PHYSICAL REVIEW E **96**, 012127 (2017)

de Almeida–Thouless instability in short-range Ising spin glasses

R. R. P. Singh
University of California Davis, Davis, California 95616, USA

A. P. Young
University of California Santa Cruz, Santa Cruz, California 95064, USA
(Received 2 May 2017; published 13 July 2017)

We use high-temperature series expansions to study the $\pm J$ Ising spin glass in a magnetic field in d-dimensional hypercubic lattices for $d = 5$–8 and in the infinite-range Sherrington-Kirkpatrick (SK) model. The expansions are obtained in the variable $w = \tanh^2 J/T$ for arbitrary values of $u = \tanh^2 h/T$ complete to order w^{10}. We find that the scaling dimension Δ associated with the ordering-field h^2 equals 2 in the SK model and for $d \geqslant 6$. However, in agreement with the work of Fisher and Sompolinsky [Phys. Rev. Lett. **54**, 1063 (1985)], there is a violation of scaling in a finite field, leading to an anomalous h-T dependence of the de Almeida–Thouless (AT) [J. Phys. A **11**, 983 (1978)] line in high dimensions, whereas scaling is restored as $d \to 6$. Within the convergence of our series analysis, we present evidence supporting an AT line in $d \geqslant 6$. In $d = 5$, the exponents γ and Δ are substantially larger than mean-field values, but we do not see clear evidence for the AT line in $d = 5$.

DOI: 10.1103/PhysRevE.96.012127

I. INTRODUCTION

One of the most striking predictions of the mean-field theory of spin glasses [1–3] is the existence of a line of transitions in the magnetic-field temperature plane first found by de Almeida and Thouless (AT) [4]. This transition is surprising since it occurs without the breaking of any "obvious" symmetry and instead marks the onset of "replica symmetry breaking" (RSB). The solution of the mean-field infinite-range Sherrington-Kirkpatrick (SK) [2] model in the RSB phase below the AT line is complicated and was obtained in a tour de force by Parisi [5,6]. Whether or not an (AT) line of instabilities occurs in the magnetic-field temperature plane for *short-range* Ising spin glasses has been an outstanding problem that has remained unresolved over the past few decades. According to the "droplet theory" [7,8] of spin glasses, the AT line is an artifact of the infinite-range nature of the interactions of the SK model and would not occur in *any* finite-dimensional short-range model. In the "RSB scenario," see, e.g., Refs. [9,10] and references therein, the behavior of real spin glasses is similar to that of the SK model and so there would be an AT line down to $d = 3$.

In recent years, numerical simulations have investigated whether or not there is an AT line in short-range spin glasses. Although some work finds evidence for an AT line only above dimension $d = 6$ [11,12], other work [9,10], which used a different method performing finite-size scaling, does find an AT line at least down to $d = 4$ and possibly $d = 3$.

In view of the importance of whether or not an AT line exists in short-range spin glasses and the fact that Monte Carlo (MC) simulations do not give an unambiguous answer, it is useful to attack the problem by other possible means, and here we use high-temperature series expansions. Some benefits of the series methods are that averaging over disorder is performed exactly, it can be performed in arbitrary dimensions, is particularly accurate in high dimensions, and that the series is an equilibrium property of the *infinite* system. In fact, Monte Carlo studies of the AT line in the range of dimension that we consider here $d \geqslant 5$ have not been performed directly but only *indirectly* using a one-dimensional model with long-range interactions [11,13].

Obtaining the series in a field is more complicated than in a zero field, so we are only able to obtain a series of moderate length. Nonetheless, the series does provide evidence for an AT line above $d = 6$. Below $d = 6$ the series does not find good evidence for an AT line, but whether this is because the line does not exist or the series is not long enough to see it is unclear. This situation is reminiscent of an early perturbative renormalization group calculation [14] which did not find a stable fixed point in a field below $d = 6$. In that case, the issue, as yet unresolved, is whether the AT line does not exist below $d = 6$ or whether there is one which is just not accessible by perturbation theory.

We consider the Hamiltonian,

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij} S_i S_j - h \sum_{i=1}^{N} S_i, \tag{1}$$

where the S_i are Ising spins which take values ± 1 and the interactions J_{ij} are *quenched* random variables with a bimodal distribution, i.e., $J_{ij} = \pm J$ with equal probability. The N spins either lie on a hypercubic lattice in which case the interactions are between nearest neighbors and have $J = 1$ or correspond to the SK [2] model in which case there is no lattice structure, every spin interacts with every other spin, and $J = 1/\sqrt{N}$. We choose a bimodal distribution because the series can be worked out much more efficiently for this case than for a general distribution [15].

The AT line is characterized by the divergence of the spin-glass susceptibility χ_{SG} where

$$\chi_{SG} = \frac{1}{N} \sum_{i,j=1}^{N} [(\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle)^2]_{av}, \tag{2}$$

where $[\cdots]_{av}$ denotes an average over the disorder. For a fixed value of h we expand the susceptibility for the hypercubic lattice in powers of

$$w = \tanh^2(J/T). \tag{3}$$

The coefficient of w^n turns out to be a polynomial of order $2n+2$ in

$$u = \tanh^2(h/T), \tag{4}$$

so

$$\chi_{SG}(w,u) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{2n+2} a_{n,m}u^m\right)w^n. \tag{5}$$

We evaluate all the coefficients $a_{n,m}$ up to order $n = 10$ for a hypercubic lattice in arbitrary dimension d [16]. The series for the SK model is obtained by setting $J = 1/2d$ and taking the $d \to \infty$ limit in which case the high-temperature expansion variable becomes $x = (1/T)^2$ rather than w. A ten-term series is only of moderate length, but, compared with a zero field, determining the series in a field requires much more computer time and memory, so it would need a very large numerical effort to *substantially* increase the number of terms beyond ten. Part of the reason for the extra complexity is that the coefficients rapidly become large, and, for a given order n, considerable cancellations occur between the coefficients with different values of m. For example, for the SK model with $u = 0.2$ the largest individual contribution to the coefficient of w^{10} is about 10^7 times greater than the final answer [16].

Equation (5) gives the high-temperature series expansion for *arbitrary* values of the ratio h/T. Fixing this ratio corresponds to expanding χ_{SG} along a diagonal line in the h-T plane ending at $h = T = 0$, which *must* intersect an AT line if one exists.

II. ZERO-FIELD SPIN-GLASS TRANSITION

It is known from earlier studies [17–20] that a ten-term series in w does not give a consistent indication of a critical temperature in $d = 3$ and is also problematic in $d = 4$ giving rather large and inconsistent values of critical exponents. Hence, we confine our analysis to $d = 5$ and higher.

For the SK model, the zero-field spin-glass series is a simple geometrical series in $x = 1/T^2$, which sums to $1/(1-x)$ showing that the exponent γ equals unity. For $d \geqslant 6$, the model's critical behavior is governed by a Gaussian fixed point with $\gamma = 1$ [21,22]. We use Padé approximants directly on the series to estimate w_c, the critical value of w. This fixes γ to unity and produces estimates for w_c shown in Table I. Note that uncertainties in the series analysis are just confidence limits [23]. These w_c values are consistent with those from large dimensionality expansions [24,25] and agree to within around a percent with those from a more sophisticated analysis taking corrections to scaling into account [19,20]. It is well known that estimates of critical points in the series analysis are correlated with estimates of critical exponents. Hence, by fixing the critical exponent to unity, we avoid some of the subtleties and get a fairly reasonable estimate of the critical point with a moderate length series.

In $d = 5$, we use standard d-log Padé approximants and differential approximants [26,27] to analyze the series. The critical point estimate $w_c = 0.1388 \pm 0.0009$ is consistent with previous studies [17,19]. Using biased approximants with the critical point fixed at the central estimate $w_c = 0.1388$, we obtain $\gamma = 1.9 \pm 0.1$, again in agreement with previous studies.

TABLE I. Estimates of critical points and exponents in various dimensions for the zero field.

d	w_c	γ	Δ
8	0.0695 ± 0.0002	1	2.0 ± 0.1
7	0.0816 ± 0.0004	1	2.0 ± 0.2
6	0.0996 ± 0.0008	1	2.0 ± 0.5
5	0.1388 ± 0.0009	1.9 ± 0.1	3.1 ± 0.4

III. SCALING DIMENSION OF THE ORDERING FIELD

Field theory predicts [21,22,28] that the scaling dimension of the ordering field h^2, or equivalently u, should be $\Delta = 2$ at the Gaussian fixed point. In other words for $d \geqslant 6$, the variable u should scale near T_c with the reduced temperature $t \equiv (T - T_c)/T_c$ in the combination u/t^2.

To study this through series expansions we consider two single variable series in w defined as

$$K_1(w) = \frac{\partial_u \chi_{SG}(w,u)|_{u=0}}{\chi_{SG}(w,0)}, \tag{6}$$

and

$$K_2(w) = \frac{\partial_u^2 \chi_{SG}(w,u)|_{u=0}}{\partial_u \chi_{SG}(w,u)|_{u=0}}. \tag{7}$$

Both quantities $K_1(w)$ and $K_2(w)$ should diverge at the critical temperature as $1/t^\Delta$ with $\Delta = 2$ for $d \geqslant 6$. Note that we consider the limit $t \to 0$ for which $t \equiv (T - T_c)/T_c \propto (w_c - w)/w_c$.

For the SK model, these quantities sum up to

$$K_1^{SK} = -2/(1-x)^2, \tag{8}$$

and

$$K_2^{SK} = 6 - 2x - (x+7)/(1-x)^2, \tag{9}$$

respectively, with $x = 1/T^2$ clearly showing that $\Delta = 2$. In fact, from an asymptotic analysis of our graphical method, one can show that the mth derivative of χ_{SG}^{SK} with respect to u, evaluated at $u = 0$ diverges as $1/t^{1+2m}$, confirming that $\Delta = 2$ is true to all perturbative orders in u.

To analyze the $K_1(w)$ and $K_2(w)$ series in finite dimensions, we use d-log Padé and differential approximants. We fix the critical point at those values estimated from the zero-field susceptibility series, see Table I. A histogram of Δ values estimated from the analysis is shown in Fig. 1. It is clear that in $d \geqslant 6$ the exponent Δ remains equal to 2. However, in $d = 5$ it is closer to 3.

IV. ANALYSIS OF A SERIES IN A FINITE FIELD

We fix a value of u and study the series in w. In the SK model the spin-glass susceptibility diverges as a simple pole at the AT line. Unlike the case of the zero field the series for the SK model are no longer simple in a field, and indeed no truncated series can reproduce exactly the violation of scaling encapsulated in the fact that, along the AT line, $T - T_c$ scales as h^θ with $\theta = 2/3$ rather than as $h^{2/\Delta}$ as expected from scaling.

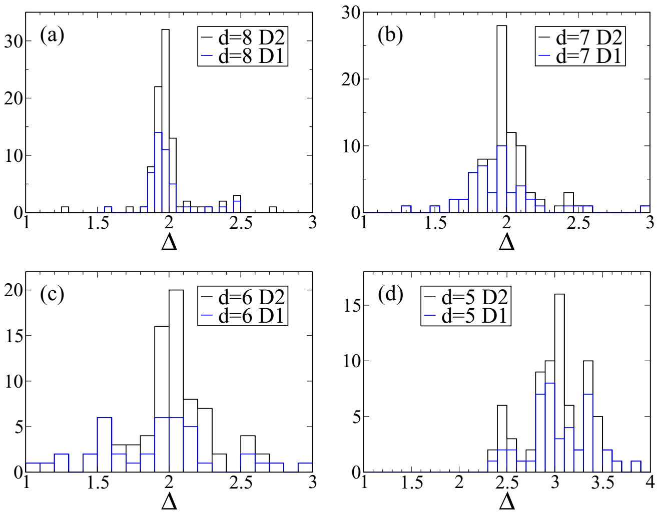

FIG. 1. Histogram of the exponent Δ in zero field obtained from various approximants in (a) $d = 8$, (b) $d = 7$, (c) $d = 6$, and (d) $d = 5$ to the series for $K_1(D_1)$ and $K_2(D_2)$ defined in Eqs. (6) and (7).

In fact, for any finite length series, at sufficiently small h, such a nonlinear relation cannot follow. Hence, our focus will be on fields which are not too small to be dominated by just the leading order field terms.

We have found that the finite-field series for the SK model do not converge well close to the AT line. The series analysis works better in the variable $w = \tanh^2 1/T$. Two diagonal Padé approximants for χ_{SG}^{-1} using the variable w for $u \equiv \tanh^2(h/T) = 0.1$ and $u = 0.2$ are shown in Fig. 2 along with the exact value computed numerically. The critical point is located reasonably well at $u = 0.1$ but not at $u = 0.2$. This is found to be true for a majority of Padé approximants, including off-diagonal ones.

For different values of u, we carry out a large number of Padé approximants and determine the critical point from the set of approximants which are bunched closest to each other. The estimated phase boundary is shown in Fig. 3. The exact value of the AT line for the SK model, determined numerically, and its asymptotic small h-t limit also are shown in the figure.

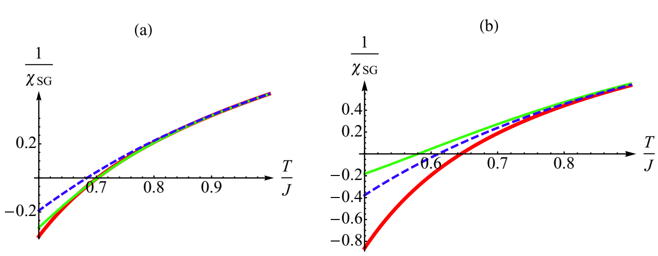

FIG. 2. The inverse of the spin-glass susceptibility in the SK model for (a) $u \equiv \tanh^2(h/T) = 0.1$ and (b) $u = 0.2$. The AT line is where $\chi_{SG}^{-1} = 0$. The exact values obtained numerically from the equations of AT are shown by the solid red (thick) lines. Also shown are the (5, 5) (green, solid lines) and (4, 4) (blue, dashed lines) Padé approximants of the high-temperature series using the transformed variable $w = \tanh^2(1/T)$. It is seen that the divergence is located reasonably well at $u = 0.1$ but less so at $u = 0.2$.

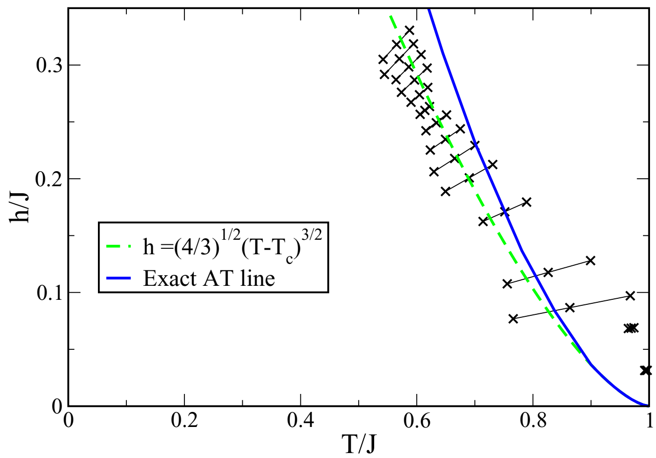

FIG. 3. The solid (blue) line is the expression for the AT line in the SK model, whereas the dashed (green) line is its asymptotic small h limit. The points joined by lines are the results of the Padé analysis of the ten-term series, including just the Padé approximants that are bunched together.

The series analysis is consistent with the correct $\theta = 2/3$ value for the AT line but overestimates the extent of the paramagnetic phase for larger u. Any significant improvement will need a substantially longer series. Note that as discussed in the previous paragraph, for very small h, the analysis is dominated by the leading h terms and shows only a small shift in the critical point. Although the convergence is not excellent, it is clear that high-temperature expansion with a moderate number of terms can capture the highly nontrivial de Almeida–Thouless instability in the SK model [4].

Fisher and Sompolinsky [28] have shown that between $d = 8$ and $d = 6$ the AT-line exponent becomes $\theta = 4/(d-2)$, which goes from the SK value of $\theta = 2/3$ in $d = 8$ to $\theta = 1$ in $d = 6$. Thus the usual scaling relation $\theta = 2/\Delta$ is restored in $d = 6$. For $d =$ 6–8, we repeat the same analysis as for the SK model. The results for the estimated AT lines are shown in Fig. 4. The uncertainties in locating the AT line are too large to

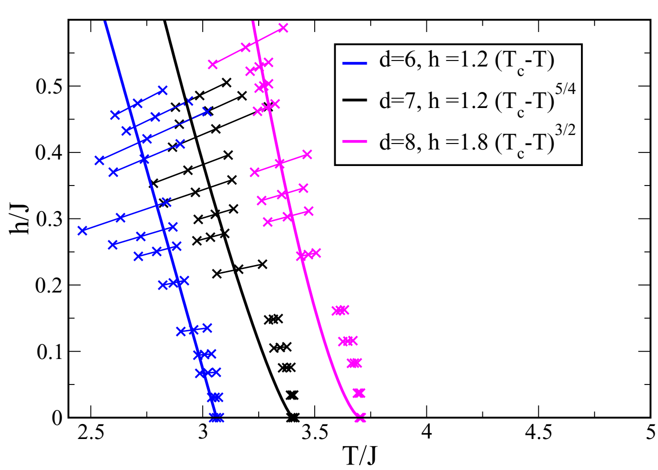

FIG. 4. Estimates of the AT line in $d = 6$, $d = 7$, and $d = 8$ obtained from Padé analyses of the series. The formulas for the lines are given in the legend and are discussed in the text.

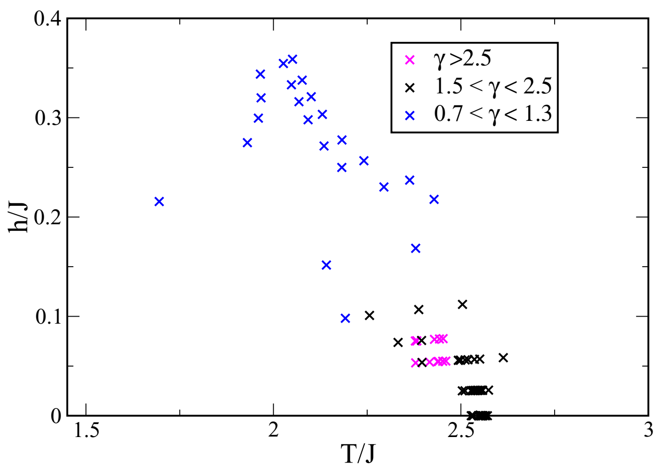

FIG. 5. Approximants showing divergence of the spin-glass susceptibility in $d = 5$ and the range of their exponent γ.

allow an unbiased fit to a power law. However, a few points can clearly be noted: (i) In both $d = 7$ and $d = 8$, the small-field behavior differs qualitatively from that at larger fields and is very similar to the behavior seen in the SK model. (ii) For $d = 7$ and 8 it is only after u exceeds a certain value that a more consistent behavior with $\theta < 1$ emerges. As a guide, we have drawn curves with $\theta = 2/3$ and $\theta = 4/5$ in $d = 8$ and $d = 7$, respectively, as expected from the analysis of Fisher and Sompolinsky [28]. (iii) In $d = 6$, we do not see the clear discrepancy at low fields, and the behavior is more consistent with $\theta = 1$ as expected from scaling, which is predicted to be restored [28] in $d = 6$. This suggests that the series analysis is capturing some of the key features of the finite-field behavior in short-range spin glasses and its changes with dimensionality and that the de Almeida–Thouless instability does exist for $d \geqslant 6$.

In $d = 5$, the system is no longer governed by a Gaussian fixed point. However, the increased Δ value shown in Table I suggests that if there is an AT line it should have a power of $\theta = 2/\Delta$, which is again close to 2/3. For this case, we analyze our series by d- log Padé approximants and differential approximants. At very small u values, we see a singularity that is very similar to what is seen in the SK model and in higher d. There is only a small shift in the critical point. But, once u exceeds a certain value most approximants do not show a consistent divergence. As seen in Fig. 5, only a handful of approximants show any divergence at all. These predict a reduced exponent of $0.7 < \gamma < 1.3$. This could imply that the series are too short to see the nontrivial critical behavior in $d < 6$ or it could mean that there is no de Almeida–Thouless instability below $d = 6$. We especially note that one difference in our analysis of the susceptibility in a field in $d = 5$ versus higher d is that in higher dimensions we biased the critical exponents to have mean-field values. The absence of such a bias contributes to the uncertainty in the $d = 5$ analysis and may be partly responsible for the lack of a more definitive answer in $d = 5$.

V. CONCLUSIONS

In conclusion, we have studied the problem of short-range Ising spin glasses in a field by high-temperature series expansion methods. We have presented evidence for the violation of scaling along the AT line in high dimensions and its restoration as $d \to 6$ as first shown by Fisher and Sompolinsky [28]. Within the convergence of our analysis, we have presented evidence for the existence of the AT line of instabilities for $d \geqslant 6$. In $d = 5$, the critical exponents γ and Δ are significantly larger than the mean-field values, but no consistent evidence for the AT line is found. Thus, our results are *consistent* with 6 being the lower critical dimension for the AT line. However it is also possible that an AT line *does* occur for $d < 6$ but the series are too short to see it.

Finally we compare our results with other work. The early renormalization group calculation of Bray and Roberts [14] did not find a stable perturbative fixed point corresponding to an AT line below the upper critical dimension of 6. Although this result is consistent with there being no AT line below $d = 6$, it is also possible that a nonperturbative fixed point is present for this range of dimension. Other studies used MC simulations on one-dimensional long-range interactions models [9,10,12]. The results were analyzed using finite-size scaling theory. Although all analyses found evidence for an AT line for interactions corresponding to d above 6, different conclusions were reached in lower dimensions [9,10,12].

Our approach is complimentary to MC in that we study short-range models directly on d-dimensional hypercubic lattices and that the series represent equilibrium property of the *infinite* system. Thus, the combined MC and series evidence provides a strong case for an AT line in short-range models at least in high enough dimensions. It would be challenging, but worthwhile, to try to extend the series approach to include higher order terms.

ACKNOWLEDGMENTS

One of us (A.P.Y.) would like to acknowledge the hospitality of the Indian Institute of Science, Bangalore and the support of a DST-IISc Centenary Chair Professorship. He is particularly grateful for stimulating discussions with H. Krishnamurthy which initiated this project. The work of R.R.P.S. was supported, in part, by U.S. NSF Grant No. DMR-1306048.

[1] S. F. Edwards and P. W. Anderson, Theory of spin glasses, J. Phys. F **5**, 965 (1975).
[2] D. Sherrington and S. Kirkpatrick, Solvable Model of a Spin Glass, Phys. Rev. Lett. **35**, 1792 (1975).
[3] K. Binder and A. P. Young, Spin glasses: Experimental facts, theoretical concepts and open questions, Rev. Mod. Phys. **58**, 801 (1986).
[4] J. R. L. de Almeida and D. J. Thouless, Stability of the Sherrington-Kirkpatrick solution of a spin glass model, J. Phys. A **11**, 983 (1978).
[5] G. Parisi, The order parameter for spin glasses: A function on the interval 0–1, J. Phys. A. **13**, 1101 (1980).
[6] G. Parisi, Order Parameter for Spin-Glasses, Phys. Rev. Lett. **50**, 1946 (1983).

[7] D. S. Fisher and D. A. Huse, Absence of many states in realistic spin glasses, J. Phys. A **20**, L1005 (1987).
[8] D. S. Fisher and D. A. Huse, Equilibrium behavior of the spin-glass ordered phase, Phys. Rev. B **38**, 386 (1988).
[9] M. Baity-Jesi *et al.* (Janus Collaboration), The three dimensional Ising spin glass in an external magnetic field: The role of the silent majority, J. Stat. Mech.: Theory Exp. (2014) P05014.
[10] R. A. Baños *et al.* (Janus Collaboration), Thermodynamic glass transition in a spin glass without time-reversal symmetry, Proc. Natl. Acad. Sci. U.S.A. **109**, 6452 (2012).
[11] H. G. Katzgraber and A. P. Young, Probing the Almeida-Thouless line away from the mean-field model, Phys. Rev. B **72**, 184416 (2005).
[12] D. Larson, H. G. Katzgraber, M. A. Moore, and A. P. Young, Spin glasses in a field: Three and four dimensions as seen from one space dimension, Phys. Rev. B **87**, 024414 (2013).
[13] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Diluted One-Dimensional Spin Glasses with Power Law Decaying Interactions, Phys. Rev. Lett **101**, 107203 (2008).
[14] A. J. Bray and S. A. Roberts, Renormalisation-group approach to the spin glass transition in finite magnetic fields, J. Phys. C **13**, 5405 (1980).
[15] R. R. P. Singh and A. P. Young, Efficient generation of series expansions for $\pm J$ Ising spin glasses in a classical or a quantum (transverse) field (unpublished).
[16] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.96.012127 for the series coefficients.
[17] R. Fisch and A. B. Harris, Series Study of a Spin-Glass Model in Continuous Dimensionality, Phys. Rev. Lett. **38**, 785 (1977).
[18] R. R. P. Singh and S. Chakravarty, Critical Behavior of an Ising Spin-Glass, Phys. Rev. Lett. **57**, 245 (1986).
[19] L. Klein, J. Adler, A. Aharony, A. B. Harris, and Y. Meir, Series expansions for the Ising spin glass in general dimension, Phys. Rev. B **43**, 11249 (1991).
[20] D. Daboul, I. Chang, and A. Aharony, Test of universality in the Ising spin glass using high temperature graph expansion, Eur. Phys. J. B **41**, 231 (2004).
[21] A. B. Harris, T. C. Lubensky, and J.-H. Chen, Critical Properties of Spin-Glasses, Phys. Rev. Lett. **36**, 415 (1976).
[22] J.-H. Chen and T. C. Lubensky, Mean field and ϵ-expansion study of spin glasses, Phys. Rev. B **16**, 2106 (1977).
[23] J. Oitmaa, C. Hamer, and W. Zheng, *Series Expansion Methods for Strongly Interacting Lattice Models* (Cambridge University, Cambridge, U.K., 2006).
[24] R. R. P. Singh and M. E. Fisher, Short-range Ising spin glasses in general dimensions, J. Appl. Phys. **63**, 3994 (1988).
[25] M. E. Fisher and R. R. P. Singh, in *Disorder in Physical Systems*, edited by G. Grimmett and D. J. A. Welsh (Oxford University Press, Oxford, 1990).
[26] D. L. Hunter and G. A. Baker, Methods of series analysis. III. Integral approximant methods, Phys. Rev. B **19**, 3808 (1979).
[27] M. E. Fisher and H. Au-Yang, Inhomogeneous differential approximants for power series, J. Phys. A **12**, 1677 (1979).
[28] D. S. Fisher and H. Sompolinsky, Ordered Phase of Short-Range Ising Spin-Glasses, Phys. Rev. Lett. **54**, 1063 (1985).