Linear and nonlinear response of the Vlasov system with nonintegrable Hamiltonian

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Linear and nonlinear response formulas taking into account all Casimir invariants are derived without use of angle-action variables of a single-particle (mean-field) Hamiltonian. This article deals mainly with the Vlasov system in a spatially inhomogeneous quasistationary state whose associating single-particle Hamiltonian is not integrable and has only one integral of the motion, the Hamiltonian itself. The basic strategy is to restrict the form of perturbation so that it keeps Casimir invariants within a linear order, and the single particle's probabilistic density function is smooth with respect to the single particle's Hamiltonian. The theory is applied for a spatially two-dimensional system and is confirmed by numerical simulations. A nonlinear response formula is also derived in a similar manner.

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I. INTRODUCTION

Long-range interaction systems show several phenomena which are out of scope of the equilibrium statistical mechanics [1,2]. One of them is that such a system is likely trapped in out-of-equilibrium quasistationary states (QSSs) whose duration increases with the number N of elements in a system and diverges when the large population limit $N \to \infty$ is taken [1–5]. Then, if the system of interest is huge enough, the relaxation time is so long that one cannot see the thermal equilibrium state. It is hence interesting to investigate the nonequilibrium statistical mechanics or thermodynamics of QSSs. In particular, the focus of this article is on the effect of external forces in the QSSs.

When N is huge enough, the temporal evolution of the long-range interaction system is well described by the Vlasov equation [6–8] (also called the collisionless Boltzmann equation [3]), which describes the evolution of a density function defined on a μ space, a single-particle phase space. The QSSs are interpreted as stable stationary solutions to the Vlasov equation [4]. The Vlasov equation has unique solutions for each given initial state [7,8], and thus the QSSs depend not only on macroscopic variables such as temperature and energy but also on mesoscopic things, details of the single-particle density function. Thus the study on responses to external forces in QSSs should be based on the Vlasov equation.

The linear response theory for the Vlasov systems has been developed for stability analysis in self-gravitating systems [3], for looking into plasma responses in magnetically confined plasmas [9], for computing the time asymptotic response to the external forces in both spatially homogeneous [10] and inhomogeneous [11] QSSs, and for a kind of fluid systems [12]. By use of this theory, critical phenomena in QSSs [13] are investigated, and some information of unforced systems is extracted by observing responses to oscillating external forces [11]. Furthermore, the nonlinear response theory has been developed to investigate the response to the finite size external forces in QSSs near or on the critical point in which the linear response theory does not work [14,15]. These response formulas have been derived only when the single-particle effective Hamiltonian is integrable and the angle-action variables are used for solving test particle dynamics.

Analyzing the linearized equation with the nonintegrable effective Hamiltonian is practically important, because systems in multidimensional spaces are more realistic (for example, self-gravitating systems in the three-dimensional (3D) space [3], and magnetically confined hot plasmas [9]) and their effective Hamiltonians are nonintegrable in general. To tackle this problem, one method to take into account constraints *a posteriori* is proposed to obtain linear response formulas approximately [16]. This method provides canonical (taking into account the normalization) and microcanonical (taking into account the normalization and the energy conservation) linear responses and other kinds of linear responses with a finite number of constraints systematically. However, one cannot obtain the linear response of the isolated systems with this method because it is practically impossible to take into account infinitely many Casimir constraints with this method. The same problem lies in the stability analysis of the Vlasov equation [17]. If the effective Hamiltonian is not integrable, it is impossible to obtain the precise stability criterion with a finite time step procedure in general, since one should obtain an infinite number of Lagrangian multipliers associated with the Casimir constraints [18].

The above problems in the linearized equation should be solved as the first step to understand the dynamics around the QSSs with nonintegrable effective Hamiltonian. After that, we are able to continue tackling more difficult problems on nonlinear Landau damping, nonlinear stability, nonlinear response, critical phenomena and their universality, and finite N effects.

In this article, we first obtain the linear response of Vlasov systems in multidimensional spaces without solving the linearized Vlasov equation and without using the angle-action variables. Let an initial state without external field be $f_0(\boldsymbol{q},\boldsymbol{p})$ and let a final state be $f_h(\boldsymbol{q},\boldsymbol{p})$ after exerting an external field h. The linear response is obtained by restricting the form of accessible perturbation by assuming smoothness of f_h with respect to h, and by taking into account the constraint conditions that the perturbation should be on a tangent "plane" of a constraint surface at f_0.

Furthermore, we mention the nonlinear response formula [14] derived via the transient (T) linearization method

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developed by Lancellotti and Dorning [19–21] to analyze the plasma oscillation and nonlinear Landau damping. In this theory, the Vlasov equation is linearized around an "unknown" asymptotic stationary state. Solving this equation, we obtain a self-consistent equation determining the asymptotic state. The asymptotic solution is obtained by redistributing the initial density function along iso asymptotic-effective-Hamiltonian sets, and this formula is called the rearrangement formula [22]. The same formula is also derived for predicting the QSSs in a one-dimensional (1D) system [23] and a 3D self-gravitating system [24] via another consideration when nonstationary initial states satisfy a (generalized) viral condition and there is no parametric resonance. It is derived that the rearrangement formula keeps Casimir invariants at the order of T linearization. Then, the nonlinear response formula is derived in a similar manner to derive the linear response formula in this article.

This article is organized as follows: The model and the dynamics in a mean-field limit $N \to \infty$ are first introduced in Sec. II, and the explicit form of the constraint condition coming from Casimir invariants is derived in Sec. III. Based on this constraint condition, the linear response formula is derived in Sec. IV and several examples are exhibited in Sec. V. We derive the nonlinear response formula, make a brief comment on problems in the T-linearized Vlasov equation for spatially multidimensional systems in Sec. VI, and summarize this article in Sec. VII.

II. MODEL AND ITS DYNAMICS

A. Model and Vlasov equation

Let us consider a system with long-range interaction whose Hamiltonian is

$$H_N = \sum_{i=1}^{N} \frac{\|\boldsymbol{p}_i\|^2}{2} + \frac{1}{2N}\sum_{i,j=1}^{N} V(\boldsymbol{q}_i - \boldsymbol{q}_j) + h(t)\sum_{i=1}^{N}\Phi(\boldsymbol{q}_i), \tag{1}$$

where $\boldsymbol{q}_i$ denotes configuration of the ith particle, $\boldsymbol{p}_i$ its conjugate momentum, V the interparticle (intersite) potential, and $h(t)\Phi(\boldsymbol{q}_i)$ the interaction between the external field h and the ith particle. Taking the mean-field limit $N \to \infty$ [6–8], the temporal evolution of this system can be described in terms of the single-particle density function $f(\boldsymbol{q},\boldsymbol{p},t)$ which is a solution to the Vlasov equation,

$$\frac{\partial f}{\partial t} + \{\mathscr{H}[f], f\} = 0, \tag{2}$$

where $\mathscr{H}[f]$ is an effective single-particle Hamiltonian,

$$\mathscr{H}[f] = \frac{\|\boldsymbol{p}\|^2}{2} + \mathscr{V}[f](\boldsymbol{q}) + h(t)\Phi(\boldsymbol{q}), \tag{3}$$

and $\{a,b\}$ is the Poisson bracket given by

$$\{a,b\} = \frac{\partial a}{\partial \boldsymbol{p}}\cdot\frac{\partial b}{\partial \boldsymbol{q}} - \frac{\partial a}{\partial \boldsymbol{q}}\cdot\frac{\partial b}{\partial \boldsymbol{p}}. \tag{4}$$

The system is initially in a QSS, f_0, and the effective Hamiltonian

$$\mathscr{H}_0(\boldsymbol{q},\boldsymbol{p}) = \mathscr{H}[f_0](\boldsymbol{q},\boldsymbol{p}) = \frac{\|\boldsymbol{p}\|^2}{2} + \mathscr{V}[f_0](\boldsymbol{q}) \tag{5}$$

has only one integral of motion, $\mathscr{H}_0$ itself. The f_0 is expressed as $f_0(\boldsymbol{q},\boldsymbol{p}) = F_0(\mathscr{H}_0(\boldsymbol{q},\boldsymbol{p}))$ by use of a monotonically decreasing function F_0. This assumption is reasonable when we are interested in the asymptotic behavior of perturbations around the formally stable solutions to the Vlasov equation [4,5,17,25]. The formal stability is defined in terms of positive or negative definiteness of the second variation of an invariant functional around f_0 which is a solution to the optimization problem:

$$\begin{aligned}
\text{maximizing} \quad & \mathscr{S}[f] = \iint s(f)\, d\boldsymbol{q}\, d\boldsymbol{p},\\
\text{subject to} \quad & 1 = \mathscr{N}[f] = \iint f\, d\boldsymbol{q}\, d\boldsymbol{p},\\
& E = \mathscr{E}[f] = \iint \frac{\|\boldsymbol{p}\|^2}{2} f\, d\boldsymbol{q}\, d\boldsymbol{p}\\
& \quad + \frac{1}{2}\iint \mathscr{V}[f] f\, d\boldsymbol{q}\, d\boldsymbol{p},
\end{aligned} \tag{6}$$

where s is a convex function. A formally stable solution is linearly stable [25]. By solving the optimization problem, we have a solution,

$$f_0(\boldsymbol{q},\boldsymbol{p}) = (s')^{-1}(\beta\mathscr{H}_0 + \alpha) \equiv F_0(\mathscr{H}_0), \tag{7}$$

where s' denotes $ds(x)/dx$, and α and β are Lagrangian multipliers with respect to the normalization and the energy conservation. Since s is convex, then the inverse of its first derivative, $(s')^{-1}$, is a strictly decreasing function. The parameter β must be positive.

B. Linear response

The external field $h(t)$ is turned on and it converges to a constant $h(t) \to h$ as $t \to \infty$. In previous studies [10–12] the asymptotic linear response δf is obtained by solving the linearized Vlasov equation around f_0,

$$\frac{\partial g_{\rm p}}{\partial t} + \{\mathscr{H}_0, g_{\rm p}\} + \{\mathscr{V}[g_{\rm p}] + h(t)\Phi, f_0\} = 0,$$

where $g_{\rm p}(t) \sim O(h)$ is a perturbation around f_0, and by taking the limit, $\delta f = \lim_{t\to\infty} g_{\rm p}(t)$. The angle-action variables of the Hamiltonian $\mathscr{H}_0$ are necessary to solve the linearized Vlasov equation, but it is impossible in general for multidimensional systems. To avoid this problem, we focus on constraint conditions restricting a form of perturbations, and we obtain the linear response δf without solving the linearized Vlasov equation.

III. CASIMIR INVARIANTS

We assume that f and its derivatives converge to zero rapidly enough as $\|\boldsymbol{p}\| \to \infty$. Furthermore, f and its derivatives are assumed to vanish on the boundary of the spatial domain or the system has a periodic boundary condition with respect to $\boldsymbol{q}$. Under these assumptions, it is shown (see Appendix A) that the Vlasov equation keeps values of Casimir functionals,

$$\mathscr{C}[f] = \iint c(f(\boldsymbol{q},\boldsymbol{p},t))\, d\boldsymbol{q}\, d\boldsymbol{p}, \tag{8}$$

for any smooth function c. The linearized Casimir conservation condition is expressed so that the accessible perturbation δf

satisfies

$$\iint c'(f_0(\boldsymbol{q},\boldsymbol{p}))\delta f(\boldsymbol{q},\boldsymbol{p})d\boldsymbol{q}d\boldsymbol{p}=0, \tag{9}$$

where $c'(x)=dc/dx$ for any smooth function c. Since $f_0(\boldsymbol{q},\boldsymbol{p})=F_0(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))$, F_0 is a monotonically decreasing function, and c is chosen arbitrarily, the constraint condition (9) is equivalent to the condition

$$0=\iint R(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))\delta f(\boldsymbol{q},\boldsymbol{p})\,d\boldsymbol{q}\,d\boldsymbol{p}=\iint R(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))\langle\delta f\rangle_{\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})}\,d\boldsymbol{q}\,d\boldsymbol{p} \tag{10}$$

for any function R [17]. The second equality is shown as follows:

$$\begin{aligned}\iint R(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))\langle\delta f\rangle_{\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})}\,d\boldsymbol{q}\,d\boldsymbol{p}&=\iint R(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))\frac{\iint\delta f(\boldsymbol{q}',\boldsymbol{p}')\delta(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})-\mathcal{H}_0(\boldsymbol{q}',\boldsymbol{p}'))\,d\boldsymbol{q}'d\boldsymbol{p}'}{\iint\delta(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})-\mathcal{H}_0(\boldsymbol{q}'',\boldsymbol{p}''))\,d\boldsymbol{q}''d\boldsymbol{p}''}d\boldsymbol{q}\,d\boldsymbol{p}\\&=\iint R(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))\left[\iint\frac{\delta f(\boldsymbol{q}',\boldsymbol{p}')\delta(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})-\mathcal{H}_0(\boldsymbol{q}',\boldsymbol{p}'))}{S(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))}\,d\boldsymbol{q}'d\boldsymbol{p}'\right]d\boldsymbol{q}\,d\boldsymbol{p}\\&=\iint\delta f(\boldsymbol{q}',\boldsymbol{p}')\left[\iint\frac{R(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))\delta(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})-\mathcal{H}_0(\boldsymbol{q}',\boldsymbol{p}'))}{S(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))}\,d\boldsymbol{q}d\boldsymbol{p}\right]d\boldsymbol{q}'\,d\boldsymbol{p}'\\&=\iint\delta f(\boldsymbol{q}',\boldsymbol{p}')\frac{R(\mathcal{H}_0(\boldsymbol{q}',\boldsymbol{p}'))S(\mathcal{H}_0(\boldsymbol{q}',\boldsymbol{p}'))}{S(\mathcal{H}_0(\boldsymbol{q}',\boldsymbol{p}'))}\,d\boldsymbol{q}'\,d\boldsymbol{p}'\\&=\iint R(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))\delta f(\boldsymbol{q},\boldsymbol{p})\,d\boldsymbol{q}\,d\boldsymbol{p},\end{aligned} \tag{11}$$

where S denotes a volume of the iso-$\mathcal{H}_0$ set,

$$S(E)=\iint\delta(E-\mathcal{H}_0(\boldsymbol{q}',\boldsymbol{p}'))\,d\boldsymbol{q}'\,d\boldsymbol{p}'. \tag{12}$$

More generally it is possible to show as in the 1D case [14] that

$$\iint\langle a\rangle_{\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})}b(\boldsymbol{q},\boldsymbol{p})\,d\boldsymbol{q}\,d\boldsymbol{p}=\iint a(\boldsymbol{q},\boldsymbol{p})\langle b\rangle_{\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})}\,d\boldsymbol{q}\,d\boldsymbol{p} \tag{13}$$

for the functions a and b when $\langle a\rangle_{\mathcal{H}_0}b$ and $a\langle b\rangle_{\mathcal{H}_0}$ are integrable. Thus, it has been shown that condition (10) is equivalent to

$$\langle\delta f\rangle_{\mathcal{H}_0}=0,\quad\text{for almost every }(\boldsymbol{q},\boldsymbol{p}). \tag{14}$$

IV. LINEAR RESPONSE FORMULA

A. Implicit form of linear response

After the external field is exerted and the limit $t\to\infty$ is taken, the effective Hamiltonian becomes

$$\mathcal{H}_h(\boldsymbol{q},\boldsymbol{p})=\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p})+\delta\mathcal{V}(\boldsymbol{q})+h\Phi(\boldsymbol{q})+O(h^2), \tag{15}$$

where the linear response $\delta\mathcal{V}\equiv\mathcal{V}[\delta f]$. Let the initial and final states be denoted, respectively, as

$$\begin{aligned}f_0(\boldsymbol{q},\boldsymbol{p})&=F_0(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))=\frac{G_0(\mathcal{H}_0(\boldsymbol{q},\boldsymbol{p}))}{\langle G_0\rangle_\mu},\\f_h(\boldsymbol{q},\boldsymbol{p})&=F_h(\mathcal{H}_h(\boldsymbol{q},\boldsymbol{p}))=\frac{G_h(\mathcal{H}_h(\boldsymbol{q},\boldsymbol{p}))}{\langle G_h\rangle_\mu},\end{aligned} \tag{16}$$

where $\langle a\rangle_\mu=\iint a\,d\boldsymbol{q}\,d\boldsymbol{p}$ and

$$\begin{aligned}G_h(\mathcal{H}_h)&=G_0(\mathcal{H}_h)+hG_1(\mathcal{H}_h)+O(h^2)\\&=G_0(\mathcal{H}_0)+G_0'(\mathcal{H}_0)(\mathcal{H}_h-\mathcal{H}_0)\\&\quad+hG_1(\mathcal{H}_0)+O(h^2).\end{aligned} \tag{17}$$

Expanding f_h around f_0, we have

$$\begin{aligned}f_h=f_0&+\frac{G_0'(\mathcal{H}_0)(\delta\mathcal{V}+h\Phi)+G_1(\mathcal{H}_0)}{\langle G_0(\mathcal{H}_0)\rangle_\mu}\\&-\frac{\langle G_0'(\mathcal{H}_0)(\delta\mathcal{V}+h\Phi)+G_1(\mathcal{H}_0)\rangle_\mu G_0(\mathcal{H}_0)}{\langle G_0(\mathcal{H}_0)\rangle_\mu^2}\\&+O(h^2).\end{aligned} \tag{18}$$

We then obtain the linear response,

$$\begin{aligned}\delta f\equiv F_0'(\mathcal{H}_0)(\delta\mathcal{V}+h\Phi)&+\frac{G_1(\mathcal{H}_0)}{\langle G_0(\mathcal{H}_0)\rangle_\mu}\\&-\langle F_0'(\mathcal{H}_0)(\delta\mathcal{V}+h\Phi)\rangle_\mu F_0(\mathcal{H}_0)\\&-\frac{\langle G_1(\mathcal{H}_0)\rangle_\mu F_0(\mathcal{H}_0)}{\langle G_0(\mathcal{H}_0)\rangle_\mu},\end{aligned} \tag{19}$$

by taking the linear order. The function G_1 is determined so that $\langle\delta f\rangle_{\mathcal{H}_0}=0$; that is,

$$\begin{aligned}\frac{G_1}{\langle G_0\rangle_\mu}&=\frac{\langle G_1\rangle_\mu}{\langle G_0\rangle_\mu}F_0-F_0'(\mathcal{H}_0)\langle\delta\mathcal{V}+h\Phi\rangle_{\mathcal{H}_0}\\&\quad+F_0(\mathcal{H}_0)\langle F_0'(\mathcal{H}_0)(\delta\mathcal{V}+h\Phi)\rangle_\mu,\end{aligned} \tag{20}$$

where $F_0'(\mathcal{H}_0)=G_0'(\mathcal{H}_0)/\langle G_0(\mathcal{H}_0)\rangle_\mu$. The response is therefore implicitly given by

$$\begin{aligned}\delta f&=F_0'(\mathcal{H}_0)(\delta\mathcal{V}(\boldsymbol{q})-\langle\delta\mathcal{V}(\boldsymbol{q})\rangle_{\mathcal{H}_0})\\&\quad+hF_0'(\mathcal{H}_0)(\Phi(\boldsymbol{q})-\langle\Phi(\boldsymbol{q})\rangle_{\mathcal{H}_0}).\end{aligned} \tag{21}$$

Solving the implicit linear response formula (21) by using a biorthogonal basis, we obtain explicitly the linear response taking into account the constraint conditions.

We make a comment on the case that there exist two integrals and f_0 depends on both of them. Let $\mathscr{L} = \iint L(\boldsymbol{q},\boldsymbol{p}) f\, d\boldsymbol{q}\, d\boldsymbol{p}$ be an additional integral (the angular momentum density, for example). We consider the optimization problem (6) and we add the additional constraint $\mathscr{L} = \text{const}$ to Eq. (6). A solution f_0 depends on $\mathscr{H}_0$ and L as $f_0 = F_0(\beta\mathscr{H}_0 + \nu L)$, where β and ν are Lagrangian multipliers. Thus, the accessible perturbation satisfies

$$\langle \delta f(\boldsymbol{q},\boldsymbol{p})\rangle_{(\beta\mathscr{H}_0(\boldsymbol{q},\boldsymbol{p})+\nu L(\boldsymbol{q},\boldsymbol{p}))} = 0, \tag{22}$$

where the bracket $\langle\cdot\rangle_{(\beta\mathscr{H}_0(\boldsymbol{q},\boldsymbol{p})+\nu L(\boldsymbol{q},\boldsymbol{p}))}$ means the average taken over the iso-$(\beta\mathscr{H}_0 + \nu L)$ set. It should be noted that a form of constrained perturbation depends on how f_0 depends on $\mathscr{H}_0$ and L. We should find ways to restrict the perturbation form for each stationary state.

If $\mathscr{H}_0$ has three independent integrals of motion, one can use angle-action variables and can solve the linearized Vlasov equation.

B. Explicit form of linear response

We introduce the biorthogonal basis [3,26,27], $\{d_i(\boldsymbol{q})\}_{i\in\mathbb{I}}$ and $\{u_i(\boldsymbol{q})\}_{i\in\mathbb{I}'}$, where the sets $\mathbb{I}'$ and $\mathbb{I}$ satisfy $\mathbb{I}' \subset \mathbb{I} \subset \mathbb{Z}$. A perturbation of spatial density is spanned by the base $\{d_i\}_i$,

$$\delta\rho(\boldsymbol{q}) = \int \delta f(\boldsymbol{q},\boldsymbol{p}) d\boldsymbol{p} = \sum_{i\in\mathbb{I}} a_i d_i(\boldsymbol{q}). \tag{23}$$

The base $\{u_i\}_i$ is introduced as

$$u_i(\boldsymbol{q}) \equiv (V * d_i)(\boldsymbol{q}) \equiv \int V(\boldsymbol{q}-\boldsymbol{q}') d_i(\boldsymbol{q}') d\boldsymbol{q}', \tag{24}$$

and the orthogonal relation

$$\int d_i(\boldsymbol{q}) \bar{u}_j(\boldsymbol{q}) d\boldsymbol{q} = \lambda_j \delta_{ij} \tag{25}$$

holds, where $\lambda_i \neq 0$ when $i \in \mathbb{I}'$ and it vanishes otherwise, and δ_{ij} is the Kronecker delta. The overbar denotes complex conjugate. Integrating the terms including δf or $\delta\mathscr{V}$ in Eq. (21) with respect to $\boldsymbol{p}$, we have

$$\begin{aligned}\int \delta f\, d\boldsymbol{p} &- \int F_0'(\mathscr{H}_0)(\delta\mathscr{V}(\boldsymbol{q}) - \langle\delta\mathscr{V}(\boldsymbol{q})\rangle_{\mathscr{H}_0}) d\boldsymbol{p} \\ &= \sum_{i\in\mathbb{I}} a_i \left[d_i - \int F_0'(\mathscr{H}_0)(u_i(\boldsymbol{q}) - \langle u_i(\boldsymbol{q})\rangle_{\mathscr{H}_0}) d\boldsymbol{p} \right]. \end{aligned} \tag{26}$$

Multiplying both sides by $\bar{u}_j$ and integrating them with respect to $\boldsymbol{q}$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}'} a_i &\left[\lambda_j\delta_{ij} - \int F_0'(\mathscr{H}_0)(u_i\bar{u}_j - \langle u_i\rangle_{\mathscr{H}_0}\langle\bar{u}_j\rangle_{\mathscr{H}_0}) d\boldsymbol{p} d\boldsymbol{q}\right] \\ &= \sum_{i\in\mathbb{I}'}\left[\lambda_j\delta_{ji} - \int F_0'(\mathscr{H}_0)(\bar{u}_j u_i - \langle\bar{u}_j\rangle_{\mathscr{H}_0}\langle u_i\rangle_{\mathscr{H}_0}) d\boldsymbol{p} d\boldsymbol{q}\right] a_i. \end{aligned} \tag{27}$$

Let $\mathbf{F} = (\mathbf{F}_{ji})_{(i,j)\in\mathbb{I}'\times\mathbb{I}'}$ be a matrix whose elements are given by

$$\mathbf{F}_{ji} = \int F_0'(\mathscr{H}_0)(\bar{u}_j u_i - \langle\bar{u}_j\rangle_{\mathscr{H}_0}\langle u_i\rangle_{\mathscr{H}_0}) d\boldsymbol{p}\, d\boldsymbol{q}. \tag{28}$$

We further assume that the term coupling with external force can be expanded as

$$\Phi(\boldsymbol{q}) = \sum_{i\in\mathbb{I}'} b_i u_i(\boldsymbol{q}). \tag{29}$$

We then have

$$\begin{aligned} h &\int F_0'(\mathscr{H}_0)(\Phi(\boldsymbol{q}) - \langle\Phi(\boldsymbol{q})\rangle_{\mathscr{H}_0}) d\boldsymbol{p} \\ &= h\sum_{i\in\mathbb{I}'} b_i \int F_0'(\mathscr{H}_0)(u_i(\boldsymbol{q}) - \langle u_i(\boldsymbol{q})\rangle_{\mathscr{H}_0}) d\boldsymbol{p}, \end{aligned} \tag{30}$$

by integrating the term coming from the external force in Eq. (21) with respect to $\boldsymbol{p}$. Multiplying it by $\bar{u}_j$ and integrating with respect to $\boldsymbol{q}$, we have (as we have already done)

$$h\sum_{i\in\mathbb{I}'} \mathbf{F}_{ji} b_i. \tag{31}$$

Combining Eqs. (27) and (31), we get the linear equation determining $\{a_i\}_{i\in\mathbb{I}'}$,

$$\sum_{j\in\mathbb{I}'} (\lambda_j\delta_{ij} - \mathbf{F}_{ij}) a_j = h\sum_{j\in\mathbb{I}'} \mathbf{F}_{ij} b_j. \tag{32}$$

Introducing the symbols $\mathbf{x} = (x_i)_{i\in\mathbb{I}'}$ for $x = a,b$ and $\Lambda = \mathrm{diag}(\lambda_i)_{i\in\mathbb{I}'}$, we can simplify the equation as follows:

$$(\mathbf{1} - \Lambda^{-1}\mathbf{F})\mathbf{a} = h\Lambda^{-1}\mathbf{F}\mathbf{b}, \tag{33}$$

and it is solved as

$$\mathbf{a} = h(\mathbf{1} - \mathbf{F})^{-1}\mathbf{F}\mathbf{b} = h\mathbf{D}^{-1}(1 - \mathbf{D})\mathbf{b}, \tag{34}$$

where $\mathbf{1}$ denotes the unit matrix and $\mathbf{D} = \mathbf{1} - \Lambda^{-1}\mathbf{F}$. The maximal eigenvalue of $\mathbf{D}$ is zero when f_0 might be marginally stable, and corresponds to the critical point. When we apply Eq. (34) to 1D systems, this explicit response formula formally coincides with what is derived in Ref. [11].

V. EXAMPLES: HAMILTONIAN MEAN-FIELD MODELS

A. One-dimensional case

Let us examine the proposed theory by use of the Hamiltonian mean-field (HMF) model whose Hamiltonian is

$$H = \sum_{i=1}^N \frac{p_i^2}{2} - \frac{1}{2N}\sum_{i\neq j}\cos(q_i - q_j) - h\sum_{i=1}^N \cos q_i, \tag{35}$$

where h is an external field, and $p_i \in \mathbb{R}$, $q_i \in [-\pi,\pi)$ for $i = 1,2,\ldots,N$. In the equilibrium state, this model shows second order phase transition at the temperature $T = 0.5$, where the Boltzmann's constant $k_{\mathrm{B}} = 1$. By use of the linear and nonlinear response formulas, the critical phenomena in the equilibrium state and QSSs for the isolated HMF model are investigated and it is shown that the Casimir constraints bring about the nonclassical critical exponents [13–15]. We here check that the present method yields the linear response formula same with the previously obtained one for the 1D HMF model. The effective Hamiltonian is

$$\mathscr{H} = \frac{p^2}{2} - \iint \cos(q - q') f(q',p',t) dq' dp' - h \cos q. \tag{36}$$

Applying Eq. (21) for the HMF model, we can derive the linear response formula

$$\delta f = (-\delta M - h) F_0'(\mathscr{H}_0)(\cos q - \langle \cos q \rangle_{\mathscr{H}_0}), \tag{37}$$

where $\delta M = \iint \cos q \delta f \, dq \, dp$. Multiplying both sides by $\cos q$ and integrating over the μ space, we obtain the linear response as

$$\delta M = \frac{1 - D}{D} h, \tag{38}$$

where

$$D = 1 + \iint F'(\mathscr{H}_0)\big(\cos^2 q - \langle \cos q \rangle_{\mathscr{H}_0}^2\big) dq \, dp. \tag{39}$$

This is equivalent to the linear response formula obtained in Ref. [13].

For more general 1D systems it is obvious that Eq. (21) is equivalent to the linear order of the nonlinear response formula derived in Ref. [15].

B. Two-dimensional case

We next examine our theory in the two-dimensional (2D) system whose Hamiltonian is

$$\begin{aligned} H = \sum_{i=1}^{N} \frac{\|\boldsymbol{p}_i\|^2}{2} - h_x \sum_{i=1}^{N} \cos x_i - h_y \sum_{i=1}^{N} \cos y_i \\ - \frac{1}{2N} \sum_{i \neq j} [\cos(x_i - x_j) + \cos(y_i - y_j) \\ + \cos(x_i - x_j)\cos(y_i - y_j)], \end{aligned} \tag{40}$$

where $\boldsymbol{q}_i = (x_i, y_i) \in [-\pi,\pi)^2$ for $i = 1,2,\ldots,N$ [28,29]. Let us assume the initial QSS f_0 with $h_x = h_y = 0$ is even with respect to both x and y. Then, the effective Hamiltonian is

$$\begin{aligned} \mathscr{H}_0 &= \frac{p_x^2 + p_y^2}{2} + \mathcal{V}(x,y), \\ \mathcal{V}(x,y) &= -M_x \cos x - M_y \sin y - P_{\rm cc} \cos x \cos y, \end{aligned} \tag{41}$$

where

$$\begin{aligned} M_x &= \iint \cos x \rho(\boldsymbol{q}) d\boldsymbol{q}, \quad M_y = \iint \cos y \rho(\boldsymbol{q}) d\boldsymbol{q}, \\ P_{\rm cc} &= \iint \cos x \cos y \rho(\boldsymbol{q}) d\boldsymbol{q}, \end{aligned} \tag{42}$$

and where $\rho(\boldsymbol{q}) = \iint f(\boldsymbol{q},\boldsymbol{p}) d\boldsymbol{p}$. The effective potential is shown in Fig. 1.

To compute the linear response of the macroscopic observables M_x, M_y, and $P_{\rm cc}$ to the external field $h_x = h_y = h$, it is necessary to compute $\langle \cos x \rangle_{\mathscr{H}_0}$, $\langle \cos y \rangle_{\mathscr{H}_0}$, and $\langle \cos x \cos y \rangle_{\mathscr{H}_0}$. We here set $M_x = M_y = M$ and $P_{\rm cc} = P$.

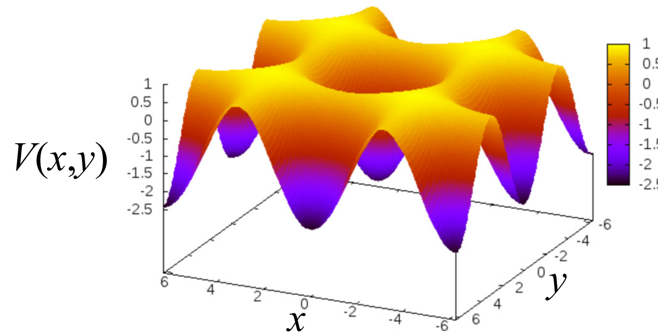

FIG. 1. The effective potential of the Hamiltonian (41). A minimum point is at (0,0) and min $\mathcal{V} = -2M - P$. Saddle points are at $(\pm\pi,0)$ and $(0,\pm\pi)$ and $\mathcal{V} = P$. A maximum point is on $(\pm\pi,\pm\pi)$ and max $\mathcal{V} = 2M - P$, where $M > P > 0$.

For any smooth function g depending only on $\boldsymbol{q}$, $\langle g \rangle_E$ is expressed as follows (see derivation in Appendix B):

$$\begin{aligned} \langle g \rangle_E &= \int_{\mathbf{R}^2} d\boldsymbol{p} \int_{[-\pi,\pi)^2} g(\boldsymbol{q}) \delta(\mathscr{H}_0(\boldsymbol{q},\boldsymbol{p}) - E) d\boldsymbol{q} \\ &= 2\pi \int_{[-\pi,\pi)^2} g(\boldsymbol{q}) \Theta(E - \mathcal{V}(x,y)) d\boldsymbol{q}. \end{aligned} \tag{43}$$

Since $\mathscr{H}_0$ is even with respect to both x and y, we have

$$\begin{aligned} \langle \sin x \rangle_{\mathscr{H}_0} &= \langle \sin y \rangle_{\mathscr{H}_0} = \langle \sin x \cos y \rangle_{\mathscr{H}_0} \\ &= \langle \sin x \sin y \rangle_{\mathscr{H}_0} = \langle \cos x \sin y \rangle_{\mathscr{H}_0} = 0. \end{aligned} \tag{44}$$

When $\mathscr{H}_0 > 2M - P = \max \mathcal{V}(x,y)$, we have

$$\langle \cos x \rangle_{\mathscr{H}_0} = \langle \cos y \rangle_{\mathscr{H}_0} = \langle \cos x \cos y \rangle_{\mathscr{H}_0} = 0. \tag{45}$$

Then we have to compute $\langle \cos x \rangle_{\mathscr{H}_0}$, $\langle \cos y \rangle_{\mathscr{H}_0}$, and $\langle \cos x \cos y \rangle_{\mathscr{H}_0}$ when $\mathscr{H}_0 < 2M - P$, and these are exhibited in Appendix C. We next derive an explicit form of linear responses,

$$\begin{aligned} \delta M_x &= \iint \cos x \delta f \, d\boldsymbol{q} \, d\boldsymbol{p}, \quad \delta M_y = \iint \cos y \delta f \, d\boldsymbol{q} \, d\boldsymbol{p}, \\ \delta P_{\rm cc} &= \iint \cos x \cos y \delta f \, d\boldsymbol{q} \, d\boldsymbol{p}. \end{aligned} \tag{46}$$

The following notations are introduced for simplicity:

$$\begin{aligned} G_1 &= -\iint F_0'(\mathscr{H}_0)\big(\cos^2 x - \langle \cos x \rangle_{\mathscr{H}_0}^2\big) d\boldsymbol{q} \, d\boldsymbol{p} \\ &= -\iint F_0'(\mathscr{H}_0)\big(\cos^2 y - \langle \cos y \rangle_{\mathscr{H}_0}^2\big) d\boldsymbol{q} \, d\boldsymbol{p}, \\ G_2 &= -\iint F_0'(\mathscr{H}_0)(\cos x \cos y - \langle \cos x \rangle_{\mathscr{H}_0} \langle \cos y \rangle_{\mathscr{H}_0}) d\boldsymbol{q} \, d\boldsymbol{p} \\ &= -\iint F_0'(\mathscr{H}_0)\big(\cos x \cos y - \langle \cos x \rangle_{\mathscr{H}_0}^2\big) d\boldsymbol{q} \, d\boldsymbol{p}, \\ G_3 &= -\iint F_0'(\mathscr{H}_0)(\cos^2 x \cos y - \langle \cos x \cos y \rangle_{\mathscr{H}_0} \langle \cos x \rangle_{\mathscr{H}_0}) \\ &\quad \times d\boldsymbol{q} \, d\boldsymbol{p} \end{aligned}$$

$$= -\iint F_0'(\mathscr{H}_0)(\cos x \cos^2 y - \langle \cos x \cos y\rangle_{\mathscr{H}_0}\langle \cos y\rangle_{\mathscr{H}_0}) \times d\boldsymbol{q}\,d\boldsymbol{p},$$
$$G_4 = -\iint F_0'(\mathscr{H}_0)\big(\cos^2 x \cos^2 y - \langle \cos x \cos y\rangle^2_{\mathscr{H}_0}\big)d\boldsymbol{q}\,d\boldsymbol{p}. \quad (47)$$

By use of them and Eq. (21), we have

$$\begin{pmatrix} 1-G_1 & -G_2 & -G_3 \\ -G_2 & 1-G_1 & -G_3 \\ -G_3 & -G_3 & 1-G_4 \end{pmatrix}\begin{pmatrix}\delta M_x\\ \delta M_y \\ \delta P_{\rm cc}\end{pmatrix} = \begin{pmatrix} h_x G_1 + h_y G_2 \\ h_x G_2 + h_y G_1 \\ (h_x+h_y)G_3\end{pmatrix}. \quad (48)$$

We therefore obtain the explicit linear response formula as follows:

$$\delta M_x = \chi_1 h_x + \chi_2 h_y, \quad \delta M_y = \chi_2 h_x + \chi_1 h_y, \quad \delta P_{\rm cc} = \chi_3(h_x+h_y), \quad (49)$$

where explicit expressions of χ_1, χ_2, and χ_3 are

$$\chi_1 = \frac{1}{\det \mathbf{G}}\big(G_1 - G_1^2 - G_1G_4 + G_2^2 + G_3^2 + G_1^2G_4 - 2G_1G_3^2 + 2G_2G_3^2 - G_2^2G_4\big), \quad (50)$$

$$\chi_2 = \frac{1}{\det \mathbf{G}}\big(G_2 + G_3^2 - G_2G_4\big), \quad (51)$$

$$\chi_3 = \frac{1}{\det \mathbf{G}}G_3(1-G_1+G_2) = \frac{G_3}{1-G_1-G_2-G_4+G_1G_4+G_2G_4-2G_3^2}, \quad (52)$$

respectively, where the determinant of $\mathbf{G}$, the matrix in the left hand side of Eq. (48), is

$$\det \mathbf{G} = (1-G_1+G_2)\big(1-G_1-G_2-G_4 + G_1G_4 + G_2G_4 - 2G_3^2\big). \quad (53)$$

A way to compute terms including $\langle \cos x\rangle_{\mathscr{H}_0}$, $\langle \cos y\rangle_{\mathscr{H}_0}$, and $\langle \cos x \cos y\rangle_{\mathscr{H}_0}$ is exhibited in Appendix D.

When $F_0(\mathscr{H}_0)$ is spatially homogeneous, that is, $M = P = 0$, we have $G_2 = G_3 = 0$ and G_1 and G_4 ($1 - G_1 < 1 - G_4$ when $M = P = 0$) do not vanish. Thus, the susceptibilities are

$$\chi_1 = \frac{G_1}{1-G_1}, \quad \chi_2 = \chi_3 = 0, \quad (54)$$

in the disordered phase.

We numerically confirm the linear response formula. The initial state is the Maxwell-Boltzmann (MB) type:

$$f_{\rm MB}(\boldsymbol{q},\boldsymbol{p}) = \frac{\exp(-\mathscr{H}_0/T)}{\langle \exp(-\mathscr{H}_0/T)\rangle_\mu}. \quad (55)$$

This system shows the first order phase transition [28] and there is no (meta)stable homogeneous state with $T < 0.5$. The initial values of order parameters for $T = 0.3$ and 0.4 are exhibited in Table I. The external field $h_x = h_y = h$ is exerted. Theoretically obtained susceptibilities are exhibited in Table I when the temperature $T = 0.3$ and 0.4, so that the initial equilibria are spatially inhomogeneous. We integrate an equation of motion derived from the Hamiltonian (40) by using a fourth order symplectic integrator [30] and compute the order parameters of N-body systems, given respectively by

TABLE I. Initial equilibria and zero-field susceptibilities.

T	M	P	$d\delta M_{x/y}/dh\vert_{h=0}$	$d\delta P_{\rm cc}/dh\vert_{h=0}$
0.3	0.90223	0.81556	0.034428	0.059089
0.4	0.84269	0.71910	0.071298	0.099709

$$M_{x\rm c}^N(t,h) = \frac{1}{N}\sum_{i=1}^N \cos x_i(t,h),$$
$$M_{y\rm c}^N(t,h) = \frac{1}{N}\sum_{i=1}^N \cos y_i(t,h),$$
$$M_{x\rm s}^N(t,h) = \frac{1}{N}\sum_{i=1}^N \sin x_i(t,h),$$
$$M_{y\rm s}^N(t,h) = \frac{1}{N}\sum_{i=1}^N \sin y_i(t,h), \quad (56)$$

$$P_{\rm cc}^N(t,h) = \frac{1}{N}\sum_{i=1}^N \cos x_i(t,h)\cos y_i(t,h),$$
$$P_{\rm cs}^N(t,h) = \frac{1}{N}\sum_{i=1}^N \cos x_i(t,h)\sin y_i(t,h),$$
$$P_{\rm sc}^N(t,h) = \frac{1}{N}\sum_{i=1}^N \sin x_i(t,h)\cos y_i(t,h),$$
$$P_{\rm ss}^N(t,h) = \frac{1}{N}\sum_{i=1}^N \sin x_i(t,h)\sin y_i(t,h), \quad (57)$$

for null amplitude $h = 0$ and nonzero h. We compare the theoretically obtained linear response δM_x, δM_y, and $\delta P_{\rm cc}$ with the numerically obtained responses given respectively by

$$\delta M_x^N(h) = \bar{M}_x^N(h) - \bar{M}_x^N(0),$$
$$\delta M_y^N(h) = \bar{M}_y^N(h) - \bar{M}_y^N(0), \quad (58)$$
$$\delta P_{xy}^N(h) = \bar{P}_{xy}^N(h) - \bar{P}_{xy}^N(0),$$

where M_x^N, M_y^N, and P_{xy}^N are given by

$$M_x^N = \sqrt{{M_{x\rm c}^N}^2 + {M_{x\rm s}^N}^2},$$
$$M_y^N = \sqrt{{M_{y\rm c}^N}^2 + {M_{y\rm s}^N}^2}, \quad (59)$$
$$P_{xy}^N = \sqrt{{P_{\rm cc}^N}^2 + {P_{\rm cs}^N}^2 + {P_{\rm sc}^N}^2 + {P_{\rm ss}^N}^2},$$

and where overbars in Eqs. (58) denote the time average:

$$\bar{M}_{x\rm c}^N(h) = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} M_{x\rm c}^N(t,h)dt. \quad (60)$$

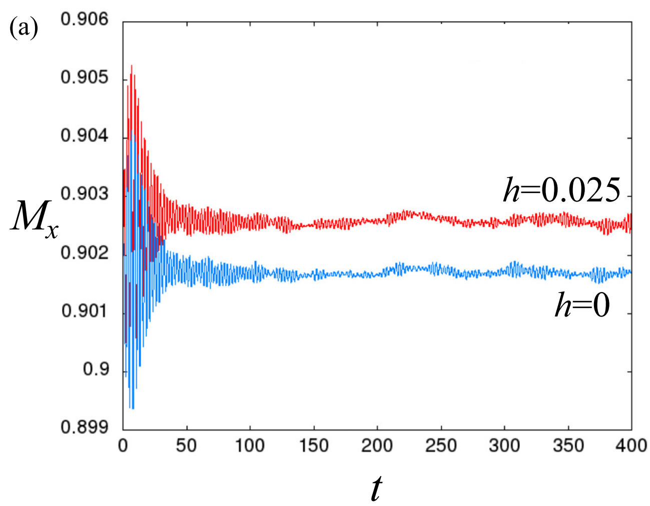

FIG. 2. Time series of order parameters for (a) M_x and (b) P_{xy}. The temperature is $T = 0.3$, the number of particles is given by $N = 4 \times 10^6$, and the time step is $\delta t = 0.05$. For each panel, the upper (red) curve is for $h = 0.025$ and the lower (blue) one is for $h = 0$.

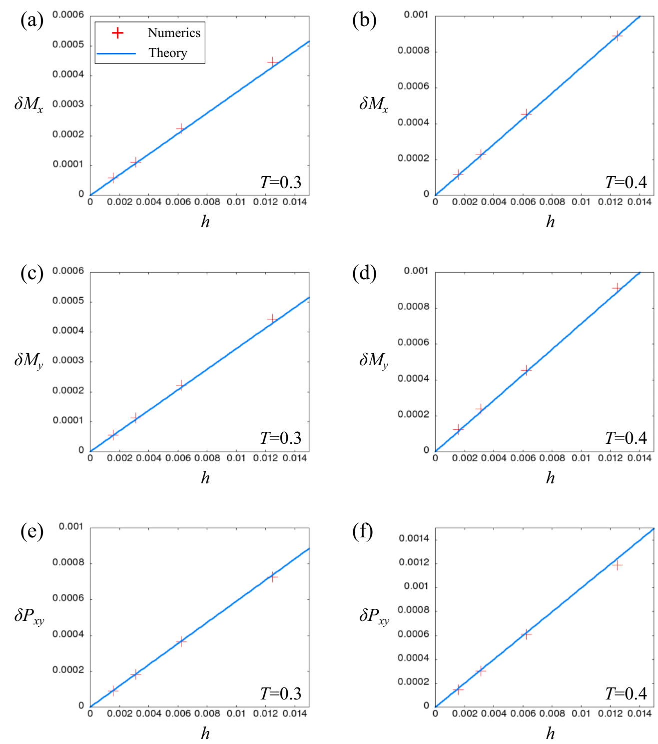

FIG. 3. δM_x, δM_y, and δP_{xy} as functions of h. The lines are the linear responses obtained theoretically and the crosses are responses obtained numerically. We set the temperature of the initial states as (a, c, e) $T = 0.3$ (left column) and (b, d, f) $T = 0.4$ (right column), the number of particles as $N = 4 \times 10^6$, and the time step as $\delta t = 0.05$.

For the 2D HMF model there is an error between $\bar{M}_x^N(0)$ and M which is a solution to the self-consistent equation. We then focus on the difference $\bar{M}_x^N(h) - \bar{M}_x^N(0)$ rather than $\bar{M}_x^N(h) - M$. We set $t_0 = 200$, $\tau = 200$, $N = 4 \times 10^6$, and the time step $\delta t = 0.05$. Figure 2 shows that these t_0 and τ are appropriate, and Fig. 3 shows the numerically obtained results confirm the theory.

VI. NONLINEAR RESPONSE FORMULA

The nonlinear response formula [14], which is called the rearrangement formula in Ref. [22], keeps the Casimir invariants within an order of the T linearization, the linearization around an asymptotic (A) state $f_{\rm A}(\boldsymbol{q},\boldsymbol{p}) = \lim_{t\to\infty} f(\boldsymbol{q},\boldsymbol{p},t)$ assumed to be stationary. We derive the nonlinear response formula via the same strategy for deriving the linear response formula in the present article. We assume that the asymptotic effective Hamiltonian

$$\mathscr{H}_{\rm A} = \|\boldsymbol{p}\|^2/2 + \mathcal{V}_{\rm A} + h\Phi, \quad \mathcal{V}_{\rm A} = \mathcal{V}[f_{\rm A}], \tag{61}$$

has only one integral of a single-particle motion and $f_{\rm A}$ depends only on $\mathscr{H}_{\rm A}$. Furthermore, $f_{\rm A} = F_{\rm A}(\mathscr{H}_{\rm A})$ is assumed to be monotonically decreasing with respect to $\mathscr{H}_{\rm A}$. Under these assumptions, expanding Eq. (8) around $f_{\rm A}$ as done in Sec. III, the constraint condition coming from Casimir invariants within an order of T linearization can be expressed as

$$\iint R(\mathscr{H}_{\rm A}(\boldsymbol{q},\boldsymbol{p}))(f_0(\boldsymbol{q},\boldsymbol{p}) - f_{\rm A}(\boldsymbol{q},\boldsymbol{p}))d\boldsymbol{q}d\boldsymbol{p} = 0, \tag{62}$$

for any smooth function R on $\mathbb{R}$. By use of Eq. (13), it is shown that Eq. (62) holds true if and only if

$$f_{\rm A} = \langle f_0 \rangle_{\mathscr{H}_{\rm A}} \tag{63}$$

for almost every $(\boldsymbol{q},\boldsymbol{p})$ in the μ space. Deriving a self-consistent equation

$$\mathcal{V}_{\rm A}(\boldsymbol{q}) = \iint V(\boldsymbol{q}-\boldsymbol{q}')\langle f_0 \rangle_{\mathscr{H}_{\rm A}(\boldsymbol{q}',\boldsymbol{p}')} d\boldsymbol{q}'d\boldsymbol{p}' \tag{64}$$

from Eq. (63) and solving it, one can obtain $\mathscr{H}_{\rm A}$ and the nonlinear response $\delta f = f_{\rm A} - f_0$.

Is it possible to derive Eq. (63) for multidimensional systems as was done in Refs. [14,15]? There is some difficulty to derive the same formula from the T-linearization method for multidimensional systems. To see this, let us exhibit a sketch of this T-linearization method (see Refs. [14,20–22] for details). We first divide $f(\boldsymbol{q},\boldsymbol{p},t)$ in two ways: One is the naive perturbation decomposition,

$$f = f_0 + g_{\rm p}, \tag{65}$$

where $g_{\rm p}$ is the perturbation around f_0, and the other one is the asymptotic-transient (AT) decomposition,

$$f = f_{\rm A} + g_{\rm T}, \tag{66}$$

where $g_{\rm T}$ is the T term satisfying $\lim_{t\to\infty} g_{\rm T} = 0$. According to the AT decomposition, the potential is also decomposed as

$$\begin{aligned} \mathcal{V}[f] + h(t)\Psi &= \mathcal{V}_{\rm A} + \mathcal{V}_{\rm T} + h\Phi, \\ \mathcal{V}_{\rm T} &= \mathcal{V}[g_{\rm T}] + (h(t)-h)\Phi. \end{aligned} \tag{67}$$

Substituting Eqs. (65) and (67) into the Vlasov equation, and omitting the nonlinear term coupling with the T-field $\mathcal{V}_{\rm T}$, we have the T-linearized Vlasov equation:

$$\frac{\partial f}{\partial t} + \{\mathcal{H}_{\rm A}, f\} + \{\mathcal{V}_{\rm T}, f_0\} = 0. \tag{68}$$

It should be noted that the nonlinearity still remains in the term $\{\mathcal{H}_{\rm A}, f\}$. A solution $f_{\rm TL}$ to the T-linearized equation is implicitly given by

$$\begin{aligned} f_{\rm TL}(\boldsymbol{q},\boldsymbol{p},t) &= f_{\rm ON}(\boldsymbol{q},\boldsymbol{p},t) + f_{\rm LA}(\boldsymbol{q},\boldsymbol{p},t), \\ f_{\rm ON}(\boldsymbol{q},\boldsymbol{p},t) &= e^{-t\{\mathcal{H}_{\rm A},\bullet\}} f_0(\boldsymbol{q},\boldsymbol{p}), \\ f_{\rm LA}(\boldsymbol{q},\boldsymbol{p},t) &= -\int_0^t e^{-(t-s)\{\mathcal{H}_{\rm A},\bullet\}} \left(\boldsymbol{F}_{\rm T} \cdot \frac{\partial f_0}{\partial \boldsymbol{p}} \right) ds, \end{aligned} \tag{69}$$

where we introduce the operator $\{\mathcal{H}_{\rm A},\bullet\}a = \{\mathcal{H}_{\rm A},a\}$ for any function $a(\boldsymbol{q},\boldsymbol{p})$, and $\boldsymbol{F}_{\rm T} = -\partial\mathcal{V}_{\rm T}/\partial\boldsymbol{q}$. The terms $f_{\rm ON}$ and $f_{\rm LA}$ are the O'Neil term and the Landau term, respectively [20,21].

When the asymptotic stationary state $f_{\rm A}$ exists, it can be picked by taking the long-time average of $f_{\rm TL}$, and we have

$$f_{\rm A}(\boldsymbol{q},\boldsymbol{p}) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau f_{\rm TL}(\boldsymbol{q},\boldsymbol{p},t)dt \tag{70}$$

within an order of the T-linearization method. Then, our next job is to compute the long-time average of $f_{\rm ON}$ and $f_{\rm LA}$, but there are several difficulties in doing this for multidimensional systems.

In 1D systems, it is shown that

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau e^{-t\{\mathcal{H}_{\rm A},\bullet\}} a(\boldsymbol{q},\boldsymbol{p})dt = \langle a\rangle_{\mathcal{H}_{\rm A}(\boldsymbol{q},\boldsymbol{p})} \tag{71}$$

by use of the angle-action variables of $\mathcal{H}_{\rm A}$ [20,21]. However, in our case, the angle-action variables cannot be constructed, so it is unclear if this ergodiclike formula holds true or not.

There is another problem: slow algebraic damping of the T force field $\boldsymbol{F}_{\rm T}$. In Ref. [14] we use the fact that $\boldsymbol{F}_{\rm T}$ damps rapidly ($\sim t^{-\nu}$ with $\nu \geqslant 2$) for the 1D systems [26] and

$$\lim_{t\to\infty} \int_t^\infty \boldsymbol{F}_{\rm T}(t)dt = \boldsymbol{0} \tag{72}$$

when we compute $\lim_{t\to 0} f_{\rm LA}$. Meanwhile, in the multidimensional Vlasov systems [27] and the 2D Euler equations [31], the T force field $\boldsymbol{F}_{\rm T}$ damps as or slower than t^{-1}, so the integral $\int_0^\infty \boldsymbol{F}_{\rm T}(t)dt$ is not defined in the L^1 meaning apparently. Sometimes, the transient part is asymptotically $\boldsymbol{F}_{\rm T} \asymp e^{-i\Omega t}t^{-\gamma}$ $(0<\gamma\leqslant 1)$ with $\Omega \neq 0$, and the integral $\int_0^\infty \boldsymbol{F}_{\rm T}(t)dt$ exists in the Riemannian meaning. In this case, one should be more careful when one computes the integrals and takes the limit. It should be remarked that there exists a case that $\Omega = 0$ [27], so it should be checked for each system.

The relation between the nonlinear response formula obtained by considering the constraint conditions and a solution to the T-linearized Vlasov equation might be an interesting future problem.

VII. SUMMARY AND PERSPECTIVE

The linear response formula has been derived without use of the analytic solution of the single-particle orbit or the angle-action variables of the effective Hamiltonian. The present method improves the generalized linear response formula obtained in Ref. [16] when the background density function is a monotonically decreasing function of the effective Hamiltonian $\mathcal{H}_0$. The response formula (21) results in the one obtained in the previous studies for 1D systems [10,11,13,15] and is numerically confirmed by use of the 2D HMF model. Furthermore, the nonlinear response formula [14] has been derived via the same strategy, when the asymptotic solution $f_{\rm A}$ to the T-linearized Vlasov equation is a monotonically decreasing function of the effective Hamiltonian $\mathcal{H}_{\rm A}$.

The nonlinear response theory based on the T-linearization method deals with the nonlinearity of order $O(h^\nu)$ with $1<\nu<2$ [14]. It should be noted that it is difficult to obtain the nonlinear response successively via the proposed method, so that the error $O(h^2)$ is unavoidable. Let δf_n be a response of order $O(h^n)$ for $n\in\mathbb{Z}$. The condition in Eq. (10) in the nonlinear regime $O(h^2)$ is written as

$$\begin{aligned} 0 &= \iint \left(R_1(\mathcal{H}_0)\delta f_2 + R_2(\mathcal{H}_0)\delta f_1^2\right) d\boldsymbol{q}\,d\boldsymbol{p} \\ &= \iint \left(R_1(\mathcal{H}_0)\langle\delta f_2\rangle_{\mathcal{H}_0} + R_2(\mathcal{H}_0)\langle\delta f_1^2\rangle_{\mathcal{H}_0}\right) d\boldsymbol{q}\,d\boldsymbol{p}, \end{aligned} \tag{73}$$

where $R_1 = c'(f_0)$ and $R_2 = c''(f_0)/2$. Unlike the linear regime, it is quite difficult to obtain explicitly δf_2 satisfying this equation for any c. Then, the error $O(h^2)$ is unavoidable in both naive perturbation and T-linearization methods.

In the present article, the form of perturbation is restricted so as to subject it to the constraint conditions coming from Casimir invariants at the linear order. By use of the form of constraint conditions (14), it is possible to take into account the Casimir constraints when we derive the formal stability criterion without use of angle-action variables, and this is the topic of a forthcoming paper [32].

In this article, we exert the uniform external force on the systems without integrability. We may also consider the case where the unperturbed system is integrable but an external force breaks its integrability. It might be an interesting future work to explore how the local chaos induced by the static external field affects meso- or macroscopic properties of systems. Such a phenomenon is found in a toy model with one charged particle confined in cylindrical or toroidal magnetic fields [33,34].

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APPENDIX A: CONSERVATION OF THE CASIMIR FUNCTIONALS (8)

It is shown that the Casimir functional (8) is conserved in the Vlasov dynamics. Taking the time derivative of $\mathscr{C}[f]$, we have

$$\frac{d\mathscr{C}[f]}{dt} = \iint \frac{\partial f}{\partial t} c'(f) d\boldsymbol{q}\, d\boldsymbol{p} = -\iint \{\mathscr{H}[f], f\} c'(f) d\boldsymbol{q}\, d\boldsymbol{p}. \tag{A1}$$

Under the conditions asserted above Eq. (8), the boundary terms vanish and the left hand side of Eq. (A1) is written as

$$-\iint \{\mathscr{H}[f], f\} c'(f) d\boldsymbol{q}\, d\boldsymbol{p} = \iint \mathscr{H}[f]\{f, c'(f)\} d\boldsymbol{q}\, d\boldsymbol{p} = 0 \tag{A2}$$

because $\{f, c'(f)\} = 0$. It is then shown that $d\mathscr{C}[f]/dt = 0$.

APPENDIX B: DERIVATION OF EQ. (43)

We derive Eq. (43). All we have to do is to perform integration with respect to $\boldsymbol{p}$ in the left hand side of Eq. (43),

$$\begin{aligned}\int_{\mathbf{R}^2} \delta(\mathscr{H}(\boldsymbol{q},\boldsymbol{p}) - E) d\boldsymbol{p} &= \int_{-\pi}^{\pi} d\theta_p \int_0^\infty p\,\delta(\mathscr{H}(\boldsymbol{q},\boldsymbol{p}) - E) dp \\ &= 2\pi \int_0^\infty p\,\delta\left(\frac{p^2}{2} + \mathscr{V}(x,y) - E\right) dp = 2\pi\,\Theta(E - \mathscr{V}(x,y)),\end{aligned}$$

where $\Theta(x) = 0$ (1) when $x < 0$ ($x \geqslant 0$) is the Heaviside step function, and we have used the relation

$$\delta(f(x)) = \sum_{x^* = f^{-1}(0)} \frac{\delta(x - x^*)}{|f'(x^*)|}, \quad f'(x^*) \neq 0.$$

Thus, we have

$$\langle g(\boldsymbol{q}) \rangle_E = \frac{\int_{(-\pi,\pi]^2} g(x,y)\Theta(E - \mathscr{V}(x,y)) dx dy}{\int_{(-\pi,\pi]^2} \Theta(E - \mathscr{V}(x,y)) dx dy}. \tag{B1}$$

APPENDIX C: COMPUTATION OF $\langle \cos x \rangle_{\mathscr{H}_0}$ AND $\langle \cos x \cos y \rangle_{\mathscr{H}_0}$

On the iso-$\mathscr{H}_0$ curve, x and y satisfy

$$\cos x = -\frac{\mathscr{H}_0 + M\cos y}{M + P\cos y}, \quad \cos y = -\frac{\mathscr{H}_0 + M\cos x}{M + P\cos x}. \tag{C1}$$

Thus,

$$\begin{aligned}\int_{(-\pi,\pi]^2} g(x,y)\Theta(\mathscr{H}_0 - \mathscr{V}(x,y)) dx dy &= 4\int_0^{\arccos(-\frac{\mathscr{H}_0+M}{M+P})} dx \int_0^{\arccos(-\frac{\mathscr{H}_0+M\cos x}{M+P\cos x})} g(x,y) dy \\ &= 4\int_0^{\arccos(-\frac{\mathscr{H}_0+M}{M+P})} dy \int_0^{\arccos(-\frac{\mathscr{H}_0+M\cos y}{M+P\cos y})} g(x,y) dx\end{aligned} \tag{C2}$$

for $\mathscr{H}_0 \in [-2M - P, P]$ and

$$\begin{aligned}\int_{(-\pi,\pi]^2} g(x,y)\Theta(\mathscr{H}_0 - \mathscr{V}(x,y)) dx dy &= 4\int_0^{\arccos(-\frac{\mathscr{H}_0-M}{M-P})} dx \int_0^{\pi} g(x,y) dy + 4\int_{\arccos(-\frac{\mathscr{H}_0-M}{M-P})}^{\pi} dx \int_0^{\arccos(-\frac{\mathscr{H}_0+M\cos x}{M+P\cos x})} g(x,y) dy \\ &= 4\int_0^{\arccos(-\frac{\mathscr{H}_0-M}{M-P})} dy \int_0^{\pi} g(x,y) dx + 4\int_{\arccos(-\frac{\mathscr{H}_0-M}{M-P})}^{\pi} \int_0^{\arccos(-\frac{\mathscr{H}_0+M\cos y}{M+P\cos y})} g(x,y) dx\end{aligned} \tag{C3}$$

for $\mathscr{H}_0 \in [P, 2M - P]$.

Let us compute $\langle \cos x\rangle_{\mathscr{H}_0} = \langle \cos y\rangle_{\mathscr{H}_0}$ and $\langle \cos x \cos y\rangle_{\mathscr{H}_0}$ as follows, respectively. When $-2M - P < \mathscr{H}_0 < P$, we have

$$\langle \cos x\rangle_{\mathscr{H}_0} = \langle \cos y\rangle_{\mathscr{H}_0} = \frac{8\pi}{\sigma(\mathscr{H}_0)} \int_0^{\arccos(-\frac{\mathscr{H}_0+M}{M+P})} \sqrt{1 - \left(\frac{\mathscr{H}_0 + M\cos x}{M + P\cos x}\right)^2}\, dx, \tag{C4}$$

$$\langle \cos x \cos y\rangle_{\mathscr{H}_0} = \frac{8\pi}{\sigma(\mathscr{H}_0)} \int_0^{\arccos(-\frac{\mathscr{H}_0+M}{M+P})} \cos x \sqrt{1 - \left(\frac{\mathscr{H}_0 + M\cos x}{M + P\cos x}\right)^2}\, dx, \tag{C5}$$

where

$$\sigma(\mathscr{H}_0) = 8\pi \int_0^{\arccos(-\frac{\mathscr{H}_0+M}{M+P})} \arccos\left(-\frac{\mathscr{H}_0 + M\cos x}{M + P\cos x}\right) dx, \tag{C6}$$

and where a range of the arccosine function is $[0,\pi]$. When $P < \mathscr{H}_0 < 2M - P$, we have

$$\langle \cos x\rangle_{\mathscr{H}_0} = \langle \cos y\rangle_{\mathscr{H}_0} = \frac{8\pi}{\sigma(\mathscr{H}_0)} \int_{\arccos(-\frac{\mathscr{H}_0-M}{M-P})}^{\pi} \sqrt{1 - \left(\frac{\mathscr{H}_0 + M\cos x}{M + P\cos x}\right)^2}\, dx, \tag{C7}$$

$$\langle \cos x \cos y\rangle_{\mathscr{H}_0} = \frac{8\pi}{\sigma(\mathscr{H}_0)} \int_{\arccos(-\frac{\mathscr{H}_0-M}{M-P})}^{\pi} \cos x \sqrt{1 - \left(\frac{\mathscr{H}_0 + M\cos x}{M + P\cos x}\right)^2}\, dx, \tag{C8}$$

where

$$\sigma(\mathscr{H}_0) = 8\pi \int_{\arccos(-\frac{\mathscr{H}_0-M}{M-P})}^{\pi} \arccos\left(-\frac{\mathscr{H}_0 + M\cos x}{M + P\cos x}\right) dx + 8\pi^2 \arccos\left(-\frac{\mathscr{H}_0 - M}{M - P}\right). \tag{C9}$$

APPENDIX D: INTEGRAL IN G_n ($n = 1,2,3,4$)

The integral $\iint F_0'(\mathscr{H}_0)\langle \cos x\rangle_{\mathscr{H}_0}^2 d\boldsymbol{q}\, d\boldsymbol{p}$ included in G_1 and G_2 is computed as follows:

$$\begin{aligned}\iint F_0'(\mathscr{H}_0)\langle \cos x\rangle_{\mathscr{H}_0}^2 d\boldsymbol{q}\, d\boldsymbol{p} &= 2\pi \int p dp \int F_0'(\mathscr{H}_0)\langle \cos x\rangle_{\mathscr{H}_0}^2 d\boldsymbol{q}\\ &= 2\pi \int_{-2M-P}^{2M-P} d\mathscr{H}_0 F_0'(\mathscr{H}_0)\langle \cos x\rangle_{\mathscr{H}_0}^2 \int \Theta(\mathscr{H}_0 + \mathscr{V}(x,y)) d\boldsymbol{q}\\ &= 2\pi \int_{-2M-P}^{2M-P} F_0'(\mathscr{H}_0)\langle \cos x\rangle_{\mathscr{H}_0}^2 \sigma(\mathscr{H}_0) d\mathscr{H}_0,\end{aligned} \tag{D1}$$

where $p = \|\boldsymbol{p}\| = \sqrt{2(\mathscr{H}_0 + \mathscr{V}(x,y))}$ and $\sigma(\mathscr{H}_0)$ is defined in Eqs. (C6) and (C9). The similar terms in G_2 and G_4 are computed in the same manner.

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