Graefe, S. Mudute-Ndumbe, and M. Taylor, Random matrix ensembles for PT-symmetric systems, J. Phys. A: Math. Theor. **48**, 38FT02 (2015).
[39] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, J. Math. Phys. **43**, 5830 (2002).
[40] P. J. Forrester, *Log-Gases and Random Matrices*, London Mathematical Society Monographs No. 34 (Princeton University Press, Princeton, NJ, 2010).
[41] G. Marinello and M. P. Pato, Pseudo-Hermitian ensemble of random Gaussian matrices, Phys. Rev. E **94**, 012147 (2016).
[42] V. L. Girko, Spectral theory of random matrices, Russ. Math. Surv. **40**, 77 (1985).
[43] V. L. Girko, The strong elliptic law. Twenty years later. Part I, Random Oper. Stochastic Equations **14**, 59 (2006).
[44] A. Mostafazadeh, Pseudo-hermiticity versus PT-symmetry III: Equivalence of pseudo-hermiticity and the presence of antilinear symmetries, J. Math. Phys. **43**, 3944 (2002).
[45] U. Magnea, Random matrices beyond the cartan classification, J. Phys. A: Math. Theor. **41**, 045203 (2008).
[46] A. M. Odlyzko, On the distribution of spacings between zeros of the zeta function, Math. Comput. **48**, 273 (1987).
[47] O. Bohigas and M. P. Pato, Decomposition of spectral density in individual eigenvalue contributions, J. Phys. A: Math. Theor. **43**, 365001 (2010).
[48] G. Szego, *Orthogonal Polynomials* (American Mathematical Society, Providence, Rhode Island, 1939).
[49] V. A. Marčenko and L. A. Pastur, Distribution of eigenvalues for some sets of random matrices, Math. USSR-Sbornik **1**, 457 (1967).
[50] U. Haagerup and S. Thorbjørnsen, Random matrices with complex gaussian entries, Expo. Math. **21**, 293 (2003).