Ensemble dynamics and the emergence of correlations in one- and two-dimensional wave turbulence

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We investigate statistical properties of wave turbulence by monitoring the dynamics of ensembles of trajectories. The system under investigation is a simplified model for surface gravity waves in one and two dimensions with a square-root dispersion and a four-wave interaction term. The simulations of decaying turbulence confirm the Kolmogorov-Zakharov spectral power distribution of wave turbulence theory. Fourth-order correlations are computed numerically as ensemble averages of trajectories. The shape, scaling, and time evolution of the correlations agree with the predictions of wave turbulence theory.

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I. INTRODUCTION

Wave turbulence [1–3] is a disordered nonequilibrium state of weakly interacting dispersive waves in nonlinear optics [4–6], fluid mechanics [7], and plasma physics [8,9]. This nonequilibrium is maintained by external driving forces and dissipation, which affect length scales that are in many cases widely separated in wave number space. Driving and damping inject and dissipate quantities like energy and wave action, which are conserved by the Hamiltonian dynamics within the inertial range between these scales. This causes flows of the conserved quantities in wave number space from the driving range to dissipation ranges at high and low wave numbers [10]. These flows are mediated by the nonlinear interaction of weakly correlated waves [1].

Wave turbulence theory provides an analytical connection between a nonlinear wave equation for a field $\psi(\mathbf{x},t)$ and statistical quantities like the wave action density $n_{\mathbf{k}}(t)$, which is the Fourier transform of the two point correlation $\langle\psi(\mathbf{x},t)\psi^*(\mathbf{x}+\mathbf{r},t)\rangle$. Wave turbulence theory is applicable to turbulence that is spatially homogeneous in the sense that ensemble averages like $\langle\psi(\mathbf{x},t)\psi^*(\mathbf{x}+\mathbf{r},t)\rangle$ depend only on the relative vector $\mathbf{r}$ but not on the location $\mathbf{x}$; $n_{\mathbf{k}}(t)$ is then independent of $\mathbf{x}$. A central step of wave turbulence theory is the derivation of closed kinetic equations for the slow time dependence of $n_{\mathbf{k}}(t)$ through an asymptotic expansion of cumulants [3,11]. Stationary solutions of the kinetic equations include turbulent nonequilibrium spectra (Kolmogorov-Zakharov spectra) and thermal equilibrium spectra (Rayleigh-Jeans spectra) [1].

In this paper we probe the predictions of wave turbulence numerically for a generic system with four-wave interactions in one and two dimensions. We compute firstly nonequilibrium spectra as time averages, and secondly fourth-order correlations as ensemble averages of trajectories; that is, we integrate the equations of motion for a large set of initial conditions and compute statistical averages over the ensemble of trajectories. In contrast to averages over time, the ensemble averages allow us to trace the time evolution of statistical quantities. The correlations that we compute are central for the interactions of waves in turbulence as they provide directed flows of conserved quantities in wave number space; we compare our numerical results to the analytical expressions that underlie the kinetic equations of wave turbulence theory.

The equation of motion under investigation (Majda-McLaughlin-Tabak equation [12]) is

$$i\frac{\partial\psi}{\partial t} = \mathcal{L}^{(1/2)}\psi + \sigma\psi|\psi|^2 + \mathcal{D}, \tag{1}$$

where $\psi(x,t)$ or $\psi(x,y,t)$ is a complex variable in one or two spatial dimensions. $\mathcal{L}^{(1/2)}$ is an operator with the eigenfunctions e^{ikx} and $e^{i\mathbf{k}\cdot\mathbf{x}}$ and the eigenvalues $\omega = \sqrt{|k|}$ and $\omega = \sqrt{|\mathbf{k}|}$ in one and two dimensions, respectively, which mimics the dispersion of gravity surface waves in fluids [13]. The nonlinearity $|\psi|^2\psi$ serves as a simplified version of the actual four-wave interaction of surface waves [14–17]; its coefficient can always be scaled to $\sigma = \pm 1$. $\mathcal{D}$ is a damping term that affects very long and short waves only. Our study is confined to decaying turbulence in order to avoid any contamination of statistical data by external driving forces.

Equation (1) without dissipation can be derived from a Hamilton function $H = \int |\mathcal{L}^{(1/4)}\psi|^2 + \sigma|\psi|^4/2dV$, with $dV = dx$ in one and $dV = dxdy$ in two dimensions. The operator $\mathcal{L}^{(1/4)}$ has again the eigenfunctions e^{ikx} or $e^{i\mathbf{k}\cdot\mathbf{x}}$, the eigenvalues are $|k|^{1/4}$ or $|\mathbf{k}|^{1/4}$. $N = \int |\psi|^2 dV$ is a second conserved quantity that is associated to the continuous phase symmetry of the system; a third conserved quantity, the momentum, follows from the translational symmetry [12].

Numerical studies of nonequilibrium spectra of this system have lead to a debate on the validity of the closure of wave turbulence. Wave turbulence theory predicts the Kolmogorov-Zakharov spectra $n_k = |k|^{-1}$ in one dimension and $n_{\mathbf{k}} = |\mathbf{k}|^{-2}$ in two dimensions. The kinetic equations are independent of the sign of σ, so the same spectra are expected for $\sigma = \pm 1$. In contrast to this prediction, numerical studies of Eq. (1) in one dimension with an external driving force have detected a spectrum $n_k \sim |k|^{-1.25}$ for $\sigma = 1$ [18–21]; the Kolmogorov-Zakharov spectrum has been found for $\sigma = -1$ [19–21]. Simulations of the fourth-order correlation function [12] have appeared to differ from the closure of wave turbulence, too.

Two different interpretations of these results have been discussed. The first one questions the validity of the wave turbulence closure and suggests alternative random wave closure [12,21] to explain the steeper spectrum. The second is that coherent structures (collapses [18–20], quasisolitons [21,22]) that coexist with wave turbulence change the spectrum. The results that we present in this paper are consistent with this

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second interpretation: By avoiding conditions under which coherent structures can emerge [22–24] we create a state of pure wave turbulence; our numerical experiments verify several central predictions of wave turbulence theory.

In Sec. II we present time averaged numerical spectra in one and two dimensions. We show that the results are in good agreement with the Kolmogorov-Zakharov spectra predicted by wave turbulence theory.

In Sec. III we study the time evolution of fourth-order correlations in one and two dimensions. The correlations are computed as ensemble averages of trajectories. We find that correlations of resonant quartets of modes emerge spontaneously from an ensemble of initial conditions with random phases. We also show that the imaginary part of fourth-order correlation scales as the third power of second-order correlations, as it is predicted by wave turbulence theory.

In Sec. IV we discuss our results in the context of instabilities and the formation of coherent structures [18,19,21,22] that can lead to deviations from pure wave turbulence.

II. WAVE TURBULENCE SPECTRA IN ONE AND TWO DIMENSIONS

We present numerical simulations of decaying turbulence that confirm the Kolmogorov-Zakharov spectra $n_k \sim |k|^{-1}$ in one dimension and $n_{\mathbf{k}} \sim |\mathbf{k}|^{-2}$ in two dimensions. Equation (1) is numerically integrated using a standard pseudospectral method that eliminates the linear part of the equation of motion [12]. The resulting non-stiff problem is integrated with an adaptive step-size Runge-Kutta solver with an accuracy that is sufficient to simulate high-k modes whose amplitudes in a turbulent spectrum are small. In one dimension the system size is $L = 2\pi$ with periodic boundary conditions. The wave numbers are integers $-2048 \leqslant k < 2048$, and we denote the Fourier modes as A_k. The wave action is related to the Fourier modes as $n_k = \langle |A_k|^2 \rangle$ with $\sum_k n_k = \int_0^{2\pi} \langle |\psi(x)|^2 \rangle dx$. The amplitude is small enough so that the linear time scales of Eq. (1) are short compared to the nonlinear time scale for all $k > 1$.

Damping is applied to the homogeneous mode $k = 0$ and to short waves with $|k| > 1024$ where the strength of damping increases linearly in $|k|$. The $k = 0$ damping absorbs the inverse cascade and the high-$|k|$ damping absorbs the direct cascade and minimizes aliasing errors caused by the interaction of short waves. No external driving force is applied. Numerical simulations are performed for a variety of initial conditions, including long waves (the power is gathered at waves with $|k| \leqslant 10$) and a white spectrum where the power is evenly distributed over all wave numbers. The equation of motion is at first integrated over a period of 20 000 time units for relaxation. Subsequently, the spectrum is computed as an average over 60 000 time units. Here, we use a time average instead of an ensemble average because of the high computational requirements for a long relaxation period. Figure 1 shows Kolmogorov-Zakharov spectra $n_k \sim |k|^{-1}$ that we find for $\sigma = -1$ and for $\sigma = 1$ for initial conditions with a broad spectrum; long-wave initial conditions where the power is gathered at wave numbers $|k| \leqslant 10$ yield very similar results (not shown here).

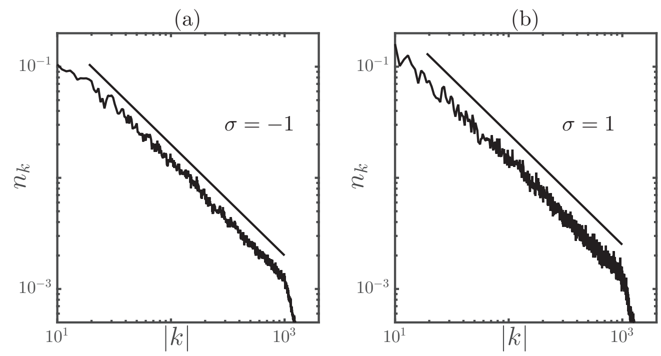

FIG. 1. Double-logarithmic plot of the wave action n_k over $|k|$ for decaying turbulence of Eq. (1) $\sigma = -1$ (a) and $\sigma = 1$ (b) in one dimension. The straight lines correspond to the direct cascade Kolmogorov-Zakharov spectrum $n_k \sim |k|^{-1}$. The spectra are time averages for decaying wave turbulence for a system that is damped both at small and at high $|k|$. The time averages are computed over an interval of 60 000 time units after an initial relaxation period of 20 000 time units.

For the two-dimensional system, we use a size of $2\pi \times 2\pi$, periodic boundary conditions, and 512×512 grid points. Damping is applied to short waves with $|\mathbf{k}| \geqslant 128$ and to the homogeneous mode $|\mathbf{k}| = 0$; again no external driving force is applied. The initial state that contains the same power in all modes is allowed to relax for 200 000 time units and then averaged over 800 000 time units. The simulations of Fig. 2 confirm the Kolmogorov-Zakharov spectrum $n_{\mathbf{k}} \sim |\mathbf{k}|^{-2}$ for either sign of the nonlinearity. Again, long wave initial conditions (the power is gathered at modes with $|\mathbf{k}| \leqslant 10$) yield the same type of spectrum after an appropriate relaxation time (not shown here).

The spectra of Figs. 1 and 2 decay slowly as dissipation reduces wave action and energy of the waves. They represent the dynamics close to the stationary solution of the kinetic

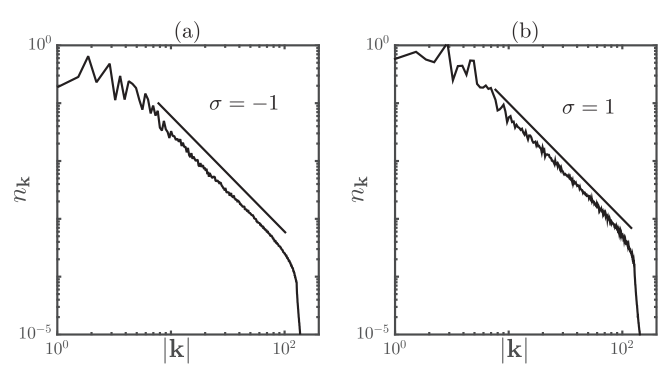

FIG. 2. Double-logarithmic plot of the wave action $n_{\mathbf{k}}$ over $|\mathbf{k}|$ for decaying turbulence of Eq. (1) $\sigma = -1$ (a) and $\sigma = 1$ (b) in two dimensions. The straight lines correspond to the direct cascade Kolmogorov-Zakharov spectrum $n_{\mathbf{k}} \sim |\mathbf{k}|^{-2}$. The spectra are averaged over time and direction for decaying wave turbulence for a system that is damped both at small and high $|\mathbf{k}|$. The time averages are computed over an interval of 800 000 time units after an initial relaxation period of 200 000 time units.

equation. Stationary Kolmogorov-Zakharov spectra can be maintained either by an external driving force that permanently supplies energy and wave action at the same rate as these quantities are dissipated, or by a small reservoir of wave action and energy on top of the spectrum at low $|k|$. In this case a direct and an inverse cascade emanate from this reservoir, so that the reservoir slowly decays but the spectrum remains unchanged until the reservoir is drained. This type of behavior can be created by starting with initial conditions where the power is gathered at $|k| \leqslant 10$ or $|\mathbf{k}| \leqslant 10$: The low-$k$ reservoir of wave action gradually decays during a transient period, and the distribution converges to a Kolmogorov-Zakharov spectrum. In conclusion, our simulations indicate that the Kolmogorov-Zakharov spectrum is attractive for a broad range of initial conditions.

The relaxation in two dimension is surprisingly slow given the fact that there are more resonant quartets of waves compared to the one-dimensional system. On the other hand, high-$|\mathbf{k}|$ waves in the Kolmogorov-Zakharov spectrum have smaller amplitudes, so the interaction of resonant quartets is weaker. This slow relaxation may be due to the absence of quasisolitons in the two-dimensional system: While quasisolitons are an efficient mechanism of energy transfer in one dimension for $\sigma = 1$ [22], the instability that triggers their formation is absent in two dimensions [23], and we have no numerical indication for the presence of quasisolitons in this case. For $\sigma = -1$ wave collapses [18–20] are possible in one and two dimensions for initial conditions with sufficiently high amplitudes.

III. ENSEMBLE DYNAMICS AND FOURTH-ORDER CORRELATIONS

In this section we study fourth-order correlations by numerically computing ensembles of trajectories. We briefly summarize their role in wave turbulence theory. While Fourier transforms A_k of arbitrary bounded physical fields $\psi(x)$ on an infinite domain do not exist as ordinary functions, cumulants [ensemble averages of products of fields at different locations like $\langle\psi(x)\psi^*(x+r)\rangle - \langle\psi(x)\rangle\langle\psi^*(x+r)\rangle$] can be Fourier transformed if the correlation of $\psi(x)$ and $\psi(x+r)$ decays at a sufficient rate as a function or r. $\langle\psi(x)\psi^*(x+r)\rangle$ and its Fourier transform n_k are independent of the base coordinate x for fields that are spatially homogeneous in a statistical sense. The Fourier modes are then correlated as $\langle A_k A^*_{k'}\rangle = n_k\delta(k-k')$, where δ is the Dirac delta. Equations of motion for lower order Fourier moments or cumulants like n_k depend on higher order moments or cumulants. For the one-dimensional example, the time evolution of n_k

$$\frac{\partial n_k}{\partial t} = 2\sigma \int \mathrm{Im} J_{123k}\delta(k_1+k_2-k_3-k)dk_1dk_2dk_3 \qquad (2)$$

depends on the fourth-order correlation

$$\langle A_{k_1}A_{k_2}A^*_{k_3}A^*_k\rangle = J_{123k}\delta(k_1+k_2-k_3-k); \qquad (3)$$

via the cubic nonlinearity of the wave equation (1) (see [1,19,21]); we abbreviate this correlation as $c_{123k} = \langle A_{k_1}A_{k_2}A^*_{k_3}A^*_k\rangle$. Similarly, the equation of motion for J_{123k} depends on the sixth-order correlation. As a result, the statistical mechanics of the field is governed by an infinite hierarchy of equations of motion.

Wave turbulence theory [1–3,11] provides an asymptotic closure for weakly interacting dispersive waves, i.e., it shows that the dynamics generates correlations in a way that higher order cumulants behave as functions of lower order cumulants. The premises of wave turbulence theory [3] are spatial homogeneity (in a statistical sense) of the field, a sufficiently rapid decay of the cumulants in the initial conditions (i.e., fields at distant points are not correlated), and a wide separation of the linear and nonlinear time scales for all k. The closure is the result of a solvability condition, i.e., a condition for avoiding secular terms in the cumulant expansion [11]. For the imaginary part of the fourth-order correlation, it follows [1] that

$$\mathrm{Im} J_{123k} \approx 2\pi\sigma f_{123k}\delta(\omega_1+\omega_2-\omega_3-\omega) \qquad (4)$$

with

$$f_{123k} = n_1n_2n_3 + n_1n_2n_k - n_1n_3n_k - n_2n_3n_k. \qquad (5)$$

$\mathrm{Im} J_{123k}$ is δ-shaped, the sign of the peak depends on σ and its size scales $\sim n^3$. The real part scales as $\mathrm{Re} J_{123k} \sim n^2$. This reduction of higher-order correlators to two correlators is a lowest-order result of the perturbation theory [11]. The slow time dependence of n_k is governed by a kinetic equation that can be written down by expressing $\mathrm{Im} J_{123k}$ in Eq. (2) as a function of the n_k using Eqs. (4) and (5). As a result, the time derivative of n_k Eq. (2) depends only on quartets n_k, n_{k_1}, n_{k_2}, n_{k_3} that satisfy the resonance condition $\omega_1+\omega_2=\omega_3+\omega$ of Eq. (4) in addition to $k_1+k_2=k_3+k$ of Eq. (2). The kinetic equation is independent of the sign of the nonlinearity as it contains the factor σ^2 only. The Kolmogorov-Zakharov spectrum ($n_k \sim |k|^{-1}$ in one dimension and $n_{\mathbf{k}} \sim |\mathbf{k}|^{-2}$ in two dimensions for $\sigma = \pm 1$) is a nonequilibrium solution of the kinetic equation and therefore a consequence of the closure of wave turbulence.

The validity of this closure has been questioned on the basis of steeper spectra that have been obtained numerically for $\sigma = 1$: Simulations of the one-dimensional system with an external driving force have generated spectra at or near $n_k \sim |k|^{-1.25}$ [12,19–21], so one might look for an alternative closure that leads to this spectrum. [12] have suggested the alternative closure

$$\mathrm{Im} J_{123k} \sim \sqrt{n_1n_2n_3n_k}(\omega_1+\omega_2-\omega_3-\omega_k)^{-1} \qquad (6)$$

from which a steeper spectrum with $n_k \sim |k|^{-1.25}$ would follow [19], have shown that the δ-function for the frequencies is a requirement for the conservation of energy. As a closure that satisfies this as well as basic symmetry properties and that leads to the steeper spectrum they have tentatively given

$$\begin{aligned}\mathrm{Im} J_{123k} \sim &\left(\frac{\partial\omega_1}{\partial k_1}+\frac{\partial\omega_2}{\partial k_2}+\frac{\partial\omega_3}{\partial k_3}+\frac{\partial\omega}{\partial k}\right)\\ &\times(n_1n_2-n_3n_k)\\ &\times\delta(\omega_1+\omega_2-\omega_3-\omega)\end{aligned} \qquad (7)$$

while stressing that this is unlike the closure of wave turbulence theory not based on any derivation.

By computing averages over an ensemble of trajectories we check central statements of the closure of Eq. (2), namely

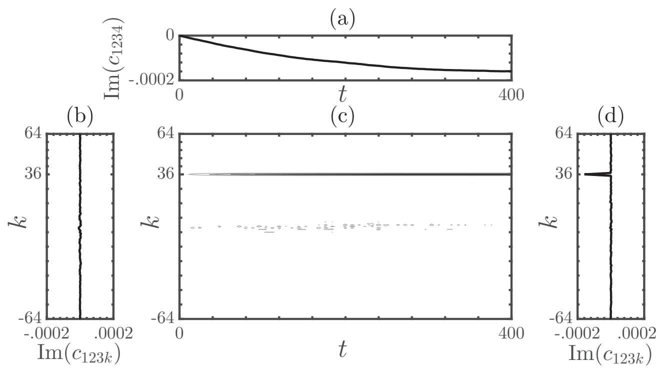

FIG. 3. Fourth-order correlation $\mathrm{Im}(c_{123k})$ with $c_{123k} = \langle A_{k_1} A_{k_2} A^*_{k_3} A^*_k \rangle$ for an ensemble of 5 120 000 trajectories of Eq. (1) with $\sigma = 1$. The modes are $k_1 = 49$, $k_2 = -4$, $k_3 = 9$ and k variable. (a) Time evolution of the correlation for the resonant quartet where $k = k_4 = 36$. (b) Correlation $\mathrm{Im}(c_{123k})$ as a function of k for the initial random-phase state. (c) Contour plot of $\mathrm{Im}(c_{123k})$ as a function of k and time showing the appearance of a correlation at $k = k_4 = 36$. (d) Correlation $\mathrm{Im}(c_{123k})$ at $t = 400$ that has a negative peak at $k = k_4 = 36$.

distribution of $\mathrm{Im}(c_{123k})$ that is sharply peaked as a function of k Eq. (3) and ω Eq. (4), its n^3-scaling Eq. (5) and its time-evolution. For the one-dimensional system we use 256 grid points. The ensemble consists of 5 120 000 trajectories. The modes A_k in the ensemble of initial conditions are random with a Gaussian distribution and a Kolmogorov-Zakharov spectrum $\langle |A_k|^2 \rangle \sim k^{-1}$ and $\langle |A_0|^2 \rangle = \langle |A_1|^2 \rangle$. The initial phases are random and uniformly distributed so that the ensemble average is $\mathrm{Im}(c_{123k}) = 0$ at $t = 0$; this distinguishes the initial conditions from wave turbulence. The equation of motion (1) is integrated numerically for each initial condition separately and the ensemble-average $\mathrm{Im}(c_{123k})(t)$ is recorded during the time-evolution. This allows us to monitor the build-up of phase-correlations as the systems converge toward wave turbulence. This is expected to occur at the intersection of the hypersurfaces in the four-dimensional space of k_1, k_2, k_3, k that are defined by Eqs. (3) and (4).

Figure 3 shows results $\mathrm{Im}(c_{123k})$ for $\sigma = 1$ in one dimension where we vary k along a path through an intersection point of these two hypersurfaces: The modes $k_1 = 49$, $k_2 = -4$, $k_3 = 9$ are fixed and k is varied; for $k = k_4 = 36$ the modes interact resonantly, i.e. $k_1 + k_2 = k_3 + k_4$, $\sqrt{|k_1|} + \sqrt{|k_2|} = \sqrt{|k_3|} + \sqrt{|k_4|}$.

Figure 3(b) shows the ensemble average $\mathrm{Im}(c_{123k})$ as a function of k for the initial state ($t = 0$). It is almost identically zero because of the random phase initial conditions. Figure 3(d) shows a negative peak in the distribution $\mathrm{Im}\, J_{123k}$ at the time $t = 400$; the peak is located at the resonance $k = k_4 = 36$. This correlation disappears for any variations of the wave numbers that do not satisfy the conditions $k_1 + k_2 = k_3 + k$ and $\omega_1 + \omega_2 = \omega_3 + \omega$. The sign of the peak agrees with Eq. (4): With the Kolmogorov-Zakharov spectrum $n_k \sim k^{-1}$ equation (5) yields $f_{1234} \sim n_4^{-1} + n_3^{-1} - n_2^{-1} - n_1^{-1} \sim |k_4| + |k_3| - |k_2| - |k_1| = 36 + 9 - 4 - 49 < 0$, so the sign of the peak is negative for $\sigma = 1$.

Figure 3(a) shows the time-evolution of $\mathrm{Im}(c_{1234})$ starting at zero and approaching a negative saturation value. In wave turbulence [1] the time-evolution of the correlation of a resonant quartet is expected to follow $\frac{d}{dt}\mathrm{Im}(c_{1234}) \sim \mathrm{Im}(c^{(\mathrm{stat})}_{1234}) - \mathrm{Im}(c_{1234})$, where $\mathrm{Im}(c^{(\mathrm{stat})}_{1234})$ is the value for a statistically stationary nonequilibrium, and our simulations are consistent with this exponential approach to the stationary state. Simulations over a longer period of time show that the correlations then decrease slowly as dissipation decreases the amplitudes of the modes.

Figure 3(c) is a contour plot of $\mathrm{Im}(c_{123k})(t)$ depending on time and k that shows the correlation emerging from a background of small residual random fluctuations due to the finite ensemble size. The fluctuations are noticeable only near $k = 0$ where the amplitudes of the modes are high.

Analogous results (not shown here) are obtained for two modified simulations. Firstly, a nonlinearity with the sign $\sigma = -1$ leads to a positive peak. This is in agreement with Eq. (5). Secondly, equivalent results are obtained for a quartet of modes $k_1 = 196$, $k_2 = -16$, $k_3 = 36$, $k_4 = 144$ in a system with four times more modes. These simulations are based on a smaller ensemble and are therefore noisier.

The correlations that we find numerically differ from the one found in [12], where the modes $k_1 = 441$, $k_2 = 81$, $k_3 = -36$ with k variable were used. This satisfies $k_1 + k_2 = k_3 + k$ for $k = 558$ and $\sqrt{|k_1|} + \sqrt{|k_2|} = \sqrt{|k_3|} + \sqrt{|k|}$ for $k = 576$, so it yields no resonant quartet of modes.

Simulations of the two-dimensional system show results that are equivalent to our findings in one dimension. In the ensemble of initial conditions the modes $A_{\mathbf{k}}$ are again random Gaussian with a Kolmogorov-Zakharov spectrum $\langle |A_{\mathbf{k}}|^2 \rangle \sim |\mathbf{k}|^{-2}$ on a grid of 256×256 modes. Because of the higher computational requirements for each trajectory in two dimensions this ensemble contains only 51 200 trajectories. We monitor collinear [25] modes $\mathbf{k}_1 = 49\hat{\mathbf{e}}_x$, $\mathbf{k}_2 = -4\hat{\mathbf{e}}_x$, $\mathbf{k}_3 = 9\hat{\mathbf{e}}_x$ $\mathbf{k} = k\hat{\mathbf{e}}_x$. The correlations in Fig. 4 are divided by the average magnitude of the amplitudes $\mathrm{Im}(\tilde{c}_{123k}) =$

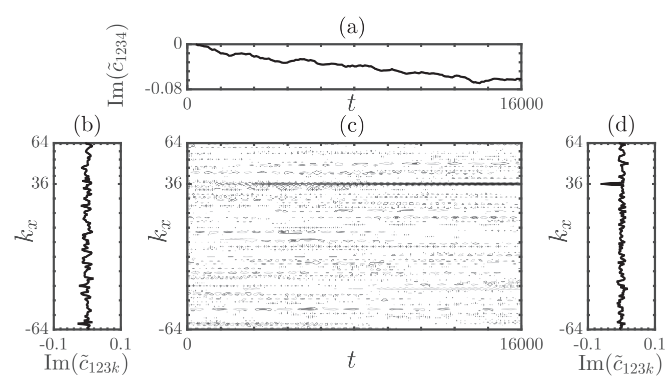

FIG. 4. Fourth-order correlation for an ensemble of 51 200 trajectories of Eq. (1) in two dimensions with $\sigma = 1$. The modes are collinear with $\mathbf{k}_1 = 49\hat{\mathbf{e}}_x$, $\mathbf{k}_2 = -4\hat{\mathbf{e}}_x$, $\mathbf{k}_3 = 9\hat{\mathbf{e}}_x$ $\mathbf{k} = k\hat{\mathbf{e}}_x$. The correlation is normalized by the magnitude of the modes as $\mathrm{Im}(\tilde{c}_{123k}) = \mathrm{Im}\langle A_{\mathbf{k}_1} A_{\mathbf{k}_2} A_{\mathbf{k}_3} A_{\mathbf{k}} \rangle / \langle |A_{\mathbf{k}_1} A_{\mathbf{k}_2} A^*_{\mathbf{k}_3} A^*_{\mathbf{k}}| \rangle$. (a) Time evolution of the correlation for the resonant quartet where $k = k_4 = 36$. (b) Correlation $\mathrm{Im}(\tilde{c}_{123k})$ as a function of k for the initial random-phase state. (c) Contour plot of $\mathrm{Im}(\tilde{c}_{123k})$ as a function of k and time showing the appearance of a correlation at $k = k_4 = 36$. (d) Correlation $\mathrm{Im}(\tilde{c}_{123k})$ at $t = 16\,000$ that has a negative peak at $k = k_4 = 36$.

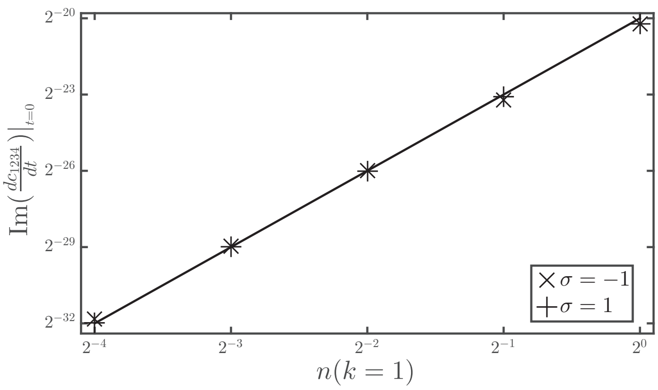

FIG. 5. Growth rate of the imaginary part of the correlation $\mathrm{Im}(c_{1234}) = \mathrm{Im}\langle A_{k_1} A_{k_2} A^*_{k_3} A^*_{k_4}\rangle$. The growth rate is determined by averaging ensembles of 512 000 trajectories of Eq. (1) with $\sigma = \pm 1$. The initial conditions have Kolmogorov-Zakharov spectral distributions with $n_k = k^{-1}$, $n_k = (2k)^{-1}$, $n_k = (4k)^{-1}$, $n_k = (8k)^{-1}$, $n_k = (16k)^{-1}$, and random phases. The line gives the wave turbulence prediction $\frac{d}{dt}\mathrm{Im}(c_{123k})|_{t=0} \sim n^3$.

$\mathrm{Im}\langle A_{\mathbf{k}_1} A_{\mathbf{k}_2} A^*_{\mathbf{k}_3} A^*_{\mathbf{k}}\rangle/\langle|A_{\mathbf{k}_1} A_{\mathbf{k}_2} A^*_{\mathbf{k}_3} A^*_{\mathbf{k}}|\rangle$ which reduces the level of noise at small k where the modes have high amplitudes.

Figure 4 shows again the formation of a peak at $k = 36$. The negative sign of the peak [Fig. 4(d)] is in accordance with $f_{1234} \sim |\mathbf{k}_4|^2 + |\mathbf{k}_3|^2 - |\mathbf{k}_2|^2 - |\mathbf{k}_1|^2 < 0$. Again we get the corresponding results with a positive peak for $\sigma = -1$. The smaller ensemble leads to the higher level of random background fluctuations in Fig. 4.

Finally, we check the scaling $\mathrm{Im}J_{1234} \sim n^3$ of Eq. (4) as opposed to $\mathrm{Im}J_{1234} \sim n^2$ of Eqs. (6) and (7). This can be tested by repeating the experiment of the formation of correlations in one dimension for initial conditions with various amounts of wave action. We measure $\frac{d}{dt}\mathrm{Im}(c_{1234})$ at $\mathrm{Im}(c_{1234}) = 0$ as a proxy for $\mathrm{Im}(c^{(\mathrm{stat})}_{1234})$. Both quantities scale $\sim n^3$ according to wave turbulence theory, but $\mathrm{Im}(c_{1234})$ has the disadvantage that it is influenced by the decay of total wave action in the damped system. Figure 5 gives the growth rate of $\mathrm{Im}(c_{1234})$ in one dimension at $t = 0$ that is derived numerically from an integration of Eq. (1) over a short period of time. The random phase initial conditions imply $\mathrm{Im}(c_{1234}(t = 0)) = 0$. Simulations are carried out for five different ensembles with initial conditions $n_k = k^{-1}$, $n_k = (2k)^{-1}$, $n_k = (4k)^{-1}$, $n_k = (8k)^{-1}$, $n_k = (16k)^{-1}$. The measured growth rates for $\sigma = \pm 1$ confirm the scaling $\sim n^3$ (Fig. 5).

IV. CONCLUSIONS

Our study confirms central predictions of wave turbulence theory in one and two dimensions:

(i) We find the Kolmogorov-Zakharov spectra $n_k \sim k^{-1}$ in one (Fig. 1) and $n_{\mathbf{k}} \sim |\mathbf{k}|^{-2}$ in two dimensions (Fig. 2). These spectra are obtained for time averages of a single trajectory.

(ii) Numerical computations of ensemble averages of states of decaying wave turbulence confirm the δ-shape Eqs. (3), (4) of the imaginary part of the fourth-order correlation [Figs. 3(d) and 4(d)] which is central to the closure of the kinetic equations.

(iii) The sign of the δ-peak in the correlation depends on σ as predicted by wave turbulence theory (5).

(iv) The time evolution of the correlations [Figs. 3(a) and 4(a)] is in agreement with $\frac{d}{dt}\mathrm{Im}(c_{1234}) \sim \mathrm{Im}(c^{(\mathrm{stat})}_{1234}) - \mathrm{Im}(c_{1234})$ as expected from wave turbulence theory. The evolution of these correlations in time is computed by tracking an ensemble of trajectories that emanate from a set of initial conditions whose statistical properties are similar to wave turbulence, but with random phases. From this initial state the system approaches the wave turbulence state exponentially in time as it builds up the correlations.

(v) The amplitude of the correlation scales $\sim n^3$ with the wave action (Fig. 5) as predicted by wave turbulence theory (5). More precisely, we infer this scaling from measuring $\frac{d}{dt}\mathrm{Im}(c_{1234})$ for a random-phase initial condition where $\mathrm{Im}(c_{1234}) = 0$. The results are obtained for decaying turbulence in Eq. (1) for either sign ($\sigma = \pm 1$) of the nonlinearity.

While this study confirms wave turbulence in one and two dimensions, significantly different behavior (in particular a steeper spectrum) has been observed in externally driven one-dimensional systems with $\sigma = 1$. We attribute this to instabilities of wave turbulence under circumstances that we have avoided in this study. First, an external driving force can create waves with high amplitudes at the driving range. A monochromatic wave of Eq. (1) with $\sigma = 1$ is unstable under two bands of modulations [24]: one where the modulations has a wavelength that is long compared to the carrier wave, and one where the wavelength of the modulation is shorter than the carrier wavelength. For $\sigma = -1$ there is no instability under long-wave modulations, but a wave is unstable under shortwave modulations [24]. Such unstable waves with sufficiently high-amplitudes could be created, e.g., by an external driving force, which we avoid in our simulations.

There is an analogous instability of states where the power is not gathered at one mode, but spread out over many random waves [23]. This instability enhances small spatial inhomogeneities of ensembles of turbulent systems. The effect of this breaking of spatial homogeneity is that the ensemble average $\langle\psi(x,t)\psi^*(x + r,t)\rangle$ and its Fourier transform $n_k(x,t)$ depend on the base coordinate x. It has been shown [23] that this type of instability can affect the state of wave turbulence for the one-dimensional version of Eq. (1) for $\sigma = 1$. As a result, wave turbulence that relies on spatial homogeneity is superseded by a coherent process, namely the formation of pulses (narrow bright quasisolitons [22]). These pulses emit long-wave radiation that transfers wave action towards small k analogously to the inverse cascade in wave turbulence. The pulse itself moves towards high k, which corresponds to a direct cascade of energy. A coherent process of evolving radiating pulses is then a new attractor that replaces wave turbulence. The spectrum derived in [22] for this process is $n_k \sim k^{1/\sqrt{2}-2} \approx k^{-1.29}$, which is in good agreement with the numerically observed steep spectra n_k near $k^{-1.25}$ [12,18–21]. Similar structures (quasiparticles) are known to be an important energy transfer mechanism between different length scales in plasma turbulence apart from conventional energy cascades [26,27].

We avoid the modulational instability [23] of random waves by applying dissipation only to the homogeneous mode $k = 0$.

This means that waves near the edge of the spectrum at $|k| = 1$ have lengths of the order of the system size; long modulations of these waves are obviously not supported by the system size so that this modulational instability is suppressed. This type of instability has been predicted not to exist in one dimension for $\sigma = -1$ and in two dimensions for any sign of σ [23], which is consistent with our simulations.

While this work relies on decaying turbulence without an external driving force, the statistically stationary nonequilibrium can be studied when the system is externally driven. A recent study [28] has shown that wave turbulence can prevail in this system when a sufficiently weak driving force is applied. Such nonequilibrium systems allow to measure the relationship between the amplitude of the wave action (the coefficient of the Kolmogorov-Zakharov spectrum) and the energy transfer rate. These quantities can be measured numerically, and compared to the Kolmogorov-constant that can be computed analytically. In a related system that does not support coherent structures this method verified the predictions of wave turbulence theory [29].

To conclude, tracking ensembles of trajectories can detect the low-dimensional dynamics of correlations [Figs. 3(a) and 4(a)] that is inherent in high-dimensional wave turbulence. This can also be a promising tool for studying interactions between nearly resonant sets of waves [25] as well as the formation of coherent structures within wave turbulence.

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