Diffusion in time-dependent random media and the Kardar-Parisi-Zhang equation

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Although time-dependent random media with short-range correlations lead to (possibly biased) normal tracer diffusion, anomalous fluctuations occur away from the most probable direction. This was pointed out recently in one-dimensional (1D) lattice random walks, where statistics related to the 1D Kardar-Parisi-Zhang (KPZ) universality class, i.e., the Gaussian unitary ensemble Tracy-Widom distribution, were shown to arise. Here, we provide a simple picture for this correspondence, directly in the continuum, which allows one to study arbitrary space dimensions and to predict a variety of universal distributions. In $d = 1$, we predict and verify numerically the emergence of the Gaussian orthogonal ensemble Tracy-Widom distribution for fluctuations of the transition probability. In $d = 3$, we predict a phase transition from Gaussian fluctuations to three-dimensional KPZ-type fluctuations as the bias is increased. We predict KPZ universal distributions for the arrival time of a first particle from a cloud diffusing in such media.

Diffusion in random media arises in numerous fields, e.g., oil exploration in porous rocks [1], spreading of pollutants in inhomogeneous flows [2], diffusion of charge carriers in conductors [3], relaxation properties of glasses [4], defect motions in solids, econophysics, and population dynamics [5,6]. Many works have studied *time-independent*, i.e., static, random environments [7], in $d = 1$ [8] or in higher dimensions, with short-range (SR) [9,10] or long-range (LR) spatial correlations [11]. It was found that static disorder with SR correlations is generically irrelevant above the upper-critical dimension $d_c = 2$, leading to normal diffusion in $d = 3$, while LR disorder can lead to anomalous diffusion in any d.

Another important class of random media are time dependent, studied, e.g., in wave propagation [12], dispersion of particles in turbulent flows [2] (Richardson's law [13]), and the passive scalar [14]. The latter cases involve long-range correlations in the flow, and lead to anomalous transport or multiscaling. The, *a priori* more benign, case of SR space-time correlations received much attention recently in probability theory, within random walks in time-dependent random environments (TD-RWREs). Although then $d_c = 0$, and the diffusion is proved to be normal (in a given sample [15]), interesting effects were shown, such as a tendency for walkers in the same sample to coalesce [16], anomalous fluctuations [17], and large deviations [18]. Note that TD-RWREs can be generated in a static environment by studying directed random walks.

An *a priori* unrelated topic is stochastic growth and the celebrated Kardar-Parisi-Zhang (KPZ) equation [19]

$$\partial_t h = \nu_0 \nabla_x^2 h + \frac{\lambda_0}{2}(\vec{\nabla}_x h)^2 + \sqrt{D_0}\eta, \qquad (1)$$

where $h(x,t) \in \mathbb{R}$ is the interface height at time t and point $x \in \mathbb{R}^d$, ν is the diffusivity, and $\eta(x,t)$ is the driving noise which, for most of our applications, will be SR space-time correlated. The nonlinearity leads to a nontrivial fixed point and exponents for the scaling of the fluctuations at large times, i.e., $h(x,t) = v_\infty t + \delta h(x,t)$, with $\delta h \sim t^{\theta_d} \sim x^{\theta_d/\zeta_d}$ and $\theta_d = 2\zeta_d - 1$ from Galilean symmetry [20]. The vast universality class of the continuum KPZ equation (1) contains discrete growth models

[21], particle transport models [22], dimer covering, directed polymers [20,23], and more, the subject in $d = 1$ of much recent progress, due to the discovery of integrable properties [24]. Beyond exponents $\zeta_{d=1} = 2/3$, the statistics of $\delta h(x,t)$ was shown to be related to the universal Tracy-Widom (TW) distributions of random matrix theory [25], with, e.g., the Gaussian unitary ensemble (GUE) [respectively Gaussian orthogonal ensemble (GOE)] TW distribution for growth starting from a droplet [26] (respectively a flat interface). For general d, little is known exactly, but exponents and universal distributions were obtained numerically in $d = 1,2,3$ [27–29] and compared with experiments [30].

Recently, Barraquand and Corwin obtained an exact solution of a discrete TD-RWRE on $\mathbb{Z}$ with SR correlated jump probabilities, the beta polymer. The sample-to-sample fluctuations of the logarithm of the cumulative [31] and transition [32] probability distribution function (PDF) *in the large-deviation regime* of the RW, i.e., away from the most probable direction, were found to be distributed with the characteristic KPZ exponent and GUE TW distribution.[1] This was followed by a proof of the universality of the one-dimensional (1D) KPZ equation for the diffusive scaling limit of TD-RWRE on $\mathbb{Z}$ with weak disorder [33].

These recent results hint at a general connection between TD-RWRE and KPZ growth. The aim of this Rapid Communication is to unveil a simple and general mechanism that explains the appearance of KPZ-type fluctuations in the TD-RWRE problem, beyond exactly solvable models, and for general d. We consider a continuum setting and we conjecture the emergence of KPZ fluctuations everywhere in the large-deviation regime of TD-RWRE in dimension $d = 1,2$, and *a phase transition* in $d \geqslant 3$ between a low-fluctuation phase for small large deviations and a phase with KPZ-class high fluctuations for large large deviations (see Fig. 1). Using this picture, we identify in $d = 1$ a setting where GOE TW-type distributions for the fluctuations of the logarithm of the PDF

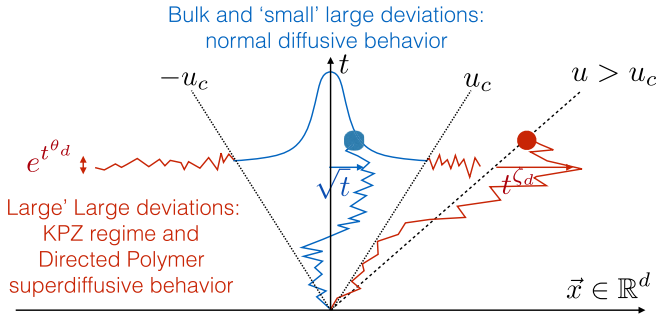[1]Note also the upcoming work [34] on the roughness of random walk paths in the beta polymer model.

FIG. 1. Conjectural picture for TD-RWRE in arbitrary dimension d. While particles typically diffuse normally as if the random environment had been averaged out, particles conditioned on arriving away from the Gaussian bulk of the distribution in the "large-deviation regime" (for $|x(t)| > u_c t$) are superdiffusive with the roughness exponent of the directed polymer in the pinned phase $\zeta_d > 1/2$. In this regime, fluctuations of the logarithm of the transition probability are large (scale with t^{θ_d} with $\theta_d = -1 + 2\zeta_d > 0$) and identical to those of the height in the rough phase of the KPZ equation. The two phases are separated by an Edward-Wilkinson regime of fluctuations when $x = u_c t + o(t)$. In $d = 1,2$, $u_c = 0$, while for $d \geqslant 3$, there is a phase transition with $u_c \neq 0$. If an external bias $\vec{f}$ is added, the bulk is around $\vec{x} \sim \vec{f} t$ and the transition occurs for $\vec{x} = \vec{u} t$ with $|(\vec{f} + \vec{u})| = u_c$.

are expected. This is checked using simulations of a discrete model. We finally discuss the emergence of KPZ universality in the extreme value statistics of $N \gg 1$ random walkers diffusing in the same time-dependent random environment: universality of the PDF of the largest distance traveled by a particle in a cloud of pollutant and of the PDF of the first arrival time in a given domain.

We consider the Langevin equation for the diffusion of a particle $\vec{x}(t) \in \mathbb{R}^d$ in a d-dimensional time-dependent random force field $\vec{\xi}(\vec{x},t) + \vec{f}$, with $\overline{\vec{\xi}(\vec{x},t)} = 0$ and $\vec{f}$ the uniform applied force,

$$\frac{d}{dt}\vec{x}(t) = \vec{\xi}(\vec{x}(t),t) + \vec{f} + \vec{\eta}(t), \qquad (2)$$

with $\vec{\eta} \in \mathbb{R}^d$ a thermal Gaussian white noise, $\langle \eta_i(t)\eta_j(t') \rangle = 2D\delta_{ij}\delta(t - t')$, and D is the bare diffusion coefficient. Here and below, $\langle \cdot \rangle$ refers to the average over thermal fluctuations $\vec{\eta}$, and $\overline{(\cdot)}$ over the disorder $\vec{\xi}(\vec{x},t)$.

In a given random environment (i.e., sample) $\vec{\xi}(\vec{x},t)$, one defines the transition probability $P(\vec{x}_2,t_2|\vec{x}_1,t_1)$ for a particle which starts at $\vec{x}_1$ at time t_1 to end up at position $\vec{x}_2$ at time t_2. It is convenient for now to consider the (backward) transition probability $Q(\vec{x},t) = P(0,0|\vec{x},-t)$ that a particle starting at position $\vec{x}$ at time $-t \leqslant 0$ ends up at the origin at time 0 (the forward is considered later). The latter obeys the following random backward Kolmogorov equation,

$$\partial_t Q = D\nabla_x^2 Q + \vec{f} \cdot \vec{\nabla}_x Q + \vec{\xi} \cdot \vec{\nabla}_x Q, \qquad (3)$$

with a final condition $Q(\vec{x},t = 0) = \delta^{(d)}(\vec{x})$. For simplicity, we focus on $\vec{\xi}(\vec{x},t)$ being a space-time Gaussian white noise [interpreting (3) in the Îto sense] with variance

$$\overline{\xi_i(\vec{x},t)\xi_j(\vec{x}',t')} = Dr_0^d \delta^{(d)}(\vec{x} - \vec{x}')\delta(t - t')\delta_{ij}, \qquad (4)$$

where the parameter r_0 has dimension of a length. Our results on the large-scale properties should hold for a more general distribution of the disorder, as long as correlations of $\vec{\xi}(\vec{x},t)$ are short ranged in space and time. When short-scale regularizations are needed, we will think of this model as an approximation of a model with a disorder of (dimensionless) magnitude σ_ξ, a finite correlation length r_c, and a finite correlation time τ_c. In that case, $\overline{\xi_i(\vec{x},t)\xi_j(\vec{x}',t')} = \frac{D}{\tau_c}\sigma_\xi^2 R_1(\frac{\vec{x}-\vec{x}'}{r_c})R_2(\frac{t-t'}{\tau_c})\delta_{ij}$, with R_1 and R_2 two dimensionless, rapidly decaying functions, and one relates r_0 to the space-time correlation volume of the noise as $r_0^d \sim \sigma_\xi^2 r_c^d \int d^d\vec{y}\,ds\,R_1(\vec{y})R_2(s)$.

In the following, we analyze this RW *locally around a given space-time direction* (moving frame velocity) $\vec{u} \in \mathbb{R}^d$, i.e., for $\vec{x} = \vec{u}t + \vec{x}'$ with $\vec{x}' = o(t)$. This is equivalent to looking around the origin $\vec{x} = o(t)$ in the model with an effective bias $\vec{f}_{\vec{u}} = \vec{f} + \vec{u}$: Using the equality in law between white noises $\vec{\xi}(t\vec{u} + \vec{x}',t) \sim_{\text{in law}} \vec{\xi}'(\vec{x}',t)$, one gets $Q_{\vec{f}}(\vec{u}t + \vec{x}',t) \sim_{\text{in law}} Q_{\vec{f}+\vec{u}}(\vec{x}',t)$. We drop the subscript $\vec{u}$ in $\vec{f}_{\vec{u}}$ unless needed, but $\vec{f}$ should thus be thought of as a control parameter analogous to the velocity of the frame of observation compared to the mean velocity of the particles. We first note that the averaged value of the transition probability is equal to the transition probability of a RW in the averaged environment,[2] hence it is Gaussian and given by $\overline{Q(\vec{x},t)} = \frac{1}{(4\pi Dt)^{\frac{d}{2}}}e^{-\frac{|\vec{x}+t\vec{f}|^2}{4Dt}}$. The regime $|\vec{x}| = o(t)$ is thus characterized by an exponential decay of the averaged probability $-\frac{|\vec{x}+t\vec{f}|^2}{4Dt} \simeq -\frac{\vec{f}\cdot\vec{x}}{2D} - \frac{t|\vec{f}|^2}{4D}$, hence it corresponds to a large-deviation regime, far away from the bulk of the probability, i.e., the optimal direction of the RW, $\vec{x} = -\vec{f}t$. To study the local fluctuations around this average profile of the probability, we introduce the *partition sum* $Z(\vec{x},t)$ and *height* $h(\vec{x},t)$ as

$$Z(\vec{x},t) := e^{\frac{\vec{f}\cdot\vec{x}}{2D}+t\frac{\vec{f}^2}{4D}}Q(\vec{x},t), \quad h(\vec{x},t) := \ln Z(\vec{x},t). \quad (5)$$

Inserting (5) in (3), we obtain

$$\partial_t Z = D\nabla_x^2 Z + \xi_{\text{DP}}Z + \vec{\xi} \cdot \vec{\nabla}_x Z, \qquad (6)$$

$$\partial_t h = D\nabla^2 h + D(\vec{\nabla}h)^2 + \xi_{\text{DP}} + \vec{\xi} \cdot \vec{\nabla}h, \qquad (7)$$

with the "droplet" initial condition $Z(\vec{x},0) = \delta(\vec{x})$. In (6) and (7) we introduced the "directed polymer (DP) noise term,"

$$\xi_{\text{DP}}(\vec{x},t) = -\frac{\vec{f} \cdot \vec{\xi}(\vec{x},t)}{2D}, \qquad (8)$$

a Gaussian white noise with $\langle \xi_{\text{DP}}(\vec{x},t)\xi_{\text{DP}}(\vec{x}',t') \rangle = \sigma_{\text{DP}}^2\delta(t - t')\delta^{(d)}(\vec{x} - \vec{x}')$ and (with $f = |\vec{f}|$ the norm of the bias)

$$\sigma_{\text{DP}}^2 = \frac{r_0^d}{4D}f^2. \qquad (9)$$

Equations (6) and (7) contain two (mutually correlated) noises. While the second source of noise (last term) is a signature

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of the RW nature of the problem [it is already present in the original backward Kolmogorov equation (3)], the first was generated by our rescaling of the transition probability (5) and is a signature of the fact that we are looking at the large-deviation regime: It is the only term in (6) and (7) that depends on f. A crucial observation is that if, in a first stage (justified below), one neglects the second source of noise, Eqs. (6) and (7) become respectively the multiplicative stochastic heat equation (MSHE) and the KPZ equation (1). The solution of the MSHE is known to be the partition sum $Z_{DP}(\vec{x},t)$ of the continuum directed polymer problem, i.e., the equilibrium statistical mechanics at temperature $T = 2D$ of directed paths of length $t, \vec{x} : \tau \in [0,t] \rightarrow \vec{x}(\tau) \in \mathbb{R}^d$ with fixed end points $\vec{x}(0) = 0$ and $\vec{x}(t) = \vec{x}$ in a quenched random potential $-2D\xi_{DP}[t',\vec{x}(t')]$. It can formally be written as a path integral,

$$Z_{DP}(\vec{x},t) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}[x] e^{-\frac{1}{2D}\int_0^t d\tau\{\frac{1}{2}(\frac{d\vec{x}}{d\tau})^2 - 2D\xi_{DP}[\tau,\vec{x}(\tau)]\}}, \quad (10)$$

while the solution of the KPZ equation with the initial condition of the droplet is given by $h_{KPZ}(\vec{x},t) = \ln Z_{DP}(\vec{x},t)$, the two problems hence being, as is well known, equivalent.

The emergence of the MSHE and KPZ equations in this problem is at the core of the connection between TD-RWRE and the KPZ universality class (KPZUC). Let us now explore some consequences of this connection in the DP language, which is more adapted to the physics of the RW problem in terms of space-time paths. It is known [20,35] that the DP exhibits a phase transition as a function of the noise strength σ_{DP} between (i) a diffusive phase at small $\sigma_{DP} < \sigma_c$ where polymer paths are diffusive $x(\tau) \sim \tau^{1/2}$ and do not feel the disorder, and (ii) a pinned phase at large $\sigma_{DP} > \sigma_c$ where directed polymer paths are superdiffusive $x(\tau) \sim \tau^{\zeta_d}$ with $\zeta_d > 1/2$ the universal (dimension-dependent) roughness exponent. In the diffusive phase, the fluctuations of the DP free energy are small, $\ln Z_{DP}(t) \sim O(1)$. In the pinned phase, the DP optimizes its energy: The partition sum is concentrated on a few optimal paths and the fluctuations of the DP free-energy scale with the length as $\ln Z_{DP}(t) \sim t^{\theta_d}$ with $\theta_d = -1 + 2\zeta_d > 0$. While for $d > 2$ there is a transition at a nontrivial value $\sigma_c > 0$,[3] $\sigma_c = 0$ in $d = 1,2$ and the system is always in the pinned phase.

We now argue, using the interface language, that the second source of noise in (6) and (7) is always *irrelevant* in the pinned phase at large time. In this phase, the KPZ field displays scale invariant fluctuations and we can rescale $h(\vec{x},t) = b^{\alpha}\tilde{h}(\vec{x}/b,t/b^z)$ with b large and $z = 1/\zeta_d$ and $\alpha = \theta_d/\zeta_d$ the dynamic and roughness exponent of the KPZUC, with $\tilde{h} = O(1)$. From the scale invariance of the Gaussian white noise, under rescaling, the second source of noise in (7) receives an additional factor $b^{\alpha-1}$ as compared to the first one. This heuristic suggests that the second source of noise is irrelevant as long as $\alpha < 1$. This condition is always satisfied in the rough phase, with $\alpha = 1/2$ in $d = 1$ and α decreases with d.

This leads us to the following conjecture. In the RW problem, looking locally[4] in the large-deviation region $\vec{x} = o(t)$, the system undergoes *a phase transition as a function of the bias* from (i) a diffusive phase for $f < f_c$, where the local fluctuations of $\ln Q(\vec{x},t)$ are $O(1)$ and the random walk paths are diffusive with the same law as the RW in an averaged environment (for $f = 0$ this was shown rigorously in Ref. [15]), and (ii) a pinned phase for $f > f_c$, where $\ln Q(\vec{x},t)$ has larger fluctuation scaling as t^{θ_d} and random walk paths are superdiffusive with the DP roughness exponent ζ_d. In addition, the full multipoint distribution of $\ln Q(\vec{x},t)$ at large t is expected to be universal and identical to those of the free energy $\ln Z_{DP}(\vec{x},t)$ of the DP problem in the pinned phase. Furthermore, $f_c = 0$ in $d = 1,2$ and in $d > 2$ we can give an estimate of the transition point. For the KPZ equation (1), $d = 2 + \epsilon$ renormalization group (RG) calculations indicate that the transition for $d > 2$ occurs for the dimensionless coupling[5] $g := K_d \Lambda^{d-2} \frac{\lambda_0^2 D_0}{8 v_0^3} = g_c$ of order ϵ, $g_c = \epsilon + O(\epsilon^2)$, with Λ^{-1} a short distance cutoff [19,36]. Translating into the RW with $\Lambda = 1/r_c$, we find $g_c = K_d \sigma_\xi^2 r_c^2 f_c^2 / (8D^2)$, which provides an estimate for f_c. As we mentioned, the bias also incorporates the effect of looking at the problem in a moving frame of velocity $\vec{u}$. The phase transition can thus be driven by $\vec{u}$ and occurs when $|\vec{f}_{\vec{u}}| = |\vec{f} + \vec{u}| = f_c$: The pinned phase occurs everywhere in space outside a "light cone" around the optimal direction of the RW (see Fig. 1). This picture agrees with known results: It was shown in Refs. [37–39] that the annealed and quenched large-deviation rate functions $I_a(u)$ and $I_q(u)$[6] of an unbiased lattice RW coincide for small u in $d \geqslant 3$, but always differ in $d = 1,2$ and for large enough u in $d \geqslant 3$. This confirms our scenario of a transition in $d \geqslant 3$, and our arguments show that the strong bias phase is in the KPZ class.

Let us now discuss the scale at which KPZUC emerges, first in the simpler one-dimensional case. To that aim, note that rescaling time, space, and height in (7) as $t = t^*t'$, $x = x^*x'$, and $h'(t',x') := \frac{1}{h^*}h(t^*t',x^*x')$ with the characteristic scales $t^* = \frac{(4D)^3}{r_0^2 f^4}$, $x^* = \frac{8D^2}{r_0 f^2}$, and $h^* = 1$, leads to a rescaled KPZ-like equation for $h'(t',x')$ identical to (1) with $\lambda_0 = D_0 = 2$, $\nu = 1$, up to the second source of noise of (7) which now involves a unit white noise multiplied by the dimensionless ratio $f r_0 / (2D\sqrt{2})$. Hence for $f r_0 / D \ll 1$ (weak bias/weak noise or large diffusivity limit) the "deformed" KPZ equation (7) becomes equivalent to the standard KPZ equation (this is reminiscent of the "weak universality" of the KPZ equation). Hence in this weak bias regime, we can apply the known results for the continuum KPZ equation (see Ref. [40]). Thus,

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[3]The existence of an upper-critical dimension d_c, where $\sigma_c = +\infty$ has not yet been settled.

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[4]Note that in a sense the conservation of probability of the random walk problem seems to be lost in the KPZ regime. This is only because the mapping to KPZ only holds locally in the large-deviation region $x = o(t)$. Everywhere in that region the probability mass escapes towards the most probable direction, where the equivalence to KPZ breaks down.

[5]$K_d = S_d/(2\pi)^d$, where S_d is the d-dimensional unit sphere area.

[6]These are defined as $I_a(u) := -\lim_{t\to\infty} \frac{\ln \overline{Q(t\vec{u},t)}}{t}$ and $I_q(u) := -\lim_{t\to\infty} \frac{\overline{\ln Q(t\vec{u},t)}}{t}$.

FIG. 2. The behavior of the RW crosses over from a diffusive to a bias dominated regime when $t \sim t_f$. The latter is also subdivided in between a EW regime and a KPZ regime for $t^* \gg t_f$ (see text for an estimation of t^* in $d = 1,2$).

for $t/t^* \gg 1$, we predict that the KPZUC appears in the RW problem. At short scale $t', x' \ll 1$, the behavior of the height in the KPZ equation becomes similar to the Edward-Wilkinson (EW) behavior [41]. In the RW problem we expect by inspection of (7) that the first source of noise (bias) dominates for $x \gg x_f = D/f$ while the second (diffusion) dominates for $x \ll x_f$ (with an associated time scale $t_f := x_f^2/D$). We conclude that for $f r_0/D \ll 1$ there is a regime $x_f \ll x \ll x^*$ and $t_f \ll t \ll t^*$ where one can already neglect the second source of noise but KPZUC-type fluctuations have not yet been built up: This should be an EW regime[7] (see Fig. 2). In general d the scale at which the bias starts to dominate remains t_f and x_f, but the scales t^* and x^* where KPZ fluctuations emerge change. For example, in $d = 2$, disorder is marginally relevant and from RG [19,36,43] one has $x^* \simeq_{g \ll 1} \Lambda^{-1} e^{1/g}$ with here (see above) $g = r_0^2 f^2/(16\pi D^2)$, $t^* = (x^*)^2/D$ and for the RW we take $\Lambda^{-1} = r_c$. For $g \ll 1$ the scales are well separated and we similarly expect an intermediate EW regime of fluctuations.

It is useful to extend our analysis to the *forward* transition probability $P(\vec{x},t) = P(\vec{x},t|0,0)$. It satisfies the Fokker-Planck equation $\partial_t P = D\nabla_x^2 P - \vec{\nabla}_x \cdot [(\vec{f} + \vec{\xi})P]$. Considering again the "partition sum variable" $Z(\vec{x},t) := e^{-\frac{\vec{f} \cdot \vec{x}}{2D} + t\frac{f^2}{4D}} P(\vec{x},t)$ generates additional noise terms in this equation and our arguments can be repeated (see Ref. [40]): The statistical properties of $Z(\vec{x},t)$ at large scale are identical to those of the DP partition sum. In fact, note that in law, $P(\vec{x},t) \sim Q(-\vec{x},t)$. We can also consider different initial (final) conditions in the forward (backward) setting. This is of great interest since the KPZUC is split in subuniversality classes [24] that depend on the boundary conditions, and we thus predict universal distributions for the fluctuations of $\ln P$ or $\ln Q$ according to the chosen boundary conditions (see Fig. 3 for examples in $d = 2$). These were determined numerically in $d = 2$ [28] and are known analytically in $d = 1$, on which we now focus. Using our argument and KPZ universality, we conjecture that the rescaled fluctuations of $\ln P(x,t)$ and $\ln Q(x,t)$ are universal in the large-deviation region and distributed as a TW GUE random variable χ_2 [24]. This has already been observed analytically and numerically for the exactly solvable beta polymer (see Refs. [31,32]). For the continuum model (2)–(4) in the absence of bias, $f = 0$, but in a moving frame, we obtain (using Ref. [26]; see Ref. [40]) a sharp prediction

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[7]We note that links between the Edward-Wilkinson universality class and the TD-RWRE have already been studied (see Refs. [17,42]). This, however, seems very different from what we discuss here.
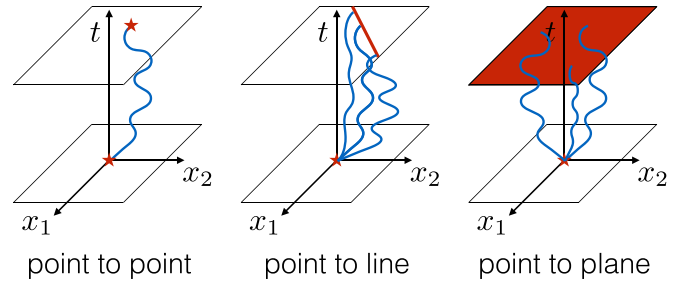

FIG. 3. Some typical polymer geometries for DP in $d = 2$. See text for applications to the RW problem.

for $t \gg t^*$,

$$\ln P(x = ut,t) \simeq -I_q(u)t + \lambda(u)t^{1/3}\chi_2, \quad (11)$$

where $I_q(u) \simeq \frac{u^2}{4D} + \frac{2r_0^2 u^4}{3(8D)^3}$ and $\lambda(u) \simeq \frac{r_0^{2/3} u^{4/3}}{4D}$, estimates valid in the weak bias limit $r_0 u/D \ll 1$. Using the equivalence (for small bias) between the RW and the KPZ equation at finite t/t^* [40], the scaling $x = y r_0 (4Dt/r_0^2)^{3/4}$ implements the crossover, as a function of $y = (t/t^*)^{1/4}$, from EW to KPZ fluctuations for $\ln P(x,t)$, the crossover to diffusion occurring for $x \sim (Dt)^{1/2}$.

We now make a prediction related to the flat KPZ subuniversality class in the TD-RWRE context. It is known that the large time fluctuations of the logarithm of the solution of the MSHE $\partial_t Z = D\nabla^2 Z + \xi_{DP} Z$ with a flat initial condition $Z(x,t = 0) = 1$, properly scaled, are distributed according to a GOE Tracy-Widom random variable χ_1. Here, it means that the initial probability of the RW[8] must be $P(x,t = 0) \sim e^{fx/(2D)}$. It is a natural and normalizable initial condition on an interval of length L with reflecting boundary conditions, $x \in [-L/2,L/2]$: It is the stationary measure of the RW in the absence of disorder. Turning on the disorder at $t = 0$, we predict that at large times [in the regime $1 \ll t/t^* \ll (L/x^*)^{3/2}$ to avoid the influence of the boundaries], $\ln P(0,t)$ fluctuates as $c(t/t^*)^{1/3}\chi_1$, where $c = 2^{-2/3}$ [44] when $t^* \gg D/f^2$. This scenario, and its universality, is checked explicitly through simulations of a one-dimensional discrete TD-RWRE (see Fig. 4).

An important application of the large-deviation regime of the RW where the KPZUC emerges is to extreme value statistics. Consider $N \gg 1$-independent walkers starting at the origin at $t = 0$ with no bias, $\vec{f} = 0$. We define $x_{max}(t) := \max_{i=1,\ldots,N}\{\vec{x}_i(t) \cdot \vec{e}_1\}$ the position of the rightmost walker in the direction of the unit vector $\vec{e}_1$. We show [40] that the KPZ universality in the fluctuations of the logarithm of the transition probability, $\ln P(\vec{x} \cdot \vec{e}_1 = ut,t|0,0)$, implies that as $N,t \to \infty$ with $\gamma = \frac{\ln N}{t}$ fixed. Then $x_{max}(t)$ grows ballistically,

$$x_{max}(t) \simeq u_\gamma^* t + c(\gamma)t^{\theta_d}\chi + o(t^{\theta_d}). \quad (12)$$

Here, θ_d is the KPZ exponent, and χ has a universal distribution [40] characteristic of the point to hyperplane (of dimension

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[8]Here, we adopt the forward setting. Indeed, observing GOE fluctuations in the backward case requires imposing the final probability $Q(x,t = 0) \sim e^{-fx/D}$ which does not seem possible.
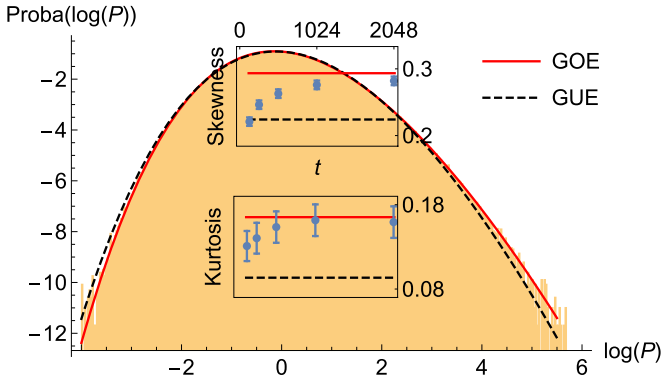
FIG. 4. Numerical observation of the GOE TW distribution for the log of the forward transition probability $P(0,t_d)$ of a TD-RWRE on $[-4096,4096] \cap \mathbb{Z}$ with a reflexive boundary in a biased random environment (see details in Ref. [40]), starting at time $t_d = 0$ with the stationary measure of the RW in the absence of disorder. Main plot: Centered and normalized histogram (in a logarithmic scale) of $\ln P(0,t_d)$ with $t_d = 2048$ compared with the GOE (red line) and GUE (black dashed line) TW distribution. The insets show the convergence of the skewness (top) and of the excess of kurtosis (bottom) of the distribution of $\ln P(0,t_d)$ to values close to those of the GOE TW distribution. Error bars are 3-sigma Gaussian estimates.

$d - 1$) subuniversality KPZUC (see, e.g., Ref. [28]). Here, $c(\gamma)$ and u_γ^* are nonuniversal, given in the continuum in Ref. [40]. This is valid if the front velocity $u_\gamma^* > u_c$, so that KPZUC appears (with $u_c = 0$ in $d = 1,2$). A formula such as (12) was rigorously shown in an exactly solvable 1D model in

Ref. [31] with $\theta_{d=1} = 1/3$ and $\chi = \chi_2$ a GUE TW random variable. Similarly, the first arrival time at $\vec{x} \cdot \vec{e}_1 = \ell$, $T_{\text{Hit}}(\ell)$, of a particle from a cloud of N-independent particles, behaves for fixed $\hat{\gamma} = \frac{\ln N}{\ell}$ as

$$T_{\text{Hit}}(\ell) \simeq \ell/v_{\hat{\gamma}}^* - d(\hat{\gamma})\ell^{\theta_d}\chi + o(\ell^{\theta_d}), \qquad (13)$$

with the same universal random variable χ [40]. Arrival times in compact domains, i.e., a ball, lead instead to a point-to-point KPZ distribution in any d.

In this Rapid Communication we investigated the origin and consequences of the emergence of universal statistics of the KPZUC in the large-deviation regime of TD-RWRE in arbitrary dimensions. We focused on short-range correlated random media, but our method readily extends to long-range (LR) *spatial* correlations [40], leading to a distinct LR space correlated KPZ universality classes [45]. Important questions for the future are how LR correlations in *time* in the medium, and interactions within a cloud of N particles, will affect the results, since those are present in many natural examples, such as the atmosphere or the ocean. We hope that this motivates further connections between the fields of growth and diffusion.

[1] P. P. Mitra, P. N. Sen, L. M. Schwartz, and P. Le Doussal, Phys. Rev. Lett. **68**, 3555 (1992).

[2] E. Balkovsky, G. Falkovich, and A. Fouxon, Phys. Rev. Lett. **86**, 2790 (2001).

[3] J. Bernasconi, H. U. Beyeler, S. Strassler, and S. Alexander, Phys. Rev. Lett. **42**, 819 (1979).

[4] J.-P. Bouchaud, J. Phys. I (France) **2**, 1705 (1992).

[5] I. Giardina, J. P. Bouchaud, and M. Mezard, J. Phys. A **34**, L245 (2001).

[6] J. P. Bouchaud and M. Mezard, Physica A (Amsterdam) **282**, 536 (2000).

[7] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).

[8] H. Kesten, M. Koslov, and F. Spitzer, Compos. Math. **30**, 145 (1975); Y. G. Sinai, Theory Probab. Appl. **27**, 256 (1982); J. P. Bouchaud, A. Comtet, A. Georges, and P. Le Doussal, Europhys. Lett. **3**, 653 (1987); Ann. Phys. (NY) **201**, 285 (1990); D. S. Fisher, P. Le Doussal, and C. Monthus, Phys. Rev. E **64**, 066107 (2001).

[9] D. S. Fisher, Phys. Rev. A **30**, 960 (1984); J. A. Aronovitz and D. R. Nelson, *ibid.* **30**, 1948 (1984); V. E. Krastsov, I. V. Lerner, and V. I. Yudson, Sov. Phys. JETP **64**, 5 (1986); J. Phys. A **18**, L703 (1985).

[10] G. Lawler, Commun. Math. Phys. **87**, 81 (1982).

[11] J. P. Bouchaud, A. Comtet, A. Georges, and P. Le Doussal, J. Phys. (Paris) **48**, 1445 (1987); **49**, 369 (1988); J. Honkonen and E. Karjalainen, J. Phys. A **21**, 4217 (1988).

[12] J. P. Bouchaud, Europhys. Lett. **11**, 505 (1990).

[13] L. F. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926); H. Hentschel and I. Procaccia, Phys. Rev. A **29**, 1461 (1984); M. C. Jullien, J. Paret, and P. Tabeling, Phys. Rev. Lett. **82**, 2872 (1999).

[14] M. Chertkov and G. Falkovich, Phys. Rev. Lett. **76**, 2706 (1996); D. Bernard, K. Gawedzki, and A. Kupiainen, Phys. Rev. E **54**, 2564 (1996).

[15] F. Rassoul-Agha and T. Seppalainen, Probab. Theory Relat. Fields **133**, 299 (2005); J.-D. Deuschel, X. Guo, and A. F. Ramirez, arXiv:1503.01964.

[16] E. Schertzer, R. Sun, and J. M. Swart, arXiv:1506.00724; Y. Le Jan and S. Lemaire, Probab. Theory Relat. Fields **130**, 109 (2004).

[17] M. Balazs, F. Rassoul-Agha, and T. Seppalainen, Commun. Math. Phys. **266**, 499 (2006).

[18] F. Rassoul-Agha, T. Seppalainen, and A. Yilmaz, Commun. Pure Appl. Math. **66**, 202 (2013); arXiv:1104.3110.

[19] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[20] M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987); D. A. Huse, C. L. Henley, and D. S. Fisher, *ibid.* **55**, 2924 (1985); T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).

[21] M. Prähofer and H. Spohn, Phys. Rev. Lett. **84**, 4882 (2000); H. Spohn, arXiv:1601.00499.

[22] For review, see, e.g., T. Kriecherbauer and J. Krug, J. Phys. A **43**, 403001 (2010).

[23] K. Johansson, Commun. Math. Phys. **209**, 437 (2000).

[24] I. Corwin, Random Matrices: Theory Appl. **01**, 1130001 (2012); arXiv:1403.6877.

[25] C. A. Tracy and H. Widom, Commun. Math. Phys. **159**, 151 (1994).

[26] T. Sasamoto and H. Spohn, Phys. Rev. Lett. **104**, 230602 (2010); P. Calabrese, P. Le Doussal, and A. Rosso, Europhys. Lett. **90**, 20002 (2010); V. Dotsenko, *ibid.* **90**, 20003 (2010); G. Amir, I. Corwin, and J. Quastel, Commun. Pure Appl. Math. **64**, 466 (2011).

[27] A. Pagnani and G. Parisi, Phys. Rev. E **92**, 010101(R) (2015).

[28] T. Halpin-Healy, Phys. Rev. Lett. **109**, 170602 (2012); T. Halpin-Healy and G. Palasantzas, Europhys. Lett. **105**, 50001 (2014).

[29] T. Halpin-Healy and Y. Lin, Phys. Rev. E **89**, 010103 (2014).

[30] T. Halpin-Healy and K. A. Takeuchi, J. Stat. Phys. **160**, 794 (2015).

[31] G. Barraquand and I. Corwin, Probab. Theory Relat. Fields **167**, 1057 (2017).

[32] T. Thiery and P. Le Doussal, J. Phys. A: Math. Theor. **50**, 045001 (2017).

[33] I. Corwin and Y. Gu, J Stat. Phys. **166**, 150 (2017).

[34] M. Balázs, F. Rassoul-Agha, and T. Seppäläinen (unpublished).

[35] See, e.g., C. Monthus and T. Garel, Eur. Phys. J. B **53**, 39 (2006), and references therein.

[36] E. Frey and U. C. Tauber, Phys. Rev. E **50**, 1024 (1994).

[37] A. Yilmaz, Ann. Inst. Henri Poincare, Sect. B **083**, 45 (2010).

[38] A. Yilmaz, Probab. Theory Relat. Fields **149**, 463 (2011).

[39] A. Yilmaz and O. Zeitouni, Commun. Math. Phys. **300**, 243 (2010).

[40] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.96.010102 for technical details, for the derivations, and for the numerical simulations.

[41] T. Gueudré, P. Le Doussal, A. Rosso, A. Henry, and P. Calabrese, Phys. Rev. E **86**, 041151 (2012).

[42] P. A. Ferrari and L. Fontes, Electron. J. Probab. **3**, 6 (1998).

[43] D. R. Nelson and P. Le Doussal, Phys. Rev. B **42**, 10113 (1990).

[44] P. Calabrese and P. Le Doussal, Phys. Rev. Lett. **106**, 250603 (2011); P. Le Doussal and P. Calabrese, J. Stat. Mech. (2012) P06001.

[45] See, e.g., S. Chu and M. Kardar, Phys. Rev. E **94**, 010101 (2016), and references therein.