Parallel random target searches in a confined space

Sunghan Ro and Yong Woon Kim*

Graduate School of Nanoscience and Technology, Korea Advanced Institute of Science and Technology, Deajeon 34141, Korea

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We study a random target searching performed by N independent searchers in a d-dimensional domain of a large but finite volume. Considering the two initial distributions of searchers where searchers are either uniformly or point distributed, we estimate the mean time for the first of the searchers to reach the target and refer to it as searching time. The searching time for the uniformly distributed searchers exhibits a universal power-law dependence on N, irrespective of dimensionality and the target-to-domain size ratio. For point-distributed searching, the searching time has a logarithmic dependence on N in the large N limit, while in the small N limit, it shows qualitatively different behaviors depending upon r_0, the initial distance of the searchers from a target. We obtain a diagram by comparing the searching times of the two initial distributions in the parameter space (r_0, N) and therein present the asymptotic lines separating three characteristic regions to explain numerical simulation results.

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I. INTRODUCTION

First-passage dynamics [1,2] provides the fundamental understanding of many physical phenomena, ranging from diffusion-limited reactions of molecules [3–7] to genetic drift in a population of organisms occurring over generations [8]. In recent years, random target searching (RTS) has received a great deal of attention in the context of first-passage dynamics [9–19], which includes abundant examples such as animal foraging, chemical reactions, and searching for a lost key or a missing child without a clue and mental map. In RTS, central quantity is the first-passage time (FPT), which is a measure of how long it takes for a random walker or a diffusing particle to first encounter a target [20–22]. The issue involves various factors such as confinement effect [11], the dependence of target location [11], mortality of searchers [19], mobility of targets [6], and size and sequence of random walk [23].

When a group of N random walkers participates in the target searching, the problem becomes more complicated than in the presence of only a single searcher. Analyzing the smallest value among FPTs recorded by N searchers is important in this case, and it requires order statistics that can be obtained through the full distribution of FPT. The order statistics has been extensively studied in one-dimensional space [10,19,24–26] in which the FPT distribution is available. In two or higher dimensions, only approximate behaviors of FPT distribution for a single searcher to reach a small target are known [27,28], and the multisearcher problem has been explored only in limited cases [9,14–16]. For instance, an asymptotic expression for the moments of survival time of a target was found in the large N limit when all searchers start from the same origin [14], and the mean searching time, also referred to as the mean target lifetime, was investigated numerically [15]. Hence the current understanding is far from complete, and it is still to be examined how the mean searching time depends on the number and initial distribution of the searchers for a wider range of N. In answering these questions, it is reasonable to suppose a searching domain of a finite size, which is relevant to most of the realistic situations: for example, chemical reactions of molecules usually occur in a vessel, and animals are foraging within their living territories.

In this work, we consider parallel searching of a target by N independent diffusing particles (called searchers) in d-dimensional ($d = 1,2,3$) confined space. Here the volume of the space is assumed much larger than the volume of the target. We estimate the searching time as the mean searching time of the searcher that arrives at the target first among the N searchers. In the case of parallel searching by multiple searchers, it is of great interest to study how the searching time depends on the initial distribution of searchers. Modulating the initial distributions of the searchers, one can in principle think of infinitely many different searching strategies. Here we focus on the two limiting searching strategies as possibly the simplest and natural choices: the initial distribution of searchers is uniform (uniformly distributed parallel searching, UPS), or the searchers start their random motion at the same point (point-distributed parallel searching, PPS). These strategies for the two extremes of possible initial distributions of searchers provide a useful starting point for the understanding of searching times of arbitrary parallel searching strategies.

The main results of this study are first the analytical behavior of the searching times. Using the dominance of the lowest eigenvalue for the small N regime and mapping onto the one-dimensional searching for the large N regime, we obtain asymptotic expressions for the searching times. For UPS, the searching time shows a universal behavior: Regardless of dimensionality and ratio of target-to-domain size, the searching time algebraically decays with increasing N. Interestingly, there is a crossover of the decay exponent, 1 for small N and 2 for large N. For PPS, the searching time for small N has a strong dependence on the initial target-searcher distance, r_0. In the large N limit, the first-passage trajectories resemble random walks in one dimension, and the searching time has a weak logarithmic dependence on N in any dimension. We also perform the Langevin dynamics simulations and find the results agree well with the analytic predictions. Finally, by comparing the searching times, we obtain a diagram in the parameter space of r_0 and N in which three regions are found to indicate which strategy is more efficient, or whether both are similar. The crossover loci determined from the asymptotic behaviors are observed to be in qualitative agreement with simulation results.

*Corresponding author: y.w.kim@kaist.ac.kr

This paper is organized as follows: In Sec. II we describe the problem to be considered in this work and introduce an effective way to calculate the mean minimum FPT, based on the lowest eigenvalue dominance in the survival probability. In Sec. III we derive the analytic expressions of the searching time for UPS and compare them with numerical simulations. In Sec. IV the searching time of PPS is estimated analytically in the asymptotic limits and also numerically. In Sec. V we discuss which strategy performs better by comparing the searching times and present a schematic diagram indicating a better strategy, plotted in the space spanned by r_0 and N. A summary follows in Sec. VI.

II. SYSTEM

Let us first consider a random target searching by a single searcher in a confined domain. More specifically, we model the random trajectories of a searcher by a Brownian motion of a diffusing particle in a d-dimensional sphere of a volume V (of a radius b), and the searcher performs a diffusive motion until it reaches a target of size a at specified position $\mathbf{r}_T$. The probability distribution of a searcher obeys the following diffusion equation:

$$\frac{\partial p(\mathbf{r},t;\mathbf{r}_0)}{\partial t} = D\nabla^2 p(\mathbf{r},t;\mathbf{r}_0)\,, \tag{1}$$

where $p(\mathbf{r},t;\mathbf{r}_0)$ is the probability density that a searcher is found in an infinitesimal volume element $d^d\mathbf{r}$ located at position $\mathbf{r}$ and time t for a given initial position $\mathbf{r}_0$. Here, D denotes the diffusion constant, which is homogeneous over the space. The presence of a target is considered as an absorbing boundary condition, $p(\mathbf{r}\in\partial\mathcal{T},t;\mathbf{r}_0)=0$ at the target ($\mathcal{T}$), and we also impose a reflecting boundary condition at the boundary of the searching domain ($\mathcal{D}$) as $\hat{n}\cdot\nabla p(\mathbf{r}\in\partial\mathcal{D},t;\mathbf{r}_0)=0$, with $\hat{n}$ being a unit outward normal at the target boundary. Throughout this work, we only consider a large volume limit [or equivalently, a small target limit, $(a/b)^d \ll 1$].

Solving Eq. (1) and getting $p(\mathbf{r},t;\mathbf{r}_0)$, we can estimate the probability that the searcher is not absorbed on the target surface until time t when the random searcher starts at $\mathbf{r}_0$ initially ($t=0$):

$$S(t;\mathbf{r}_0) = \int_{\mathcal{D}\setminus\mathcal{T}} d^d\mathbf{r}\, p(\mathbf{r},t;\mathbf{r}_0), \tag{2}$$

where the integration is performed over the space unoccupied by the target, $\mathcal{D}^* = \mathcal{D}\setminus\mathcal{T}$. This quantity, also known as the target survival probability, is related to the probability distribution of FPT, $F(t;\mathbf{r}_0)$, as

$$S(t;\mathbf{r}_0) = \int_t^\infty dt'\, F(t';\mathbf{r}_0)\,, \tag{3}$$

provided that the target is certainly found in the limit $t\to\infty$, i.e., $\lim_{t\to\infty} S(t;\mathbf{r}_0)=0$. Here $F(t;\mathbf{r}_0)$ is the probability that a single searcher arrives at a target site for the first time at a specified time t, and the mean first-passage time is given by $\overline{t}(\mathbf{r}_0) \equiv \int dt\, t\, F(t;\mathbf{r}_0)$, which depends on the initial position of the single searcher. If averaging $\overline{t}(\mathbf{r}_0)$ over distribution of $\mathbf{r}_0$, one obtains the global mean first-passage time, τ.

Now we consider the parallel searching process where N searchers are performing independent random searches; they neither physically interact nor communicate with one another. Each searcher first reaches the target site and records its FPT. We then have N random FPTs, say, $(t^{(1)},t^{(2)},\ldots,t^{(N)})$. The minimum first-passage time, $t_{\min}\equiv\min[t^{(1)},t^{(2)},\ldots,t^{(N)}]$, should satisfy the probability distribution of minimum FPT, F_N, given as [24]

$$F_N\big(t_{\min};\{\mathbf{r}_0^{(i)}\}\big) = \sum_{i=1}^N F\big(t_{\min};\mathbf{r}_0^{(i)}\big)\prod_{j\neq i}^N S\big(t_{\min};\mathbf{r}_0^{(j)}\big)\,, \tag{4}$$

where $\{\mathbf{r}_0^{(i)}\} = (\mathbf{r}^{(1)},\mathbf{r}^{(2)},\ldots,\mathbf{r}^{(N)})$ are the initial positions of N searchers. This expression can be easily understood by looking at, for example, the summand given by $i=1$. It amounts to the probability that the first passage event is hosted by the first searcher with the probability $F(t_{\min};\mathbf{r}_0^{(1)})$, while other searchers do not yet locate the target, as signified by the product of the target survival probability, $S(t_{\min};\mathbf{r}_0^{(j)})$. When the initial distribution of N searchers is given as $p_0(\{\mathbf{r}_0^{(i)}\})$, the average of minimum FPT for N-parallel searching, $\langle t_N\rangle$, is obtained by [15]

$$\langle t_N\rangle = \prod_{i=1}^N \int_{\mathcal{D}^*} d^d\mathbf{r}_0^{(i)} \int_0^\infty dt\, t\, F_N\big(t;\{\mathbf{r}_0^{(i)}\}\big)\, p_0\big(\{\mathbf{r}_0^{(i)}\}\big). \tag{5}$$

Let us introduce the quantities of our interest to compare the two opposite searching strategies mentioned in the Introduction. For PPS, we suppose that all searchers are introduced into the searching space at the same position $\mathbf{r}_0$, i.e., $p_0(\{\mathbf{r}_0^{(i)}\}) = \prod_i \delta(\mathbf{r}_0^{(i)}-\mathbf{r}_0)$. The average of searching time, defined as minimum FPT, $\langle t_{p,N}\rangle$, is determined by Eq. (5) as

$$\begin{aligned}\langle t_{p,N}\rangle &= \int_0^\infty dt\, t\, F_N(t;\mathbf{r}_0)\\ &= N\int_0^\infty dt\, t\, F(t;\mathbf{r}_0)[S(t;\mathbf{r}_0)]^{N-1},\end{aligned} \tag{6}$$

which upon using Eq. (3) and integrating by parts becomes

$$\langle t_{p,N}\rangle = \int_0^\infty dt\, [S(t;\mathbf{r}_0)]^N. \tag{7}$$

As a second parallel searching strategy, we consider the situation where the initial distribution of searchers is uniform as $p_0(\{\mathbf{r}_0^{(i)}\}) = (1/V^*)^N$, with V^* being the volume of the entire domain, except for the region occupied by the target. The average of minimum FPT, $\langle t_{u,N}\rangle$, for UPS can be obtained from Eq. (5) as

$$\langle t_{u,N}\rangle = \prod_i \int_{\mathcal{D}^*}\left(\frac{d^d\mathbf{r}_0^{(i)}}{V^*}\right)\int_0^\infty dt\, t\, F_N\big(t;\{\mathbf{r}_0^{(i)}\}\big). \tag{8}$$

Using Eqs. (3) and (4) and again integrating Eq. (8) by parts, we obtain an expression for the average minimum FPT,

$$\langle t_{u,N}\rangle = \int_0^\infty dt\, \big[\overline{S}(t)\big]^N, \tag{9}$$

where $\overline{S}(t)$ is the *global* survival probability, the survival probability averaged over all possible initial positions:

$$\overline{S}(t) = \frac{1}{V^*}\int_{\mathcal{D}^*} d^d\mathbf{r}_0\, S(t;\mathbf{r}_0). \tag{10}$$

The searching times, Eqs. (7) and (9), are the key quantities of our analysis on the parallel searching strategy, and we are going to scrutinize their dependence on N, initial location, and dimensionality. For the sake of analytic treatment, it is necessary to obtain the explicit expression of the survival probability, an essential quantity to determine the searching times. However, since it cannot be solved exactly, we must consider an approximation to effectively describe the behavior of the survival probability.

In general, the probability distribution function $p(\mathbf{r},t;\mathbf{r}_0)$, the solution to the diffusion equation (1), is expressed in terms of the eigenfunction expansion [29],

$$p(\mathbf{r},t;\mathbf{r}_0) = \sum_{n=1}^{\infty} \phi_n^*(\mathbf{r})\phi_n(\mathbf{r}_0)e^{-\lambda_n D t}, \tag{11}$$

where $\phi_n(\mathbf{r})$ and λ_n are the nth eigenfunction and eigenvalue of the Laplace operator, respectively. Each eigenmode exponentially decays in time t with respective decay time $1/(\lambda_n D)$. The survival probability $S(t;\mathbf{r}_0)$ then reads as

$$\begin{aligned} S(t;\mathbf{r}_0) &= \sum_{n=1}^{\infty} \int_{\mathcal{D}^*} d^d\mathbf{r}\phi_n^*(\mathbf{r})\phi_n(\mathbf{r}_0)e^{-\lambda_n D t} \\ &\equiv (1-q(\mathbf{r}_0))e^{-\lambda_1 D t} + q(\mathbf{r}_0)f(t;\mathbf{r}_0), \end{aligned} \tag{12}$$

where λ_1 is the smallest eigenvalue, $\lambda_1 < \lambda_n (n \geqslant 2)$. In the above, terms are separated into the slowest decaying one with λ_1 and the other terms decaying faster with $\lambda_n (n \geqslant 2)$. They have the respective statistical weights of $1 - q(\mathbf{r}_0)$ and $q(\mathbf{r}_0)$. In the long-time limit, the eigenmode with the longest decay time characterizes the diffusive dynamics, and the asymptotic survival probability decays as a single exponential function of time. For a random walk on dimensions greater than one ($d \geqslant 2$), the lowest eigenvalue gives the global mean first-passage time τ of a single searcher as $\tau = 1/(D\lambda_1)$ [20], leading to the following expression of the survival probability:

$$S(t;\mathbf{r}_0) = [1-q(\mathbf{r}_0)]e^{-t/\tau} + q(\mathbf{r}_0)f(t;\mathbf{r}_0). \tag{13}$$

For $d \geqslant 2$, this approximation, based on a time-scale separation of the searching process, is useful and gives results consistent with previous studies. For example, in Ref. [27] it was found that in the large volume limit, the first-passage time distribution shows a universal behavior such as

$$F(t;\mathbf{r}_0) = \Pi(\mathbf{r}_0)e^{-t/\tau} + [1-\Pi(\mathbf{r}_0)]\delta\left(\frac{t}{\tau}\right), \tag{14}$$

where the probability distribution is well described at the time scales of τ by a long-time tail given by a single exponential decay and by a short-time δ function. The so-called *geometric factor* $\Pi(\mathbf{r}_0)$ represents the statistical weight of trajectories touching the domain boundary before reaching the target. Since the FPT distribution $F(t;\mathbf{r}_0)$ is related to the survival probability through Eq. (3), the factor of $1 - q(\mathbf{r}_0)$ represents the statistical weight of the same trajectories in the survival probability and $f(t;\mathbf{r}_0)$ is given by a step function.

Using the results presented above, we will put forward analytic formulations for minimum FPT, asymptotically valid in the limiting cases of small or large N. For quantitative analysis we will also perform numerical simulations in the following sections, which explain well the analytic results. From now on, we let the target position be at the origin, i.e., $\mathbf{r}_T = 0$.

III. PARALLEL SEARCHING BY UNIFORMLY DISTRIBUTED SEARCHES

Consider a parallel search with N searchers uniformly distributed over the searching space at $t = 0$. For analytic treatments, we consider the asymptotic behaviors of survival probability and averaged minimum FPT in the small or large N limit, separately.

In the small N limit [$N \approx \mathcal{O}(1)$], the searching events are expected to occur at the time scale comparable to τ, the global mean FPT of a single searcher. For $d \geqslant 2$, all searchers are expected to touch domain boundary before arriving at the target and thus neglect the short-time contribution given by the step function, $f(t;\mathbf{r}_0)$, by taking $q(\mathbf{r}_0) \to 0$ in Eq. (13). This allows an approximation,

$$\overline{S}(t) = \frac{1}{V^*}\int_{\mathcal{D}_*} d\mathbf{r}_0\, S(t;r_0) \approx e^{-t/\tau}, \tag{15}$$

with $\tau = 1/(D\lambda_1)$, which leads to the minimum FPT for UPS at the small N regime (for a more detailed derivation, see Appendix A):

$$\langle t_{u,<}\rangle = \int_0^{\infty} dt\big[\overline{S}(t)\big]^N \simeq \frac{\tau}{N}. \tag{16}$$

Using the expressions for the global mean first-passage time τ in terms of the system parameters given in Ref. [20], the searching time [Eq. (16)] is explicitly written for each dimension as

$$\langle t_{u,<}\rangle \simeq \begin{cases} \frac{V}{2\pi DN}\ln\left(\frac{b}{a}\right) & \text{for} \quad d=2 \\ \frac{V}{4\pi DaN} & \text{for} \quad d=3\,, \end{cases} \tag{17}$$

where V is the volume of searching domain in the respective dimension, e.g., $V = \pi b^2$ for $d = 2$ with radius b.

For one-dimensional searching, unlike the case of $d \geqslant 2$, the condition that all searchers touch the domain boundary before reaching the target is not valid, even if the target size is negligible compared to the domain volume. We should use the whole series of Eq. (11), instead of using only the lowest eigenvalue term, in order to evaluate global mean FPT for $d = 1$. Locating the target at the origin of the searching domain, we obtain $\tau = (2b)^2/12D$, where $2b$ is the length of the one-dimensional searching space. Inspired by the Eq. (16), we suggest an approximate expression for the searching time in the small N limit as

$$\langle t_{u,<}\rangle \approx \frac{b^2}{3DN} \quad \text{for} \quad d=1, \tag{18}$$

which is exact when $N = 1$ and is supposed to give a reasonable estimation of the searching time if $N \approx \mathcal{O}(1)$. The validity of Eq. (18) for one-dimensional searching will be confirmed shortly through numerical simulations.

On the other hand, in the large $N (\gg 1)$ limit, the searching time is much smaller than τ, and the previous approximation, Eq. (15), neglecting the short-time contribution is no longer valid even for $d \geqslant 2$. Furthermore, when $N \to \infty$, there are always a finite number of searchers very close to the target and almost instantaneously absorbed, in which only short trajectories directly connecting the initial searcher positions to the target, without touching a boundary, are dominant. As a consequence, the short-time contribution to the survival probability becomes significant, and the first-passage trajectories must resemble random walks in one dimension. Based on this argument, we attempt to describe the searching dynamics for large N in an arbitrary dimension as the first-passage problem in one dimension. When searchers are uniformly distributed in an infinite one-dimensional space with a given density $\rho \equiv N/V$, the global survival probability in Eq. (10) in the presence of N searchers is approximated by the survival probability of a target [30]

$$[\overline{S}(t)]^N = \exp\left(-4\rho\sqrt{\frac{Dt}{\pi}}\right) \tag{19}$$

for the target presenting an absorbing boundary at the center. In obtaining Eq. (19), we used the survival probability in an infinite space, which is a valid approach despite the finite searching domain considered in our work, because the contribution of boundary-touching trajectories is negligible. Using Eq. (9), we obtain the minimum FPT in one dimension for large $N \gg 1$,

$$\langle t_{u,>}\rangle = \frac{\pi}{8D\rho^2}\,, \tag{20}$$

which is in agreement with the result of Ref. [24]. For other dimensions, ρ in Eq. (20) should be replaced with the effective one-dimensional density of searchers near the target when $N \to \infty$. The total number of searchers absorbed through the target boundary per unit time is given by integrating the flux over the target surface as $-2D(\partial\rho/\partial x)_{\partial\mathcal{T}}$ for $d = 1$ and $-AD(\partial\rho/\partial\hat{n})_{\partial\mathcal{T}}$ for $d = 2,3$, respectively. Here $\hat{n}$ is a unit outward normal at the target boundary and A is the target surface area in d dimension, e.g., $A = 2\pi a$ for $d = 2$. Assuming a uniform initial distribution, this defines the effective one-dimensional density as $\rho = (N/V)(A/2)$. The factor of 2 appears because we consider the situation in Eq. (19) where the target is at the origin and the searchers can be absorbed from both sides. According to Eq. (20), the searching time is now expressed more explicitly for $N \gg 1$ as

$$\langle t_{u,>}\rangle \approx \begin{cases} \frac{\pi}{2D}\left(\frac{b}{N}\right)^2 & \text{for} \quad d = 1 \\ \frac{\pi}{8D}\left(\frac{b}{a}\right)^2\left(\frac{b}{N}\right)^2 & \text{for} \quad d = 2 \\ \frac{\pi}{18D}\left(\frac{b}{a}\right)^4\left(\frac{b}{N}\right)^2 & \text{for} \quad d = 3\,. \end{cases} \tag{21}$$

Comparing $\langle t_{u,<}\rangle$ and $\langle t_{u,>}\rangle$ in Eqs. (17) and (21), we can estimate the number of searchers where the N dependence of average searching time is changed from the small N behavior ($\sim 1/N$) into the large N behavior ($\sim 1/N^2$). This crossover number N_1 for the uniform searcher distribution is given by

$$N_1 = \begin{cases} \frac{3\pi}{2} & \text{for} \quad d = 1 \\ \frac{\pi}{4}\left(\frac{b}{a}\right)^2\frac{1}{\ln(b/a)} & \text{for} \quad d = 2 \\ \frac{\pi}{6}\left(\frac{b}{a}\right)^3 & \text{for} \quad d = 3\,. \end{cases} \tag{22}$$

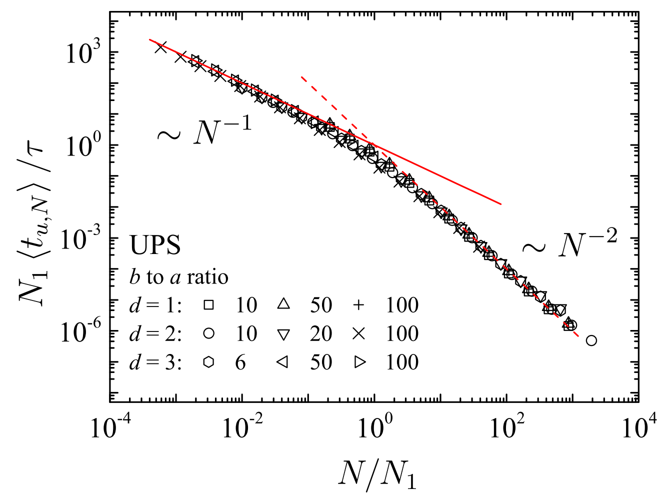

FIG. 1. The searching times, defined as mean minimum FPT, for UPS as a function of rescaled searcher number, N/N_1. Symbols show the simulation data for various ratios of domain-to-target size, b/a, in d dimension ($d = 1,2,3$). All the simulation data coalesce to a single curve, displaying the universal behaviors in the small (solid) or large (dotted line) N limit given by Eq. (23).

For the estimation of N_1, we assumed that for the small N regime, $\langle t_{u,<}\rangle$ follows τ/N even in the case of $d = 1$, with the assumption of sharp time-scale separation, i.e., a single exponential approximation for the survival probability is not strictly valid in compact random walks. For $d \geqslant 2$, the crossover number N_1 roughly corresponds to the number of lattices when the entire searching volume is divided by a unit cell with the target volume. In one dimension, $N_1 \approx \mathcal{O}(1)$, and the large N regime is realized for relatively small $N(> N_1)$. Interestingly, when expressed in terms of dimensionless variables that are normalized by τ and N_1, the mean searching time $\langle t_{u,N}\rangle$ shows a universal behavior, irrespective of dimensionality, as follows:

$$N_1\frac{\langle t_{u,N}\rangle}{\tau} \approx \begin{cases} \frac{N_1}{N} & \text{for} \quad N < N_1 \\ \left(\frac{N_1}{N}\right)^2 & \text{for} \quad N > N_1\,. \end{cases} \tag{23}$$

In Fig. 1 we present the numerical results from Langevin dynamics simulations where the average of minimum FPT is measured in a d-dimensional sphere with a target at the origin. The simulations are performed for different values of the ratio of the target-to-domain size in various dimensions ($d = 1,2,3$). When rescaled by N_1 and τ, the simulation data for the searching time all fall on a single curve, clearly demonstrating the universal behavior of Eq. (23). When N is much smaller than the threshold value N_1, the searching time is indeed inversely proportional to N. As N increases, the searching time shows crossover to $1/N^2$ behavior around $N \simeq N_1$, as it is described by an effective one-dimensional searching of the searchers close to the target. Despite the absence of the clear time-scale separation, the numerical results obtained for one-dimensional random searching show a good agreement with the universal behavior, which motivated us to introduce the approximate expression of Eq. (18) even for the one-dimensional case.

IV. PARALLEL SEARCHING BY POINT-DISTRIBUTED SEARCHERS

Now we consider the opposite searching strategy where all searchers start their random searching from the same position $\mathbf{r}_0$. The minimum FPT of PPS is then described by Eq. (7) using the survival probability.

In the small N limit, we consider the approximate expression of the survival probability $S(t;\mathbf{r}_0)$, Eq. (13), and neglect the short-time contribution $f(t;\mathbf{r}_0)$ to $S(t;\mathbf{r}_0)$ when the target size is sufficiently small compared to the size of the searching domain, which is valid for $d \geqslant 2$. For the case of PPS, the survival probability has a strong initial position dependence through the factor of $1-q(\mathbf{r}_0)$ in Eq. (13) because the trajectories of all the searchers start at the same initial position. In fact, the factor $q(\mathbf{r}_0)$ has asymptotic behaviors, $q(\mathbf{r}_0) \to 1$ for $r_0 \to a$, and $q(\mathbf{r}_0) \to 0$ for $r_0 \gg a$, which will be explicitly shown later. Here, $r_0 = |\mathbf{r}_0 - \mathbf{r}_T| = |\mathbf{r}_0|$. Due to the asymptotic behaviors of $q(\mathbf{r}_0)$, it is reasonable to conjecture the survival probability for the two limiting cases as

$$S^N(t;\mathbf{r}_0) \approx \begin{cases} e^{-Nt/\tau} & \text{for} \quad r_0 \gg a \\ [1-q(\mathbf{r}_0)]^N e^{-Nt/\tau} & \text{for} \quad r_0 \to a. \end{cases} \tag{24}$$

This suggests that depending upon the quality of the initial guess, i.e., how close the initial searcher position is to the target location, the mean searching time shows two qualitatively different N dependences (for a more detailed derivation, see Appendix A):

$$\langle t_{p,<}\rangle \simeq \begin{cases} \langle t_{u,<}\rangle & \text{for} \quad r_0 \gg a \\ (1-q(\mathbf{r}_0))^N \langle t_{u,<}\rangle & \text{for} \quad r_0 \to a\,. \end{cases} \tag{25}$$

We note that the above expression is derived by assuming the dominance of the lowest eigenvalue in the survival probability, which is valid only in dimensions of $d \geqslant 2$.

In the large N limit, we first consider the one-dimensional case. When N searchers are initially located at r_0 away from the target in an infinite one-dimensional space, the average minimum FPT is given as [19]

$$\langle t_{p,>}\rangle = \int_0^\infty dt\, \mathrm{erf}^N\left(\frac{r_0}{\sqrt{4Dt}}\right). \tag{26}$$

It can further be approximated for large N as [19]

$$\langle t_{p,>}\rangle \simeq \frac{\tau_D}{\ln N}, \tag{27}$$

where $\tau_D \equiv r_0^2/4D$ is a characteristic diffusion time, which is defined as a time scale for a particle to diffuse over the initial target-searcher distance, r_0. For the three-dimensional case, an explicit form of $f(t;\mathbf{r}_0)$ based on a pseudopotential approximation by Isaacson and Newby [18] is given in the limit of small target size as

$$\begin{aligned} f(t;\mathbf{r}_0) &\simeq 1 - \frac{D}{H(\mathbf{0}|\mathbf{r}_0)}\int_0^\infty dt' K(\mathbf{0},t'|\mathbf{r}_0,0) \\ &= \mathrm{erf}\left(\frac{r_0}{\sqrt{4Dt}}\right), \end{aligned} \tag{28}$$

where the free space form of the propagator is given by $K(\mathbf{0},t'|\mathbf{r}_0,0) = \exp[-r_0^2/(4Dt)]/(4\pi Dt)^{3/2}$, and the pseudo-Green's function $H(\mathbf{r}|\mathbf{r}_0)$ satisfies

$$-\nabla^2 H(\mathbf{r}|\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0) - \frac{1}{V}. \tag{29}$$

In the large N regime, the survival probability is approximated as

$$S(t;\mathbf{r}_0) \approx 1 - q(\mathbf{r}_0)\,\mathrm{erfc}(r_0/\sqrt{4Dt}), \tag{30}$$

and we obtain (details are shown in Appendix B)

$$\langle t_{p,>}\rangle \approx \frac{\tau_D}{\ln[Nq(\mathbf{r}_0)]}, \tag{31}$$

which with $Nq(\mathbf{r}_0)$ replaced by N turns out to be identical to the searching time in one dimension, Eq. (27). Consistently with our previous conjecture, this result suggests that at the large N regime, the first-passage trajectories are described by the one-dimensional random walk, except that the effective number of searchers is renormalized into $Nq(\mathbf{r}_0)$. We regard this weak logarithmic N dependence of searching time as a common feature of PPS for $N \gg 1$ and assume that the searching time in two dimension follows the same equation, Eq. (31). A similar result was obtained for the escaping problem of $N(\gg 1)$ random walkers starting from the same place: the mean escape time that it takes one of the random walkers to touch the boundary first is given by Eq. (27) up to leading order, inversely proportional to $\ln N$, showing the one-dimensional feature [31–34]. This similarity can be understood by considering that in the large N limit, the minimum FPT is determined by the shortest path connecting the initial searcher position and the absorbing boundary, irrespective of whether the absorbing boundary is a pointlike target or multidimensional hypersphere boundary.

The explicit form of $q(\mathbf{r}_0)$ can be obtained from the approximation adopted by Condamin *et al.* [35], in which the authors approximate the first-passage time density $F(t;\mathbf{r}_0)$ as a summation of a δ function and an exponential function. Evaluating the survival probability, Eq. (3), by the use of Eq. (53) of Ref. [35] and comparing the result with Eq. (13), we obtain

$$q(\mathbf{r}_0) = \frac{H(\mathbf{r}_T|\mathbf{r}_0)}{G_0(a) + H^*(\mathbf{r}_T|\mathbf{r}_T)}, \tag{32}$$

while $G_0(r)$ is Green's function in free space, and $H^*(\mathbf{r}|\mathbf{r}_0) = H(\mathbf{r}|\mathbf{r}_0) - G_0(|\mathbf{r}-\mathbf{r}_0)$ is the regular part of the pseudo-Green's function. The vector $\mathbf{r}_T$ refers the target position, which is the origin for our case. Especially for the spherical space with a small target of our interest, we get

$$q(\mathbf{r}_0) \simeq \begin{cases} \ln(b/r_0)/\ln(b/a) & \text{for} \quad d=2 \\ a/r_0 & \text{for} \quad d=3. \end{cases} \tag{33}$$

The validity of decomposition of $F(t;\mathbf{r}_0)$ into a short-time δ function and a long-time exponential function depends on the dimensionality. In one dimension, all the eigenvalues contribute to the probability distribution in Eq. (11) and the approximation of clear time-scale separation becomes poor, which makes it difficult to define $q(\mathbf{r}_0)$ for $d=1$ properly.

In Fig. 2, the simulation results are compared with analytic predictions. If the initial starting position of searchers is far distant from the target [see Fig. 2(a) for $a \ll r_0 \approx b$], the searching time shows the same behavior with UPS ($\langle t_{p,<}\rangle \simeq$

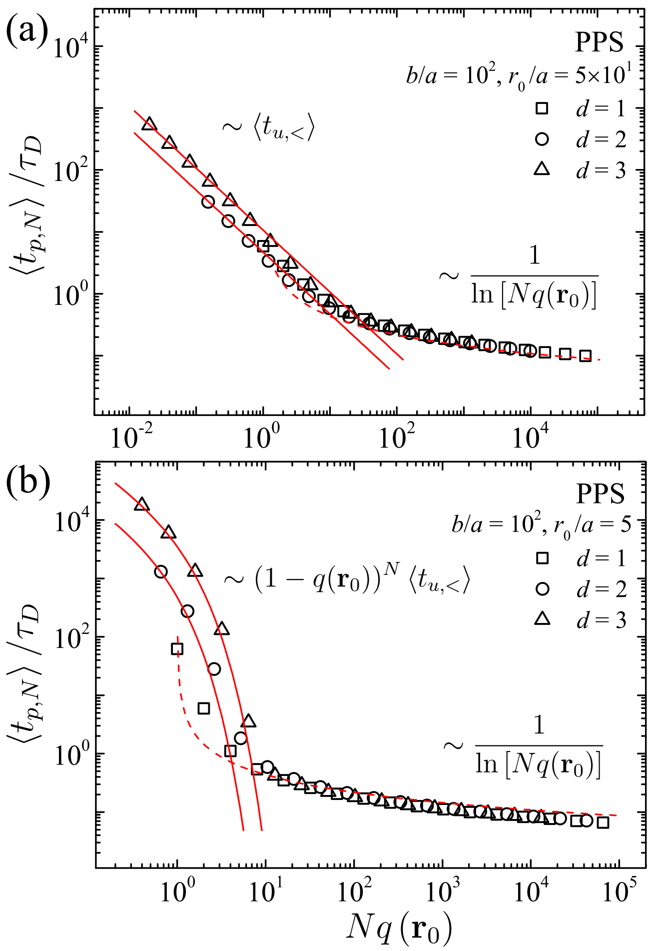

FIG. 2. The minimum searching times of parallel searching performed by point-distributed searchers as functions of (rescaled) searcher number $Nq(\mathbf{r}_0)$ for $d = 2,3$ and N for $d = 1$: (a) when the initial position of the searcher is far from the target, $a \ll r_0 \approx b$, and (b) when the searchers are initially close to the target, $a \approx r_0 \ll b$. Simulation data (symbols) are in good agreement with the asymptotic behaviors given by Eq. (25) for the small N limit (solid lines) and Eq. (31) for the large N limit (dotted lines). The theoretical (solid) lines for small N are presented only for $d = 2,3$, because Eq. (25) is valid for $d \geqslant 2$ (see the text).

$\langle t_{u,<}\rangle$) at small N, but as N increases, the unique feature of PPS manifests itself as the logarithmic dependence on N. If the starting position of searchers happens to be close to the target [see Fig. 2(b) for $a \approx r_0 \ll b$], the minimum first-passage time has a strong initial position dependence, $\langle t_{p,<}\rangle \simeq [1 - q(\mathbf{r}_0)]^N \langle t_{u,<}\rangle$, at small N, before entering into the weak logarithmic N dependence at the large N regime. As mentioned, $q(\mathbf{r}_0)$ cannot be defined for $d = 1$ in the same way as for higher dimensions. However, in order to emphasize the universal feature of logarithmic N dependence for large N [compare Eq. (27) and Eq. (31)], we present the simulation data for all dimensions together using a common x axis, i.e., as a function of $Nq(\mathbf{r}_0)$, which should be read as N for $d = 1$. It is interesting to notice that for the large N regime, the simulation data for $\langle t_{p,N}\rangle$ indeed collapse on a single curve and show the universal behavior, regardless of dimensionality.

V. DISCUSSION

Using the searching times, defined as the mean minimum FPTs, obtained in the previous sections, we compare the two parallel searching strategies for a given number of searchers and initial searcher-target distance and discuss which one outperforms in terms of the searching time. In this section, we focus our discussion on searching in two or three dimensions, relevant to most of the practical applications of random target searching. Let us first summarize the results obtained in the previous sections: As N increases, the searching time of UPS shows a crossover from $1/N$ to $1/N^2$ behavior [Eq. (23)]. The crossover occurs at N_1 given by Eq. (22). On the other hand, as shown in Eq. (25), the searching time of PPS displays qualitatively different behaviors, depending upon r_0, along with a universal behavior as $\langle t_{p,>}\rangle \approx \tau_D/\ln[Nq(\mathbf{r}_0)]$ for large N.

We first consider the regime where the initial searcher position is far distant from the target ($r_0 \gg a$). In this regime, solving $\langle t_{p,<}\rangle|_{N_2} = \langle t_{p,>}\rangle|_{N_2}$, we can determine the crossover point N_2 at which the small N behavior changes into the large N behavior. Neglecting the logarithmic N dependence of $\langle t_{p,>}\rangle$, it is found that $N_2 \approx \tau/\tau_D$, or more explicitly,

$$N_2 \approx \begin{cases} \left(\frac{b}{r_0}\right)^2 \ln\left(\frac{b}{a}\right) & \text{for} \quad d = 2 \\ \left(\frac{b}{r_0}\right)^2 \left(\frac{b}{a}\right) & \text{for} \quad d = 3, \end{cases} \tag{34}$$

from which one can find that $(N_2/N_1)^2 \approx (a/r_0)^2 \ll 1$, and hence N_2 much smaller than N_1, the crossover number for UPS, in the considered regime of $r_0 \gg a$. Figure 3(a) is the schematic presentation to compare the searching times: For $N < N_2$, because $\langle t_{p,<}\rangle \approx \langle t_{u,<}\rangle$ [see Eq. (25) for $r_0 \gg a$], both strategies are indistinguishable in terms of searching time. For $N > N_2$, $\langle t_{p,>}\rangle$ is a logarithmically decaying function, and therefore the searching time of UPS, algebraically decreasing, is shorter than PPS.

Next, we consider the opposite regime where the searchers are initially located close to the target ($r_0 \approx a$). In this regime, since $\langle t_{p,<}\rangle \approx [1 - q(\mathbf{r}_0)]^N \langle t_{u,<}\rangle$ for small N, $\langle t_{p,<}\rangle$ has strong dependence on N and rapidly decreases far below $\langle t_{u,<}\rangle$ as N increases. However, if N increases above N_2, PPS enters the large N regime, which is universally characterized by the weak dependence of $\langle t_{p,>}\rangle$ on N as described by Eq. (31). This behavior of searching time of PPS is presented in Fig. 3(b). The crossover point of N_2 can be determined by solving $\langle t_{p,<}\rangle = \langle t_{p,>}\rangle$ for N, and one can also show that $N_2 \ll \ln N_1 \ll N_1$. As increasing N further above N_2, $\langle t_{p,>}\rangle$ becomes larger than $\langle t_{u,<}\rangle$ at a certain number of searchers, say, N_3, which can be obtained from $\langle t_{p,>}\rangle|_{N_3} = \langle t_{u,<}\rangle|_{N_3}$ [and by assuming $\langle t_{p,>}\rangle \sim \tau_D$] as

$$N_3 \sim \begin{cases} \left(\frac{b}{r_0}\right)^2 \ln\left(\frac{b}{a}\right) & \text{for} \quad d = 2 \\ \left(\frac{b}{r_0}\right)^2 \left(\frac{b}{a}\right) & \text{for} \quad d = 3. \end{cases} \tag{35}$$

We note that N_3 is given by the same equation for N_2 for $r_0 \gg a$ and gives the upper bound of N at which PPS is the better strategy. We can also determine the lower bound N_4, above which $\langle t_{p,<}\rangle$ is smaller than $\langle t_{u,<}\rangle$ by a certain factor α.

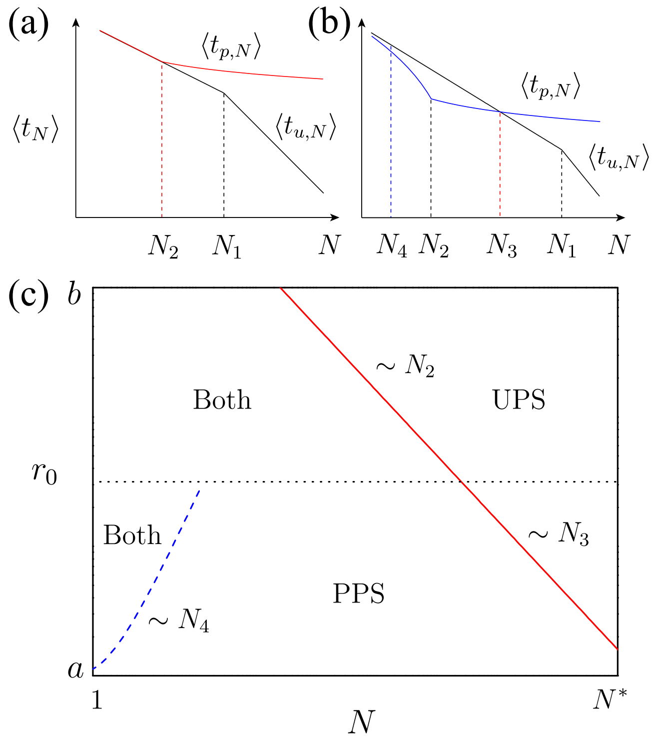

FIG. 3. Schematic diagram of $\langle t_{p,N} \rangle$ and $\langle t_{u,N} \rangle$ as a function of N when (a) $r_0 \gg a$ or (b) $r_0 \approx a$. N_1 is the crossover searcher number from the small N to the large N regime for UPS, while N_2 is for PPS. N_3 and N_4 correspond to the upper and lower boundaries of the region where PPS gives smaller searching times than UPS. (c) Schematic diagram indicating which parallel searching strategy performs better in parameter space of r_0 and N. Lines represent the analytic predictions on the domain boundaries, i.e., Eqs. (34)–(36).

Solving $\langle t_{p,<} \rangle|_{N_4} = \langle t_{u,<} \rangle|_{N_4}/\alpha$ with $\alpha > 1$, we get

$$N_4 = -\frac{\ln \alpha}{\ln[1 - q(\mathbf{r}_0)]}. \quad (36)$$

The above discussions can be summarized using a schematic diagram in Fig. 3(c), presenting the efficient searching strategy yielding shorter searching times. In the upper plane of $r_0 \to b$, both PPS and UPS are comparable for $N < N_2$, while UPS gives the better result for $N > N_2$. The boundary between different regions is given by the locus of N_2. In the lower plane of $r_0 \to a$, both are almost equivalent for $N < N_4$. PPS is superior for $N_4 < N < N_3$, and UPS is the method of choice for $N > N_3$. The crossover loci are described by N_3 and N_4 lines.

For quantitative comparisons, we perform Langevin dynamics simulations and measure the average of searching times of the two searching strategies. The simulation results are shown in Fig. 4, where the average searching times are compared in the parameter space of the number of searchers and the initial target-searcher distance. Here, we adopt three categories to judge effective searching strategies: if $\langle t_{u,N} \rangle/\alpha < \langle t_{p,N} \rangle < \alpha \langle t_{u,N} \rangle$ is satisfied, the searching times are considered to be similar and both strategies are equivalent (triangles). If $\langle t_{p,N} \rangle < \langle t_{u,N} \rangle/\alpha$, PCS outperforms UCS (circles), while if $\langle t_{u,N} \rangle < \langle t_{p,N} \rangle/\alpha$, UCS surpasses PCS (crosses). In Fig. 4, we use different symbols (in the parenthesis) to indicate which category corresponds to the search times calculated at given locations r_0 and N. The boundary of different regions in the numerical phase diagram can be seen to be well explained by the analytic lines of Eqs. (34)–(36).

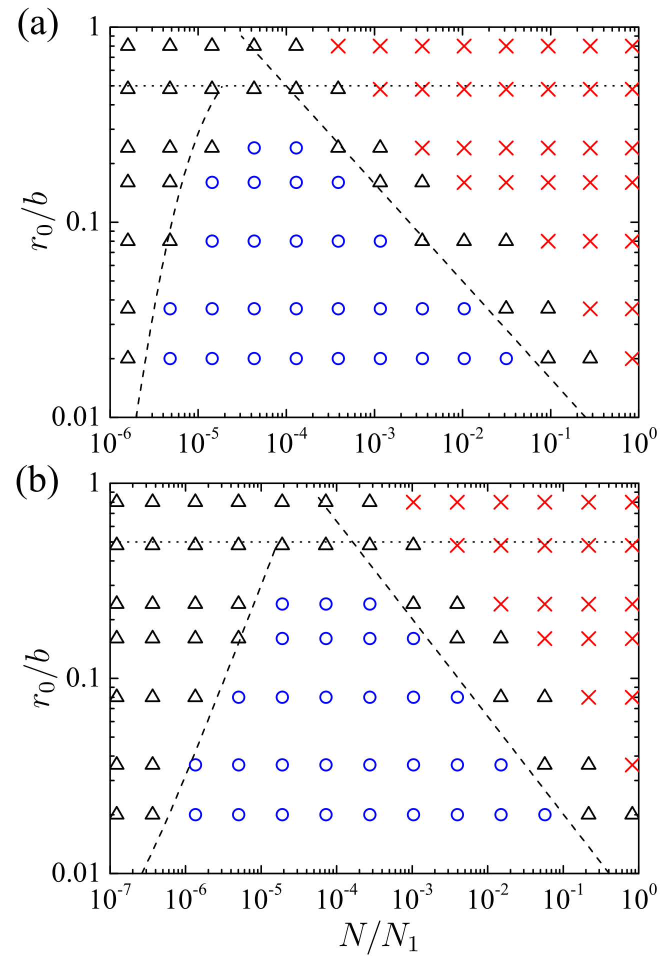

FIG. 4. Diagrams comparing $\langle t_{u,N} \rangle$ and $\langle t_{p,N} \rangle$, obtained from simulations, as a function of N/N_1 and r_0/b for (a) $d = 2$ and (b) $d = 3$. The symbols represent three different regions where $\langle t_{u,N} \rangle/\langle t_{p,N} \rangle < 1/\alpha$ (red crosses), $1/\alpha \leqslant \langle t_{u,N} \rangle/\langle t_{p,N} \rangle \leqslant \alpha$ (black triangles), and $\alpha < \langle t_{u,N} \rangle/\langle t_{p,N} \rangle$ (blue circles) with $\alpha = 3$. Lines indicate boundaries between different regions, analytically predicted based on the asymptotic expressions, i.e., using Eqs. (34)–(36). Small target size is realized by letting $b/a = 2.5 \times 10^3$ for $d = 2$ and 2.5×10^2 for $d = 3$. The data are obtained from 10^1–10^8 ensembles, depending upon N, of Langevin dynamic simulations.

Using the results of analysis shown above, we can determine which strategy should be employed when the number of available searchers is given. Overall, PPS is better if the target position can be precisely predicted, while UPS outperforms if N is substantially large. However, information on the target position is not available in advance of performing the searches. In this case, if we have N less than $N_2(r_0 = b)$, PPS should be the method of choice, because either it yields shorter searching times or the difference in expected searching times is insignificant for any initial target-searcher distance. On the other hand, if $N > N_3(r_0 = a) \approx N_1$, UPS is always better than PPS. In the intermediate range of N $[N_2(r_0 = b) < N < N_3(r_0 = a)]$, it depends on r_0, a strategy which performs better. At higher dimensions, the region where both searching strategies are comparable expands since the initial position dependence of searching time becomes weak in noncompact searches.

VI. SUMMARY

We have addressed the problem of parallel random target searching where a set of N particles perform independent searching by diffusion for a small target in d-dimensional ($d = 1,2,3$) bounded space. The searching time is defined as the mean time for the first of N searchers to reach the target. For parallel random target searching, it is one of the important questions, especially in the context of the searching strategy to be chosen, how the searching time depends on the initial searcher distribution. In answering the question, we have compared two prototypes of searching strategies: uniformly distributed parallel searching (UPS) and point-distributed parallel searching (PPS). When rescaled into a dimensionless quantity, the searching time of UPS, $\langle t_{u,N}\rangle$, shows a universal behavior, i.e., crossover from $\sim 1/N$ for small N to $\sim 1/N^2$ for large N regimes. The universal behavior is in excellent agreement with the Langevin dynamics simulation results obtained for various domain-to-target sizes in different dimensions. On the other hand, for PPS, the searching time $\langle t_{p,N}\rangle$ shows a logarithmic dependence on N in the large N limit, while in the small N limit, it has a stronger dependence on the initial target-searcher distance r_0 as r_0 becomes of the order of a. These analytic predictions were also confirmed by the numerical results. Finally, we have drawn a diagram where the ratio of searching times of UPS to PPS is plotted in terms of r_0 and N. The parameter spaces are divided into three regions in which $\langle t_{u,N}\rangle$ is much smaller than, similar to, or much larger than $\langle t_{p,N}\rangle$. Based on the asymptotic expressions, we analytically determined boundaries between the regions, which qualitatively explain the simulation results.

There are a number of open problems related to the target searching by multiple searchers. For example, it would be intriguing to study the order statistics of PPS and UPS where one is interested in estimating the first-passage time of the jth walker that arrives at the target. What was considered in this work is a special case (with $j = 1$) of the general problem. Because the one-dimensional mapping of the first-passage trajectory is no longer valid even for large N when j is not a small number, the extension of the present study to the general problem of the order statistics is nontrivial and poses an important question to be explored in the future.

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APPENDIX A: DERIVATION OF $\langle t_{u,<}\rangle$ AND $\langle t_{p,<}\rangle$ FOR TWO OR THREE DIMENSIONS

Here we present a derivation of the searching times for $d \geqslant 2$ in small target and small N limits ($a/b \to 0$ and $N \approx 1$). Consider parallel searching of N searchers with an initial position distribution given as $p_0(\mathbf{r}_0)$. The average of minimum FPT of this N parallel searching is evaluated from Eq. (5) as

$$\langle t_N\rangle = \int_0^\infty dt\left[\int_{\mathcal{D}_*} d^d\mathbf{r}_0 S(t;\mathbf{r}_0)p_0(\mathbf{r}_0)\right]^N, \tag{A1}$$

where we performed the integration by parts. The quantity in the square brackets of Eq. (A1) corresponds to average survival probability and will be denoted as $\tilde{S}(t)$. Using the eigenfunction expansion of Eq. (12), $\tilde{S}(t)$ can be written in series form as

$$\tilde{S}(t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n Dt}, \tag{A2}$$

where the coefficient c_n is given as

$$c_n \equiv \int_{\mathcal{D}^*} d^d\mathbf{r}_0 \int_{\mathcal{D}^*} d^d\mathbf{r}\phi_n^*(\mathbf{r})\phi_n(\mathbf{r}_0)p_0(\mathbf{r}_0). \tag{A3}$$

The searching time $\langle t_N\rangle$ is now obtained by inserting Eq. (A2) into (A1):

$$\begin{aligned}\langle t_N\rangle &= \int_0^\infty dt\left[\sum_{n=1}^{\infty} c_n e^{-\lambda_n Dt}\right]^N\\ &= \int_0^\infty dt\sum_{\{k_n\}}\left(N!\prod_{n=1}^{\infty}\frac{1}{k_n!}c_n^{k_n}e^{-k_n\lambda_n Dt}\right)\\ &= \frac{N!c_1^N}{\lambda_1 D}\sum_{\{k_n\}}\left(\prod_{n=1}^{\infty}\frac{c_n^{k_n}}{k_n!c_1^{k_n}}\right)\frac{1}{\sum_{n=1}^{\infty}k_n\frac{\lambda_n}{\lambda_1}},\end{aligned} \tag{A4}$$

where multinomial theorem is used in the second equality and summation with $\{k_n\}$ runs over all possible combinations of non-negative integers k_n with a constraint of $\sum_{n=1}^{\infty} k_n = N$. It can be shown in the small target limit that when $d \geqslant 2$, the principal eigenvalue converges to zero as [20]

$$\lim_{a/b\to 0}\lambda_1 = 0, \tag{A5}$$

while the other eigenvalues remain finite. Accordingly, the ratio $\lambda_{n\neq 1}/\lambda_1$ diverges, which implies that the only nonvanishing term in Eq. (A4) satisfies $k_1 = N$ and $k_{n\neq 1} = 0$. Therefore, the searching time is obtained as

$$\langle t_N\rangle = \frac{c_1^N}{N\lambda_1 D}. \tag{A6}$$

For UPS, the coefficient c_1 can be evaluated by putting $n = 1$ and $p_0(\mathbf{r}_0) = (V^*)^{-1}$ in Eq. (A3). When $d \geqslant 2$, the asymptotic behavior of the principal eigenfunction is given in the small target limit as [20]

$$\lim_{a/b\to 0}\phi_1(\mathbf{r}) = \frac{1}{\sqrt{V}}. \tag{A7}$$

Note that V^* converges to V in the small target limit, which leads to $\lim_{a/b\to 0} c_1 = 1$. Now, we obtain the mean minimum FPT for UPS in the small N limit as

$$\langle t_{u,<}\rangle = \frac{1}{N\lambda_1 D}, \tag{A8}$$

which becomes Eq. (16) with the relation $\tau = 1/(D\lambda_1)$.

For PPS, the initial position distribution is given as the δ function $p_0(\mathbf{r}_0) = \delta(\mathbf{r}_0 - \mathbf{r}_i)$, and the coefficient c_1 attains an initial position dependence as $c_1(\mathbf{r}_i)$. Comparing Eqs. (12) and (A2), we rewrite $c_1(\mathbf{r}_i)$ as

$$c_1(\mathbf{r}_0) = 1 - q(\mathbf{r}_0), \tag{A9}$$

where we substitute $\mathbf{r}_0$ for $\mathbf{r}_i$. Combining Eqs. (A6) and (A9), the mean minimum FPT for PPS in the small N limit is obtained as

$$\langle t_{p,<}\rangle = (1 - q(\mathbf{r}_0))^N \langle t_{u,<}\rangle. \tag{A10}$$

Similarly to the case of UPS, the value of $q(\mathbf{r}_0)$ in a small target limit can be evaluated by using the asymptotic behavior of the principal eigenfunction shown in Eq. (A7), which leads to $\lim_{a/b\to 0} q(\mathbf{r}_0) = 0$ for $r_0 \gg a$. If r_0 is of the order of a, however, $q(\mathbf{r}_0)$ does not necessarily converge to zero but remains finite [e.g., see Eq. (33)]. Taking these observations into consideration, we write the equation for $\langle t_{p,<}\rangle$ as follows:

$$\langle t_{p,<}\rangle \simeq \begin{cases} \langle t_{u,<}\rangle & \text{for} \quad r_0 \gg a \\ (1 - q(\mathbf{r}_0))^N \langle t_{u,<}\rangle & \text{for} \quad r_0 \to a\,. \end{cases}$$

APPENDIX B: DERIVATION OF $\langle t_{p,>}\rangle$ FOR A THREE-DIMENSIONAL SEARCH

The three-dimensional searching time for PPS in the large N regime can be written as

$$\langle t_{p,>}\rangle = \int_0^\infty dt\, S^N(t;\mathbf{r}_0) = \int_0^\infty dt \left[1 - q(\mathbf{r}_0)\mathrm{erfc}\left(\frac{|\mathbf{r}_0|}{\sqrt{4Dt}}\right)\right]^N,$$

by using Eqs. (7) and (30). In the small target size limit, $q(\mathbf{r}_0)$ is a very small number since it is proportional to a. Using the smallness of $q(\mathbf{r}_0)$, we convert $S^N(t;\mathbf{r}_0)$ as follows:

$$\begin{aligned} S^N(t;\mathbf{r}_0) &\simeq \exp\left[N \ln\left\{1 - q(\mathbf{r}_0)\mathrm{erfc}\left(\frac{|\mathbf{r}_0|}{\sqrt{4Dt}}\right)\right\}\right] \\ &\simeq \exp\left[-Nq(\mathbf{r}_0)\mathrm{erfc}\left(\frac{|\mathbf{r}_0|}{\sqrt{4Dt}}\right)\right] \\ &\simeq \exp\left[Nq(\mathbf{r}_0)\ln\left\{\mathrm{erf}\left(\frac{|\mathbf{r}_0|}{\sqrt{4Dt}}\right)\right\}\right] \\ &= \mathrm{erf}^{Nq(\mathbf{r}_0)}\left(\frac{|\mathbf{r}_0|}{\sqrt{4Dt}}\right). \end{aligned} \tag{B1}$$

Comparing Eqs. (26) with Eq. (B1), we find that a three-dimensional searching time can be obtained by replacing N with $Nq(\mathbf{r}_0)$ from the one-dimensional searching time, Eq. (27). As a result, we obtain Eq. (31):

$$\langle t_{p,>}\rangle \approx \frac{\tau_D}{\ln[Nq(\mathbf{r}_0)]}.$$

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