Continuous-time random walk under time-dependent resetting

V. P. Shkilev*

Chuiko Institute of Surface Chemistry, National Academy of Sciences of Ukraine, 17 General Naumov Street, 03164 Kyiv, Ukraine

(Received 20 February 2017; revised manuscript received 15 June 2017; published 12 July 2017)

Continuous-time random walks of a particle that is randomly reset to an initial position are considered. The distribution of the waiting time between the reset events is represented as a sum of an arbitrary number of exponentials. The governing equation of this stochastic process is established. The mean first-passage time to a particular position is calculated. It is shown that anomalous subdiffusion has a significant impact on the shape of the stationary state.

DOI: 10.1103/PhysRevE.96.012126

I. INTRODUCTION

Recently, intermittent stochastic processes, in which the diffusion of a particle is interrupted by random resetting to an initial position, have been extensively studied. Such processes have attracted considerable attention because random resetting fundamentally changes the properties of the diffusion process. In particular, with resetting, the distribution of the particle position does not expand indefinitely, but evolves towards a nonequilibrium steady state. Another important consequence is that the mean first passage time, which is infinite in the case of simple diffusion, becomes finite with resetting.

Examples of intermittent stochastic processes are found in many fields such as chemistry [1], biology [2], ecology [3], and computer science [4].

The investigation of diffusion processes with stochastic resetting was initiated in Refs. [5–7]. Later, several specific models describing particular intermittent processes were considered [8–17]. In recent works [18–20], the effect of the distribution of the waiting time between the reset events on the properties of the intermittent stochastic process was studied. In these studies, it was assumed that a particle performs a simple random walk between the reset events. However, in many real-world processes, the particle moves in complex disordered media; therefore, the simple random walk model is not applicable. In this study, we consider a further generalization of the model [18–20] considering medium disorder. We assume that in a given disordered medium, the motion of a particle has a subdiffusive (slowing down) character, and it can be described by the continuous-time random walk (CTRW) model. As a starting point, we use the Markov representation of the CTRW model [21–23]. As independent variables, we use the spatial coordinate of a particle, time elapsed since the beginning of the observation, time elapsed since the last diffusion jump, and time elapsed since the last reset event. First, we found the propagator. Then, we represented the distribution of the waiting time as a sum of an arbitrary number of exponentials. With such a representation, we found an equation satisfied by the propagator and calculated the mean first-passage time to a particular position. We also showed that the medium disorder affects the properties of the intermittent stochastic process. In particular, the stationary probability distribution shape is different in the cases of normal diffusion (i.e., diffusion in a homogeneous medium) and anomalous subdiffusion (i.e., diffusion in a disordered medium).

II. PROPAGATOR

We start our consideration with the balance equation, which describes the CTRW under time-dependent resetting on a discrete one-dimensional lattice. Let $\xi_i(t,\tau,\sigma)$ be the probability density of finding a particle at site i at time t, whose residence time in this site is equal to τ and time elapsed since the last reset event is equal to σ. The balance equation is [21–24]

$$\frac{\partial \xi_i(t,\tau,\sigma)}{\partial t}+\frac{\partial \xi_i(t,\tau,\sigma)}{\partial \tau}+\frac{\partial \xi_i(t,\tau,\sigma)}{\partial \sigma} = -\omega(\tau)\xi_i(t,\tau,\sigma)-\lambda(\sigma)\xi_i(t,\tau,\sigma). \quad (1)$$

Here, the first term on the right-hand side describes the decrease of the probability $\xi_i(t,\tau,\sigma)$ due to transitions of the particle to neighboring sites. $\omega(\tau)$ is the corresponding rate. The second term describes the decrease of the probability $\xi_i(t,\tau,\sigma)$ due to transitions of the particle to the initial position. $\lambda(\sigma)$ is the corresponding rate.

Let us suppose that at the initial time the particle is in site 0, the residence time is equal to zero, and the time elapsed since the last reset event is equal to zero. Then, the initial condition is expressed as

$$\xi_i(0,\tau,\sigma)=\delta(\tau)\delta(\sigma)\delta_{i0}. \quad (2)$$

Here, $\delta(\tau)$ is the Dirac delta function and δ_{i0} is the Kronecker symbol.

In order for the solution of Eq. (1) to be uniquely determined, it is necessary to additionally define the boundary conditions at $\tau=0$ and at $\sigma=0$. The first boundary condition should express the fact that the residence time becomes zero at the moment of particle transition from one site to another. If the particle jumps to the left and right neighboring sites with equal probability, then the boundary condition at $\tau=0$ has the form

$$\xi_i(t,0,\sigma)=\int_0^\infty \frac{\omega(\tau)}{2}[\xi_{i-1}(t,\tau,\sigma)+\xi_{i+1}(t,\tau,\sigma)]d\tau. \quad (3)$$

The second boundary condition should express the fact that the waiting time of the next reset event becomes zero at the moment of resetting. In this section, we will assume that at the moment of the reset event, the residence time also becomes zero. In this case, the boundary condition at $\sigma=0$ has the

*shkilevv@ukr.net

form

$$\xi_i(t,\tau,0) = \delta(\tau)\delta_{i0}\sum_{j=-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\lambda(\sigma)\xi_j(t,\tau,\sigma)d\sigma\, d\tau. \quad (4)$$

The CTRW model gives a rough description of reality and does not determine the described process unambiguously. As a consequence, the physical interpretation of the presented initial and boundary conditions can be different, depending on the process described. For example, in the multiple-trapping model, which reduces to the CTRW model in the transition to a contracted description [25], boundary condition (4) indicates that after resetting the particle enters the transport state. In the mean-field approximation of the random trap model, which is also reduced to the CTRW model [26], this boundary condition indicates that after resetting the particle enters any trap with equal probability.

We perform the Laplace transform in the time variable t and the Fourier transform in the spatial variable i:

$$\hat{\bar{\xi}}(u,\tau,\sigma,k) = \sum_{i=-\infty}^{\infty}\int_0^{\infty}\exp(jihk - ut)\xi_i(t,\tau,\sigma)dt, \quad (5)$$

where j is the imaginary unit and h is the lattice constant. As a result of the transformations, Eqs. (1), (3), and (4) become

$$\frac{\partial\hat{\bar{\xi}}(u,\tau,\sigma,k)}{\partial\tau} + \frac{\partial\hat{\bar{\xi}}(u,\tau,\sigma,k)}{\partial\sigma} = -[\omega(\tau)+\lambda(\sigma)+u]\hat{\bar{\xi}}(u,\tau,\sigma,k) + \delta(\tau)\delta(\sigma), \quad (6)$$

$$\hat{\bar{\xi}}(u,0,\sigma,k) = \cos(hk)\hat{\bar{F}}(u,\sigma,k), \quad (7)$$

$$\hat{\bar{\xi}}(u,\tau,0,k) = \delta(\tau)\bar{\Sigma}(u), \quad (8)$$

where

$$\hat{\bar{F}}(u,\sigma,k) = \int_0^{\infty}\omega(\tau)\hat{\bar{\xi}}(u,\tau,\sigma,k)d\tau, \quad (9)$$

$$\bar{\Sigma}(u) = \int_0^{\infty}\int_0^{\infty}\lambda(\sigma)\hat{\bar{\xi}}(u,\tau,\sigma,0)d\sigma\, d\tau. \quad (10)$$

Equation (6) with boundary conditions of Eqs. (7) and (8) has the solution

$$\hat{\bar{\xi}}(u,\tau,\sigma,k) = \delta(\tau-\sigma)[1+\bar{\Sigma}(u)] \times \exp\{-u\sigma\}\Psi_r(\sigma)\frac{\Psi_d(\tau)}{\Psi_d(\tau-\sigma)} \quad (11)$$

for $\tau \geqslant \sigma, \sigma > 0$,

$$\hat{\bar{\xi}}(u,\tau,\sigma,k) = \delta(\tau)\bar{\Sigma}(u) \quad (12)$$

for $\tau \geqslant \sigma, \sigma = 0$, and

$$\hat{\bar{\xi}}(u,\tau,\sigma,k) = \cos(hk)\hat{\bar{F}}(u,\sigma-\tau,k) \times \exp\{-u\tau\}\Psi_d(\tau)\frac{\Psi_r(\sigma)}{\Psi_r(\sigma-\tau)} \quad (13)$$

for $\tau < \sigma, \tau \geqslant 0$. Here,

$$\Psi_r(\sigma) = \exp\left\{-\int_0^{\sigma}\lambda(y)dy\right\} \quad (14)$$

is the survival probability of the reset process and

$$\Psi_d(\tau) = \exp\left\{-\int_0^{\tau}\omega(y)dy\right\} \quad (15)$$

is the survival probability of the diffusion (jump) process. Note that at $\sigma = 0$ the solution has a discontinuity, which is a consequence of the presence of the inhomogeneous term in Eq. (6).

By dividing $\hat{\bar{\xi}}(u,\tau,\sigma,k)$ by $\Psi_r(\sigma)$ and integrating with respect to τ from zero to infinity, we obtain

$$\frac{\hat{\bar{\eta}}(u,\sigma,k)}{\Psi_r(\sigma)} = [1+\bar{\Sigma}(u)]\exp\{-u\sigma\}\Psi_d(\sigma) + \cos(hk)\int_0^{\sigma}\exp\{-u\tau\}\Psi_d(\tau)\frac{\hat{\bar{F}}(u,\sigma-\tau,k)}{\Psi_r(\sigma-\tau)}d\tau, \quad (16)$$

where $\hat{\bar{\eta}}(u,\sigma,k) = \int_0^{\infty}\hat{\bar{\xi}}(u,\tau,\sigma,k)d\tau$. By performing the Laplace transform of Eq. (16) in σ ($f(\sigma) \to L[f(\sigma)](s)$), we find

$$L\left[\frac{\hat{\bar{\eta}}(u,\sigma,k)}{\Psi_r(\sigma)}\right] = [1+\bar{\Sigma}(u)]\bar{\Psi}_d(u+s) + \cos(hk)L\left[\frac{\hat{\bar{F}}(u,\sigma,k)}{\Psi_r(\sigma)}\right]\bar{\Psi}_d(u+s). \quad (17)$$

By dividing $\hat{\bar{\xi}}(u,\tau,\sigma,k)$ by $\Psi_r(\sigma)$, multiplying by $\omega(\tau)$, and integrating with respect to τ from zero to infinity, we obtain

$$\frac{\hat{\bar{F}}(u,\sigma,k)}{\Psi_r(\sigma)} = [1+\bar{\Sigma}(u)]\exp\{-u\sigma\}\psi_d(\sigma) + \cos(hk)\int_0^{\sigma}\exp\{-u\tau\}\psi_d(\tau)\frac{\hat{\bar{F}}(u,\sigma-\tau,k)}{\Psi_r(\sigma-\tau)}d\tau, \quad (18)$$

where $\psi_d(\tau) = \omega(\tau)\Psi_d(\tau)$ is the waiting time distribution of the diffusion process. The Laplace transform of this relation in σ is

$$L\left[\frac{\hat{\bar{F}}(u,\sigma,k)}{\Psi_r(\sigma)}\right] = [1+\bar{\Sigma}(u)]\bar{\psi}_d(u+s) + \cos(hk)L\left[\frac{\hat{\bar{F}}(u,\sigma,k)}{\Psi_r(\sigma)}\right]\bar{\psi}_d(u+s). \quad (19)$$

By solving Eqs. (17) and (19) and performing some simple manipulations, one can write

$$\hat{\bar{\eta}}(u,\sigma,k) = \Psi_r(\sigma)[1+\bar{\Sigma}(u)]\exp\{-u\sigma\}\hat{P}_d(\sigma,k), \quad (20)$$

where $\hat{P}_d(\sigma,k)$ is the propagator of the CTRW without resetting, which has the following Laplace transform ($\sigma \to s$):

$$\hat{\bar{P}}_d(s,k) = \frac{1}{s+\bar{\Theta}(s)[1-\cos(hk)]}. \quad (21)$$

$\bar{\Theta}(s)$ is the memory function defined as

$$\bar{\Theta}(s) = \frac{s\bar{\psi}_d(s)}{1-\bar{\psi}_d(s)}. \quad (22)$$

By substituting Eq. (20) into Eq. (10), we find

$$\bar{\Sigma}(u) = \frac{\bar{\psi}_r(u)}{1 - \bar{\psi}_r(u)}. \tag{23}$$

Here, $\bar{\psi}_r(u)$ is the Laplace transform of the waiting time distribution of the reset process $[\psi_r(\sigma) = -\frac{d\Psi_r(\sigma)}{d\sigma} = \omega(\sigma)\Psi_r(\sigma)]$. Thus, we can write the propagator of the CTRW under time-dependent resetting defined as $\hat{\bar{P}}_{dr}(u,k) = \int_0^\infty \int_0^\infty \hat{\bar{\xi}}(u,\tau,\sigma,k)d\sigma\, d\tau$ in the form

$$\hat{\bar{P}}_{dr}(u,k) = \frac{1}{1 - \bar{\psi}_r(u)} \int_0^\infty \exp\{-u\sigma\}\Psi_r(\sigma)\hat{P}_d(\sigma,k)d\sigma. \tag{24}$$

An expression similar to that was previously obtained for the Brownian motion under time-dependent resetting in Ref. [20].

In the physical domain, the propagator in Eq. (24) satisfies the first renewal equation [20]:

$$P_{dr}(t,x) = \Psi_r(t)P_d(t,x) + \int_0^t \psi_r(\tau_f)P_{dr}(t-\tau_f,x)d\tau_f. \tag{25}$$

This equation differs from the analogous equation obtained in Ref. [20] only by the fact that here the CTRW propagator is used instead of the diffusion propagator. The terms of this equation on the right-hand side have the same meaning as in Ref. [20]: the first term corresponds to the trajectories with no resets and the second term corresponds to the trajectories with at least one reset. τ_f is the time of the first reset [20].

We found the propagator corresponding to initial condition (2) and boundary conditions (3) and (4). However, the approach used here makes it possible to obtain propagators corresponding to other initial and boundary conditions. In particular, one can consider nonsymmetric random walks, resetting several points, the case when the residence time does not necessarily become zero after resetting, and other cases.

III. GOVERNING EQUATION

To obtain specific results, one must introduce an assumption about the probability of resetting. In previous works, such assumptions were introduced in various ways. In Refs. [18] and [19], the assumption was made about the function $\psi_r(\sigma)$. In Ref. [18] it was assumed that this function is a truncated power law, and in Ref. [19], it was fixed as the gamma distribution and Weibull distribution. In Ref. [20], the assumption was made about the function $\omega(\sigma)$. This function was given by some simple analytic expressions. In this study, we assume that the survival probability $\Psi_r(\sigma)$ has the form of a sum of exponentials:

$$\Psi_r(\sigma) = \sum_{i=1}^{N} \alpha_i \exp\{-\nu_i\sigma\} \tag{26}$$

with $\nu_i > 0$, $\sum_{i=1}^N \alpha_i = 1$. Such an assumption has two advantages: first, sum (26) can describe any reasonable function with any given accuracy in the interval $(0,\infty)$ [27,28]; second, it allows us to derive a governing equation for the process under consideration. Note that the function $\psi_r(\sigma)$ corresponding to Eq. (26) has the form $\psi_r(\sigma) = \sum_{i=1}^N \alpha_i\nu_i \exp\{-\nu_i\sigma\}$.

We now present the derivation of the governing equation. If the survival probability has form (26), then propagator (24) becomes

$$\hat{\bar{P}}_{dr}(u,k) = \sum_{i=1}^{N} \hat{\bar{\rho}}_i(u,k), \tag{27}$$

where functions $\hat{\bar{\rho}}_i(u,k)$ are given by

$$\hat{\bar{\rho}}_i(u,k) = \frac{1}{1 - \bar{\psi}_r(u)} \times \frac{\alpha_i}{u + \nu_i + \bar{\Theta}(u+\nu_i)[1-\cos(hk)]} \tag{28}$$

with $\bar{\psi}_r(u) = \sum_{i=1}^N \frac{\alpha_i\nu_i}{u+\nu_i}$. By direct substitution, one can verify that these functions satisfy the system of equations

$$u\hat{\bar{\rho}}_i(u,k) - \alpha_i = -\bar{\Theta}(u+\nu_i)[1-\cos(hk)]\hat{\bar{\rho}}_i(u,k) - \nu_i\hat{\bar{\rho}}_i(u,k) + \alpha_i\sum_{l=1}^N \nu_l\hat{\bar{\rho}}_l(u,0), \quad (i = 1,2,\ldots,N). \tag{29}$$

By taking the continuum limit [i.e., replacing $1-\cos(hk)$ by $\frac{h^2}{2}k^2$] and returning to the physical variables, we get

$$\frac{\partial\rho_i(t,x)}{\partial t} = \frac{h^2}{2}\int_0^t \Theta(\tau)\exp(-\nu_i\tau)\frac{\partial^2\rho_i(t-\tau,x)}{\partial x^2}d\tau - \nu_i\rho_i(t,x) + \alpha_i\delta(x)\sum_{l=1}^N \nu_l\int_{-\infty}^{\infty}\rho_l(t,x)dx, \quad (i = 1,2,\ldots,N). \tag{30}$$

The initial conditions are $\rho_i(0,x) = \alpha_i\delta(x)$.

Equation (30) is valid when the original function $\Theta(t)$ exists. However, in the model of anomalous subdiffusion (where $\bar{\Theta}(u) = \text{const} \times u^{1-\gamma}$ with $\gamma \in (0,1)$ [29,30]) it does not exist, so the integral operator is written in a different way with the use of a fractional derivative [31].

For $N = 1$, system of equations (30) reduces to the non-Markovian diffusion equation with source and sink [32]:

$$\frac{\partial\rho(t,x)}{\partial t} = \frac{h^2}{2}\int_0^t \Theta(\tau)\exp(-\nu\tau)\frac{\partial^2\rho(t-\tau,x)}{\partial x^2}d\tau - \nu\rho(t,x) + \delta(x)\nu\int_{-\infty}^{\infty}\rho(t,x)dx. \tag{31}$$

Here, the source and sink terms correspond to the resetting process. The sink term modifies the integral operator by means of appearance of an exponential factor, as in the case of the subdiffusion-reaction equations [31,32]. In the case of a homogeneous medium, the memory function $\Theta(t)$ is equal to the delta function [$\bar{\Theta}(u)$ is equal to a constant], and Eq. (31) reduces to the usual equation of diffusion with resetting [6]:

$$\frac{\partial\rho(t,x)}{\partial t} = \bar{\Theta}\frac{h^2}{2}\frac{\partial^2\rho(t,x)}{\partial x^2} - \nu\rho(t,x) + \delta(x)\nu\int_{-\infty}^{\infty}\rho(t,x)dx. \tag{32}$$

If all the parameters α_i are positive [33], they can be considered as probabilities. In this case, the system of equations (30) admits the following interpretation. A particle can be in one of N different internal states. These states influence how the particle resets. $\rho_i(t,x)$ is the probability of finding a particle in state i at time t at point x. Initially, the particle is in the ith state with probability α_i. Being in this state, it exhibits subdiffusive motion and resets with a constant rate ν_i. As a result of resetting, the particle falls into state i with probability α_i, regardless of what state it had before resetting.

Let us comment that the obtained equations can straightforwardly be extended for an arbitrary spatial dimension. In d dimensions, the second derivative is replaced by a d-dimensional Laplacian and simple integral is replaced by a d-fold integral.

IV. SURVIVAL PROBABILITY

The system of equations (30) plays a role of the master equation. It allows us to solve different boundary value problems. As an example, we compute the survival probability in the presence of the absorbing center. Let is suppose that, at the initial time, the particle is at x_0 (to be specific, we assume that $x_0 > 0$). The absorption center is at the origin. The equation for $\bar{\rho}_i(u,x)$ and its boundary condition are written as

$$u\bar{\rho}_i(u,x) - \alpha_i\delta(x-x_0) = \frac{h^2}{2}\bar{\Theta}(u+\nu_i)\frac{\partial^2\bar{\rho}_i(u,x)}{\partial x^2} - \nu_i\bar{\rho}_i(u,x) + \alpha_i\delta(x-x_0) \times \sum_{l=1}^{N}\nu_l\int_0^\infty \bar{\rho}_l(u,x)dx,$$

$$\bar{\rho}_i(u,0) = 0. \tag{33}$$

In the considered case, the probability that the particle survives until time t without being absorbed is defined as

$$Q(t,x_0) = \sum_{i=1}^{N} q_i(t,x_0), \tag{34}$$

where $q_i(t,x_0) = \int_0^\infty \rho_i(t,x;x_0)dx$. We explicitly indicated the dependence of ρ_i on the initial position x_0. It is possible to show that functions $\bar{q}_i(u,x_0)$ satisfy the backward boundary value problem (where the initial position x is the variable) [34]:

$$u\bar{q}_i(u,x) - \alpha_i = \frac{h^2}{2}\bar{\Theta}(u+\nu_i)\frac{\partial^2\bar{q}_i(u,x)}{\partial x^2} - \nu_i\bar{q}_i(u,x) + \alpha_i\sum_{l=1}^{N}\nu_l\bar{q}_l(u,x_0),$$

$$\bar{q}_i(u,0) = 0. \tag{35}$$

This problem can easily be solved. If the last term on the right-hand side is assumed to be known, we get N independent equations. The solution of the ith equation has the form

$$\bar{q}_i(u,x) = \left[1 + \sum_{l=1}^{N}\nu_l\bar{q}_l(u,x_0)\right]\alpha_i\bar{p}_i(u,x), \tag{36}$$

where

$$\bar{p}_i(u,x) = \frac{1-\exp(-\kappa_i x)}{u+\nu_i}, \quad \kappa_i = \sqrt{\frac{2(u+\nu_i)}{h^2\bar{\Theta}(u+\nu_i)}}. \tag{37}$$

By multiplying Eq. (36) by ν_i, setting $x = x_0$, and summing, we obtain the equation for the expression $\sum_{l=1}^{N}\nu_l\bar{q}_l(u,x_0)$. By solving this equation and substituting the result into Eq. (36), we obtain $\bar{q}_i(u,x_0) = \alpha_i\bar{p}_i(u,x_0)/[1-\sum_{l=1}^{N}\alpha_l\nu_l\bar{p}_l(u,x_0)]$. Thus, the Laplace transform of survival probability (34) can be expressed as

$$\bar{Q}(u,x_0) = \frac{\sum_{i=1}^{N}\alpha_i\bar{p}_i(u,x_0)}{1-\sum_{i=1}^{N}\alpha_i\nu_i\bar{p}_i(u,x_0)}. \tag{38}$$

This expression is consistent with that previously obtained for the Brownian motion under time-dependent resetting in Ref. [20] (see [35]).

The mean first-passage time, $T(x_0)$, may be obtained from $T(x_0) = \bar{Q}(u=0,x_0)$, which yields

$$T(x_0) = \frac{\sum_{i=1}^{N}\frac{\alpha_i}{\nu_i}[1-\exp(-\nu_i x_0)]}{\sum_{i=1}^{N}\alpha_i\exp(-\nu_i x_0)}, \tag{39}$$

where $\nu_i = \sqrt{\frac{2\nu_i}{h^2\bar{\Theta}(\nu_i)}}$. We note that $T(x_0)$ is finite even in the case of anomalous subdiffusion [where $\bar{\Theta}(\nu) = \text{const} \times \nu^{1-\gamma}$], despite the fact that in this case the mean residence time of the process without resetting is equal to infinity. This result is a consequence of boundary condition (4). If we consider another boundary condition—in particular, if we assume that the resetting does not affect the "age" (residence time) of the particle—then $T(x_0)$ will be infinite. Such a dependence of the results on the microscopic details of the process is characteristic of non-Markovian diffusion with reaction [36].

V. EFFECT OF ANOMALOUS SUBDIFFUSION

We now turn to the effect of anomalous subdiffusion on the shape of the stationary state. Let the survival probability $\Psi_r(t)$ have the integral representation

$$\Psi_r(t) = \int_0^\infty \alpha(\nu)\exp\{-\nu t\}d\nu \tag{40}$$

and let kernel $\alpha(\nu)$ be the gamma distribution

$$\alpha(\nu) = \frac{m^n}{\Gamma(n)}\nu^{n-1}\exp\{-m\nu\}, \quad n > 0, m > 0. \tag{41}$$

In such a case, the stationary probability distribution, $\hat{P}_{dr}^{st}$, takes the form

$$\hat{P}_{dr}^{st}(k) = \frac{m^n}{t_0\Gamma(n)}\int_0^\infty \frac{\nu^{n-1}\exp\{-m\nu\}}{\nu+\bar{\Theta}(\nu)[\cos(hk)-1]}d\nu \tag{42}$$

in the Fourier space [$\hat{P}_{dr}^{st}(k) = \lim_{u\to 0} u\hat{\bar{P}}_{dr}(u,k)$]. The mean waiting time t_0 [$t_0 = \lim_{u\to 0}\frac{1-\bar{\psi}_r(u)}{u} = \bar{\Psi}_r(u=0)$] is equal to $\frac{m^n}{\Gamma(n)}\int_0^\infty \nu^{n-2}\exp\{-m\nu\}d\nu$.

Distribution (42) exists when $t_0 < \infty$, i.e., when $n > 1$. The variance, M_2, of the corresponding real-space distribution ($M_2 = -[\partial_{kk}\hat{P}_{dr}^{st}(k)]_{k=0}$) is

$$M_2 = \frac{h^2 m^n}{t_0\Gamma(n)}\int_0^\infty \bar{\Theta}(\nu)\nu^{n-3}\exp\{-m\nu\}d\nu. \tag{43}$$

The convergence of the integral on the right is determined by the behavior of the function $\bar{\Theta}(\nu)\nu^{n-3}$ in a neighborhood of $\nu = 0$. In the case of normal diffusion, $\bar{\Theta}(\nu)$ is constant and the integral converges for $n > 2$. Hence, we have the infinite variance for $n \in (1,2]$. In the case of anomalous subdiffusion [where $\bar{\Theta}(\nu) = \text{const} \times \nu^{1-\gamma}$ with $\gamma \in (0,1)$], the integral converges for $n > 1+\gamma$. In this case, the variance is infinite for $n \in (1,1+\gamma]$. Thus, for $n \in (1+\gamma,2]$, the variance is infinite in the case of normal diffusion and finite in the case of anomalous subdiffusion; i.e., anomalous subdiffusion leads to a significant narrowing of the stationary distribution. This example illustrates the effect of diffusion slowing on the shape of the stationary state.

VI. STATIONARY STATE SPLITTING

In this section, we show that in the model under consideration, a qualitatively new effect can occur: the steady state can split into regular (continuous) and singular (sharply peaked) parts. This effect will be observed if the residence time does not become zero after resetting, as assumed in Sec. II. As an example, we will consider the case when resetting does not affect the residence time; i.e., the residence time after resetting is equal to the residence time before resetting. For simplicity, we assume that the function $\lambda(\sigma)$ is equal to a constant λ. In such a case, boundary condition (8) should be replaced by the following condition:

$$\hat{\bar{\xi}}(u,\tau,0,k) = \lambda \bar{\Sigma}(u,\tau) \tag{44}$$

where

$$\bar{\Sigma}(u,\tau) = \int_0^\infty \hat{\bar{\xi}}(u,\tau,\sigma,0)d\sigma. \tag{45}$$

The solution to Eq. (6) with boundary conditions of Eqs. (7) and (44) is

$$\hat{\bar{\xi}}(u,\tau,\sigma,k) = [\delta(\tau-\sigma) + \lambda\bar{\Sigma}(u,\tau-\sigma)] \times \exp\{-(u+\lambda)\sigma\}\frac{\Psi_d(\tau)}{\Psi_d(\tau-\sigma)} \tag{46}$$

for $\tau \geqslant \sigma, \sigma > 0$,

$$\hat{\bar{\xi}}(u,\tau,\sigma,k) = \lambda \bar{\Sigma}(u,\tau) \tag{47}$$

for $\tau \geqslant \sigma, \sigma = 0$, and

$$\hat{\bar{\xi}}(u,\tau,\sigma,k) = \cos(hk)\hat{\bar{F}}(u,\sigma-\tau,k) \times \exp\{-(u+\lambda)\tau\}\Psi_d(\tau) \tag{48}$$

for $\tau < \sigma, \tau \geqslant 0$.

As a result of calculations similar to those performed in Sec. II, we get the following expression for the propagator:

$$\hat{\bar{P}}_{dr}(u,k) = \frac{u+\lambda+[1-\cos(hk)][\bar{\Theta}(u+\lambda)-\bar{\Theta}(u)]}{u\{u+\lambda+\bar{\Theta}(u+\lambda)[1-\cos(hk)]\}}. \tag{49}$$

This expression can be represented as the sum of a k-independent term and a k-depending term. The k-independent term is expressed as

$$\bar{Q}(u) = \frac{1}{u}\left(1 - \frac{\bar{\Theta}(u)}{\bar{\Theta}(u+\lambda)}\right). \tag{50}$$

The k-dependent term has the form (in the continuum limit)

$$\hat{\bar{R}}(u,k) = \frac{(u+\lambda)\bar{\Theta}(u)}{u\bar{\Theta}(u+\lambda)}\frac{1}{u+\lambda+\frac{h^2k^2}{2}\bar{\Theta}(u+\lambda)}. \tag{51}$$

Returning to the x variable, we get the singular part of the propagator,

$$\bar{S}(u,x) = \delta(x)\bar{Q}(u), \tag{52}$$

and the regular part of the propagator,

$$\bar{R}(u,x) = \frac{\bar{\Theta}(u)}{u\bar{\Theta}(u+\lambda)}\frac{\kappa(u)}{2}\exp[-\kappa(u)|x|], \tag{53}$$

where

$$\kappa(u) = \sqrt{\frac{2(u+\lambda)}{h^2\bar{\Theta}(u+\lambda)}}. \tag{54}$$

The singular part of the propagator is the probability that the particle is in the immobile state, i.e., that it did not jump after arriving at the point $x=0$. The regular part of the propagator is the probability that the particle is in a mobile state. The realizations of random walks where the particle remains immobile do not contribute to this part of the propagator. Splitting of the propagator into the regular and singular parts is also observed in the model without resetting [37]. In this case, splitting occurs if at the initial moment the residence time is different from zero. Over time, the singular part disappears. In the case considered here, the residence time is zero at the initial moment. Splitting occurs because the residence time is different from zero after resetting. Since resetting continues all the time, the singular part of the propagator never disappears.

In the stationary state, the probability that the particle is in the immobile state (let us denote it by p_{im}) is equal to $1 - \frac{\bar{\Theta}(0)}{\bar{\Theta}(\lambda)}$. For anomalous subdiffusion, $\bar{\Theta}(0)$ is equal to 0, so p_{im} is equal to 1; i.e., the whole probability is concentrated at the point $x=0$. For transient subdiffusion, the relations $0 < \bar{\Theta}(0) < \bar{\Theta}(\lambda)$ are satisfied; therefore, the probability p_{im} is in the interval $(0,1)$. Let us show that this probability can take any value in this interval. For this, we consider the case when the function $\psi_d(\tau)$ has the form

$$\psi_d(\tau) = \alpha_1\nu_1\exp(-\nu_1\tau) + \alpha_2\nu_2\exp(-\nu_2\tau) \tag{55}$$

with

$$\nu_1,\nu_2 > 0, \quad \alpha_1 \in (0,1), \quad \alpha_2 = 1-\alpha_1. \tag{56}$$

In such a case, the function $\bar{\Theta}(u)$ has the form

$$\bar{\Theta}(u) = \frac{1}{\xi}\frac{1+a\sigma u}{1+\sigma u}, \tag{57}$$

where

$$\xi = \frac{\alpha_1}{\nu_1} + \frac{\alpha_2}{\nu_2}, \quad \sigma = \frac{1}{\nu_1\nu_2\xi}, \quad a = \xi(\alpha_1\nu_1 + \alpha_2\nu_2). \tag{58}$$

The probability p_{im} is expressed as

$$p_{im} = \frac{(a-1)\sigma\lambda}{1+a\sigma\lambda}. \tag{59}$$

The parameters ξ, σ, and a can change independently in the ranges $(0,\infty)$, $(0,\infty)$, and $(1,\infty)$, respectively [26]. Thus, from formula (59) we see that, when the parameters σ and a are varied, the probability p_{im} can take any value between 0 and 1. In addition, for any values of the parameters σ, a, and λ, the characteristic width of the regular component of the propagator

$$\frac{1}{\kappa(0)} = \sqrt{\frac{h^2(1+a\sigma\lambda)}{2\xi\lambda(1+\sigma\lambda)}}. \quad (60)$$

will be a macroscopic quantity if the parameter ξ is sufficiently small.

VII. SUMMARY

In this study, we generalized the model of diffusion under time-dependent resetting by considering medium disorder. We used the CTRW model as a model of slowing diffusion. Starting with the Markov representation of the considered process, we found a propagator. Then, we represented the distribution of the waiting time as a sum of exponentials. With such a representation, we found an equation satisfied by the propagator. This equation allows us to solve a variety of boundary value problems for diffusion under time-dependent resetting in disordered media. As an example, we calculated the mean first-passage time to a particular position. We also showed that the medium disorder has a significant impact on the shape of the stationary state. Our findings can be extended in different directions. In particular, one can consider the case when the waiting time distribution of the first diffusive jump is different from that of the second and subsequent jumps. In this case, the steady state will remain the same, but the first passage times can be changed significantly.

[1] S. Reuveni, M. Urbachm, and J. Klafter, Proc. Natl. Akad. Sci. U.S.A. **111**, 4391 (2014).
[2] E. Kussell and S. Leibler, Science **309**, 2075 (2005).
[3] G. M. Viswanathan, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, *The Physics of Foraging: An Introduction to Random Searches and Biological Encounters*, (Cambridge University Press, Cambridge, 2011).
[4] A. Montanari and R. Zecchina, Phys. Rev. Lett. **88**, 178701 (2002).
[5] S. C. Manrubia and D. H. Zanette, Phys. Rev. E **59**, 4945 (1999).
[6] M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. **106**, 160601 (2011).
[7] M. R. Evans and S. N. Majumdar, J. Phys. A: Math. Theor. **44**, 435001 (2011).
[8] J. Whitehouse, M. R. Evans, and S. N. Majumdar, Phys. Rev. E **87**, 022118 (2013).
[9] M. R. Evans and S. N. Majumdar, J. Phys. A: Math. Theor. **47**, 285001 (2014).
[10] A. Pal, Phys. Rev. E **91**, 012113 (2015).
[11] M. Montero and J. Villarroel, Phys. Rev. E **87**, 012116 (2013).
[12] C. Christou and A. Schadschneider, J. Phys. A: Math. Theor. **48**, 285003 (2015).
[13] S. N. Majumdar, S. Sabhapandit, and G. Schehr, Phys. Rev. E **92**, 052126 (2015).
[14] T. Rotbart, S. Reuveni, and M. Urbakh, Phys. Rev. E **92**, 060101(R) (2015).
[15] S. Gupta, S. N. Majumdar, and G. Schehr, Phys. Rev. Lett. **112**, 220601 (2014).
[16] X. Durang, M. Henkel, and H. Park, J. Phys. A: Math. Theor. **47**, 045002 (2014).
[17] V. Méndez and D. Campos, Phys. Rev. E **93**, 022106 (2016).
[18] A. Nagar and S. Gupta, Phys. Rev. E **93**, 060102(R) (2016).
[19] S. Eule and J. Metzger, New J. Phys. **18**, 033006 (2016).
[20] A. Pal, A. Kundu, and M. R. Evans, J. Phys. A: Math. Theor. **49**, 225001 (2016).
[21] M. O. Vlad and J. Ross, Phys. Rev. E **66**, 061908 (2002).
[22] Y. Nec, V. A. Volpert, and A. A. Nepomnyashchy, Discrete Contin. Dyn. Sysrt., Ser. A **27**, 827 (2010).
[23] S. Fedotov, H. Al-Shamsi, A. Ivanov, and A. Zubarev, Phys. Rev. E **82**, 041103 (2010).
[24] Equation (1) follows from the obvious relation $\xi_i(t+\Delta,\tau+\Delta,\sigma+\Delta)=\xi_i(t,\tau,\sigma)-\omega(\tau)\xi_i(t,\tau,\sigma)\Delta-\lambda(\sigma)\xi_i(t,\tau,\sigma)\Delta+o(\Delta)$. By adding and subtracting the quantities $\xi_i(t,\tau+\Delta,\sigma+\Delta)$ and $\xi_i(t,\tau,\sigma+\Delta)$ on the left-hand side, subtracting $\xi_i(t,\tau,\sigma)$ from both sides, dividing by Δ, and letting Δ go to zero, we obtain the required result.
[25] J. W. Haus and K. W. Kehr, Phys. Rep **150**, 263 (1987).
[26] V. P. Shkilev, Exp. Theor. Phys. (JETP) **101**, 562 (2005).
[27] R. W. Hamming, *Numerical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1962).
[28] A. Feldmann and W. Whitt, Perform. Eval. **31**, 245 (1998).
[29] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[30] *Anomalous Transport: Foundations and Applications*, edited by R. Klages, G. Radons, and I. M. Sokolov (Wiley-VCH, Weinheim, 2008).
[31] I. M. Sokolov, M. G. W. Schmidt, and F. Sagués, Phys. Rev. E **73**, 031102 (2006).
[32] A. Yadav and W. Horsthemke, Phys. Rev. E **74**, 066118 (2006).
[33] All parameters α_i are positive if and only if the function $\Psi_r(\sigma)$ is completely monotone [28].
[34] By multiplying the ith equation of system (33) by $\bar{q}_i(u,x)$ and integrating from $x=0$ to $x=\infty$, one can see that the functions $\bar{q}_i(u,x_0)$ are equal to the integrals $\int_0^\infty \bar{\rho}_i(u,x;x_0)dx$, provided that these functions satisfy boundary value problem (35).
[35] With a constant $\bar{\Theta}$ (this is the case for Brownian motion) and a function $\Psi_r(t)$ of form (26), the numerator of (38) [let us denote it by $\bar{q}_r(u,x_0)$] can be written as $\bar{q}_r(u,x_0)=\int_0^\infty \exp(-ut)\Psi_r(t)q(t,x_0)dt$, where $q(t,x_0)=\mathrm{erf}[\frac{x_0}{\sqrt{4Dt}}]$ is the survival probability of a free Brownian particle with the diffusion constant $D=\bar{\Theta}h^2/2$. The denominator can be expressed as $u\bar{q}_r(u,x_0)-\bar{k}_r(u,x_0)$, where $\bar{k}_r(u,x_0)=\int_0^\infty \exp(-ut)\Psi_r(t)\frac{d}{dt}q(t,x_0)dt$. This coincides with the result obtained in Ref. [20].
[36] V. P. Shkilev, J. Exp. Theor. Phys. (JETP) **112**, 711 (2011).
[37] J. H. P. Schulz, E. Barkai, and R. Metzler, Phys. Rev. X **4**, 011028 (2014).