Revoking amplitude and oscillation deaths by low-pass filter in coupled oscillators

Wei Zou,[1,*] Meng Zhan,[2] and Jürgen Kurths[3,4,5]

[1]*Department of Physics, Hong Kong Baptist University, Kowloon Tong, Hong Kong SAR, People's Republic of China*
[2]*State Key Laboratory of Advanced Electromagnetic Engineering and Technology, School of Electrical and Electronic Engineering, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China*
[3]*Potsdam Institute for Climate Impact Research, Telegraphenberg, Potsdam D-14415, Germany*
[4]*Institute of Physics, Humboldt University Berlin, Berlin D-12489, Germany*
[5]*Department of Control Theory, Nizhny Novgorod State University, Gagarin Avenue 23, 606950 Nizhny Novgorod, Russia*

(Received 6 April 2017; published 12 June 2017)

When in an ensemble of oscillatory units the interaction occurs through a diffusion-like manner, the intrinsic oscillations can be quenched through two structurally different scenarios: amplitude death (AD) and oscillation death (OD). Unveiling the underlying principles of stable rhythmic activity against AD and OD is a challenging issue of substantial practical significance. Here, by developing a low-pass filter (LPF) to track the output signals of the local system in the coupling, we show that it can revoke both AD and OD, and even the AD to OD transition, thereby giving rise to oscillations in coupled nonlinear oscillators under diverse death scenarios. The effectiveness of the local LPF is proven to be valid in an arbitrary network of coupled oscillators with distributed propagation delays. The constructive role of the local LPF in revoking deaths provides a potential dynamic mechanism of sustaining a reliable rhythmicity in real-world systems.

DOI: 10.1103/PhysRevE.95.062206

Modeling coupled nonlinear oscillators constitutes a powerful and popular paradigm to study the dynamics of various real-life systems. This paradigm provides a rich source of ideas and insights into understanding the emergence of self-organized behaviors in diverse fields such as physics, chemistry, biology, and engineering [1,2]. Quenching of oscillations in systems of coupled sustained oscillators can happen via two distinct manifestations: amplitude death (AD) and oscillation death (OD) [3,4]. In AD, oscillations are suppressed when coupled oscillators are entrained to the same homogeneous steady state (HSS) [3]. In contrast, OD occurs due to a stabilization of an inhomogeneous steady state (IHSS), where the individual units occupy different branches of the IHSS [4]. The circumstances with the tendency to facilitate AD and OD are inevitable and prevalent in many natural systems, such as a distribution of frequencies introduced by a diversity or an inhomogeneity of subsystems [5–7], time delays due to a finite transmission speed of signals [8–11], and many innovative forms of interaction [12–15]. Even so, stable oscillations are always reliably sustained in order to ensure the normal functional evolutions of systems [16]. Thus, it is of practical importance to unravel the potential principles of rhythmicity against AD and OD [17–20]. How to revoke deaths to efficiently revive a stable rhythmic activity is a challenging issue of practical significance [3]. Here we propose an approach by developing a local low-pass filter (LPF) in coupled dynamical networks.

A LPF passes low-frequency signals and attenuates signals with high frequencies. Examples of LPFs have been widely found in acoustics, optics, and electronics [21]. A stiff physical barrier acts as a LPF for transmitting sound with the tendency to reflect sound with higher frequencies. Radio transmitters use a LPF to impede emissions of harmonic waves that interfere with other communications. The tone knob of an electric guitar plays the role of a LPF to depress the sound's treble. Generally, a LPF produces a smooth form of the incoming signals, which can remove the short-term fluctuations. Mathematically, a conventional LPF is described by a linear ordinary differential equation (ODE). The difference between the actual and filtered output signals has been utilized as an adaptive feedback controller capable of automatically locating and stabilizing unknown steady states of single uncoupled dynamical systems [22–24], which has received a great deal of attention in the field of controlling chaos [25–28].

Practically, signals may be deformed to some extent during the transmission due to diverse channel effects, such as bandwidth limitation, phase distortion, amplitude attenuation, and channel noise [29,30]. Consequently, it is reasonable to take into account the frequency-selective, filterlike properties of the coupling. Such effects can be well captured by introducing a LPF in the communication channel. The coupling via a LPF exerts a frequency-dependent influence on the dynamics of coupled systems. Hitherto, by incorporating a LPF in a communication channel, there have been some experimental investigations confined to issues of synchrony of coupled systems. In particular, the entrainment behavior of relaxation oscillators coupled by LPFs has been investigated by conducting a series of experiments [31]. Synchronization of two Mackey-Glass analog circuits coupled via a LPF has been studied both numerically and experimentally [32]. Coupled semiconductor lasers subject to filtered optical feedback have been shown to achieve a better quality of synchronization with respect to the conventional feedback [33].

In this paper, we apply a local tracking LPF in coupled oscillator networks, and we perform a systematic study of its dynamic effects on oscillation quenching. Specifically, by implementing a LPF to track the outputs of local system in the coupling, we find that it can effectively revoke both AD and OD by destabilizing the stable HSS and IHSS in coupled paradigmatic oscillators under diverse death scenarios. The local LPF in the coupling serves as a generator of oscillatory behavior against both AD and OD in coupled dynamical networks.

*weizou83@gmail.com

Let us start with two coupled Stuart-Landau oscillators to illustrate our scheme of a local LPF:

$$\dot{Z}_j = (1 + iw_j - |Z_j|^2)Z_j + K[Z_k(t-\tau) - \mu_j(t)], \quad (1)$$

where $Z_j = x_j + iy_j$ is the complex amplitude, w_j is the intrinsic frequency of the jth uncoupled oscillator ($j,k = 1,2$, $j \neq k$), K measures the strength of coupling, and τ is the propagation delay. For $K = 0$, both uncoupled Stuart-Landau oscillators move along the unit cycle with the eigenfrequency w_j. The dynamics of augmentation μ_j in Eq. (1) is governed by a linear ODE,

$$\alpha\dot{\mu}_j = -\mu_j + Z_j, \quad (2)$$

which represents a conventional LPF (RC circuit) with a time constant $\alpha > 0$ and a cutoff frequency $1/\alpha$. The LPF passes signals from the local node Z_j if its frequency is lower than the cutoff frequency $1/\alpha$, otherwise it attenuates outputs of Z_j. In the limiting case of $\alpha = 0$, μ_j is exactly equal to Z_j, resulting in the normal form of traditional diffusive coupling.

From the point of view of dynamics, the outputs of dynamic agent μ_j in Eq. (2) play a role in tracking and filtering the signals of local node Z_j in the coupling, where the parameter $\alpha \geqslant 0$ is a characteristic adaptation time modeling how quickly μ_j adapts to the state of Z_j. The smaller the value of α is, the faster the μ_j tracks to Z_j. For $\alpha \to 0$, the filtered signal of μ_j approaches the actual state of Z_j. The emergence of AD in the coupled system (1) with $\alpha = 0$ has been previously explored by Aronson *et al.* for $\tau = 0$ [5] and by Reddy *et al.* for $\tau > 0$ [8], respectively. Aronson *et al.* reported that AD occurs only for coupled oscillators having sufficiently disparate frequencies [5], whereas Reddy *et al.* showed that the propagation delay $\tau > 0$ can induce AD even in identical oscillators [8]. Here, we will reveal that implementing a local LPF with $\alpha > 0$ in the diffusive coupling can revoke AD by destabilizing the stable HSS under both death scenarios.

To unveil the role of the local LPF in revoking AD, one needs to examine the onset conditions of AD in the coupled system (1) with $\alpha > 0$, which can be obtained from a standard linear stability analysis around $Z_1 = Z_2 = 0$ and $\mu_1 = \mu_2 = 0$. Assuming all linear perturbations to vary as $e^{\lambda t}$ yields the characteristic equation

$$\begin{vmatrix} 1 + iw_1 - \lambda & -K & Ke^{-\lambda\tau} & 0 \\ \frac{1}{\alpha} & -\frac{1}{\alpha} - \lambda & 0 & 0 \\ Ke^{-\lambda\tau} & 0 & 1 + iw_2 - \lambda & -K \\ 0 & 0 & \frac{1}{\alpha} & -\frac{1}{\alpha} - \lambda \end{vmatrix} = 0. \quad (3)$$

AD emerges due to stabilization of HSS, $Z_1 = Z_2 = 0$, which requires that all eigenvalues of Eq. (3) are located in the left half-plane $\mathrm{Re}(\lambda) < 0$.

For $\alpha = 0$ and $\tau = 0$, Aronson *et al.* have analytically derived that the coupled system (1) experiences AD within the coupling interval of $1 < K < (1 + \Delta^2/4)/2$ if $\Delta > 2$ [5]. Interestingly, for $\Delta > 2$, we find that the stable coupling interval of AD decreases monotonically and eventually disappears for increasing α from zero beyond the certain threshold α_c; this is directly verified in Fig. 1(a) with $\Delta = 10$, where

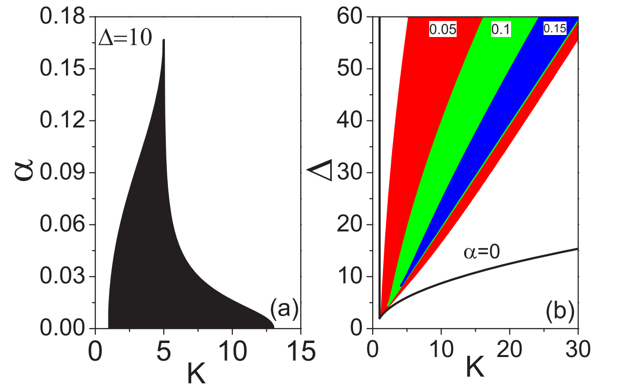

FIG. 1. Revoking AD (stable HSS) in the coupled system (1) with $\tau = 0$, $w_1 = 10 - \Delta/2$, and $w_2 = 10 + \Delta/2$. (a) The AD interval vs α with $\Delta = 10$. (b) AD regions in the parameter space of (K,Δ) for $\alpha = 0$, 0.05 (red region), 0.1 (green region), and 0.15 (blue region).

the frequencies are $w_1 = 10 - \Delta/2$ and $w_2 = 10 + \Delta/2$, and $\tau = 0$ is fixed. For a global picture, Fig. 1(b) further depicts the spread of stable AD regions in the parameter space of (K,Δ) for $\alpha = 0$ (bounded by two black lines), 0.05 (red region), 0.1 (green region), and 0.15 (blue region), respectively. The stable AD region shrinks and no longer exists as α increases gradually from zero. Increasing the value of α tends to erase the stability region of HSS, implying that the local LPF revokes AD under the death scenario of frequency mismatch.

Incorporating a propagation delay $\tau > 0$ into the coupling, Reddy *et al.* found that the coupled system (1) with $\alpha = 0$ experiences AD even for identical oscillators [8]. The characteristic equation (3) is then simplified as

$$(1 + iw \pm Ke^{-\lambda\tau} - \lambda)\left(\frac{1}{\alpha} + \lambda\right) - \frac{K}{\alpha} = 0 \quad (4)$$

for $w_1 = w_2 = w$. From the report of Reddy *et al.*, a pronounced AD island can be formed in the parameter space of (τ,K) for $\alpha = 0$; this is reproduced in Fig. 2(a) with $w = 10$. Figure 2(a) also plots AD islands for $\alpha = 0.01$, 0.02, and 0.03, respectively. Surprisingly, we observe that the

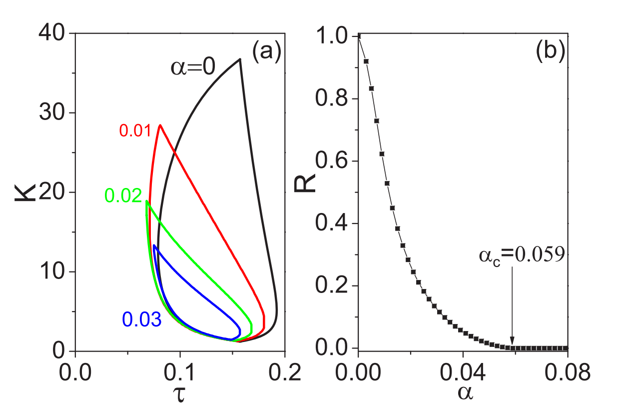

FIG. 2. Revoking AD (stable HSS) in the coupled system (1) with $\tau > 0$ and $w_1 = w_2 = w = 10$. (a) AD islands in the parameter space of (τ,K) for $\alpha = 0$, 0.01, 0.02, and 0.03. (b) The AD island ratio $R = S(\alpha)/S(\alpha = 0)$ vs α.

AD island reduces strictly as α is increased. To quantify the spread of the AD island versus α, a normalized size ratio $R = S(\alpha)/S(\alpha = 0)$ is introduced, where $S(\alpha)$ represents the area of the AD island with α. The dependence of R on α is shown in Fig. 2(b). Clearly, R decreases monotonically as α increases, and it is acquired at $R = 0$ for all $\alpha > \alpha_c = 0.059$, which indicates that the stabilization of unstable HSS leading to AD is impossible for any K and τ. Hence, the local LPF in the coupling revokes AD induced by the propagation delay in two coupled oscillators.

The local LPF in revoking AD is not limited to two oscillators, which can carry over to an arbitrary number of oscillators. Let us validate its generality in networks of N coupled Stuart-Landau oscillators with distributed propagation delays:

$$\dot{Z}_j = (1 + iw - |Z_j|^2)Z_j + \frac{K}{d_j}\sum_{\substack{k=1\\k\neq j}}^{N} g_{jk}\left[\int_0^\infty f(\tau')Z_k(t-\tau')d\tau' - \mu_j(t)\right], \quad (5)$$

where μ_j $(j = 1,2,\ldots,N)$ is the LPF as in Eq. (2). The topology of the coupled network is characterized by g_{jk} as follows: if the jth and kth nodes are linked, $g_{jk} = g_{kj} = 1$, otherwise $g_{jk} = g_{kj} = 0$, $g_{jj} = 0$, and $d_j = \sum_{k=1}^{N} g_{jk}$ gives the degree of the jth node. The function f is an integral kernel describing a distribution of propagation delays, which is assumed to be positive-definite and normalized to unity. If f is the Dirac delta function $f(\tau') = \hat{\delta}(\tau' - \tau)$, it recovers the discrete propagation delay considered in the coupled system (1). Here, we discuss a uniformly distributed delay kernel: $f(\tau') = 1/(2\beta)$ if $|\tau' - \tau| < \beta$ and zero elsewhere as in [10], where Atay found that coupled oscillators experience AD for a much larger set of parameters if the propagation delays are distributed over an interval; in particular, when the variance of the distribution β exceeds a threshold, the AD islands merge into an unbounded region along the τ direction.

By performing a linear stability analysis, the characteristic equation that determined the stability of the HSS (AD) in the coupled system (5) with $\alpha > 0$ reads

$$\left(1 + iw + K\rho_j e^{-\lambda\tau}\frac{\sinh(\lambda\beta)}{\lambda\beta} - \lambda\right)\left(\frac{1}{\alpha} + \lambda\right) - \frac{K}{\alpha} = 0, \quad (6)$$

where ρ_j's are the eigenvalues of $G = (\frac{g_{jk}}{d_j})_{N\times N}$ ordered as $1.0 = \rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant -\frac{1}{N-1} \geqslant \rho_N \geqslant -1.0$ [34]. AD is stable if and only if all the roots of Eq. (6) with each ρ_j have negative real parts. In fact, the stability condition of AD depends only on the two ends of eigenvalues $\rho_1 = 1$ and ρ_N. Figure 3 depicts stable regions of AD in the parameter space of (τ, K) for different values of α with $\rho_N = -1$ as an illustration, where $\beta = 0.02$ and $w_j = 30$ are fixed. For $\alpha = 0$, AD persists in a pronounced region [Fig. 3(a)]. Upon a minute increment of α from zero, we find astonishingly that the AD domain shrinks drastically, as shown in Figs. 3(b) and 3(c) for $\alpha = 0.007$ and 0.008, respectively. The AD region splits into three disconnected and bounded islands for $\delta = 0.01$

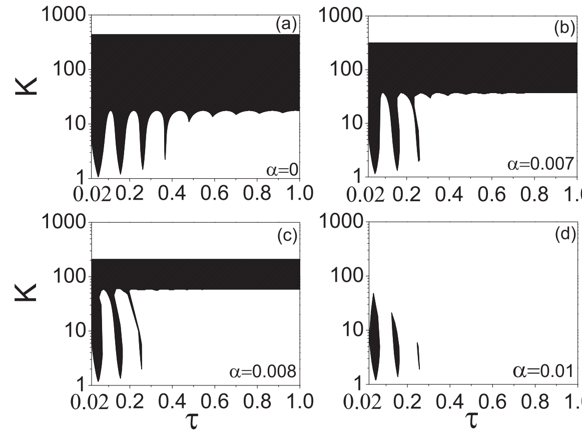

FIG. 3. Revoking AD (stable HSS) in networks of coupled Stuart-Landau oscillators with the propagation delays uniformly distributed over $\tau \pm 0.02$, $w_j = w = 30$, and $\rho_N = -1$. (a)–(d) The stability regions of AD (stable HSS) for $\alpha = 0$, 0.007, 0.008, and 0.01, respectively.

[Fig. 3(d)]. Increasing α further completely wipes off the stable region of AD from the whole parameter space. Hence, the local LPF in the coupling revokes AD in an arbitrary number of coupled oscillators even when the propagation delays are distributed over a certain interval.

Furthermore, the local LPF in the coupling is capable of revoking not only AD, but also OD, and even the AD to OD transition. Consider a system of two Stuart-Landau oscillators with symmetry-breaking coupling [35],

$$\dot{Z}_j = (1 + iw_j - |Z_j|^2)Z_j + K[\mathrm{Re}(Z_k) - \mu_j], \quad (7)$$

$$\alpha\dot{\mu}_j = -\mu_j + \mathrm{Re}(Z_j), \quad (8)$$

where $j,k = 1,2$ and $j \neq k$. Here, the coupling involving only the real parts breaks the rotational symmetry of the system, which is deemed to be a necessary condition for OD in coupled Stuart-Landau oscillators [36]. AD and OD in the coupled system (7) with $\alpha = 0$ have been well investigated by Koseska *et al.* [35], where they even observed the transition from AD to OD due to the interplay between the coupling strength K and the heterogeneity of both coupled oscillators $\delta = w_2/w_1$. As an exemplary illustration, the AD to OD transition in the case of $\alpha = 0$ is shown in Fig. 4(a) by depicting the bifurcation diagram of the steady states, where $w_1 = 2$ and $w_2 = 8$ are fixed [37]. The local LPF with $\alpha > 0$ cannot perturb the location of these steady states, but it may switch their stability. These assertions are confirmed in three typical bifurcation diagrams of steady states plotted in Figs. 4(b)–4(d). With a small increment of α from zero, OD is destabilized from large coupling strengths [Fig. 4(b) for $\alpha = 0.03$], whose stable region totally vanishes at $\alpha = 0.051$ [Fig. 4(c)], whereas the stable AD interval seems to be unaffected. However, as α increases further, the stable AD interval then shrinks simultaneously from both its upper and lower bounds [Fig. 4(d) for $\alpha = 0.07$]. Figure 4(e) plots both stable coupling intervals of AD and OD as a function of α. Increasing α from zero first destabilizes OD until it is completely revoked at $\alpha_{c1} = 0.051$, and then the stable AD interval begins to decrease

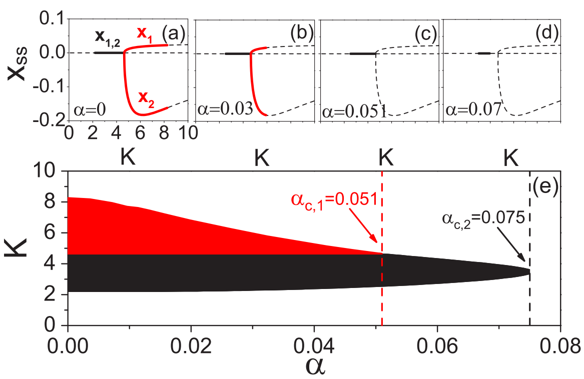

FIG. 4. Revoking AD (stable HSS) and OD (stable IHSS) in the coupled system (7) with $w_1 = 2$ and $w_2 = 8$. (a)–(d) The bifurcation diagrams of steady-state solutions for $\alpha = 0$, 0.03, 0.051, and 0.07, respectively. Solid black (dark gray) and solid red (light gray) lines mark the stable HSS (AD) and the stable IHSS (OD). Thin dashed lines denote the unstable steady states. (e) The stable interval of coupling for AD (black region) and OD (red region) vs α.

and vanishes at $\alpha_{c2} = 0.075$. The local LPF in the coupling revokes first OD and then AD step by step in the AD to OD transition.

To gain an overall view of the local LPF on revoking AD and OD in the coupled system (7), we depict the stability diagrams of both HSS and IHSS in the parameter space of (K,δ) for different values of α in Fig. 5, where $w_1 = 2$ and $w_2 = \delta w_1$ are used as in Ref. [35]. Figure 5(a) reproduces the stability diagram of AD and OD for coupled system (7) with $\alpha = 0$ (the same as obtained by Koseska *et al.* [35]). For a small $\alpha = 0.02$ in Fig. 5(b), the structures of stable HSS (AD) and stable IHSS (OD) in the (K,δ) plane remain quite similar to, but become a little smaller than, that for $\alpha = 0$. However, both AD and OD regions strongly shrink for $\alpha = 0.065$ [Fig. 5(c)]. Only two tiny

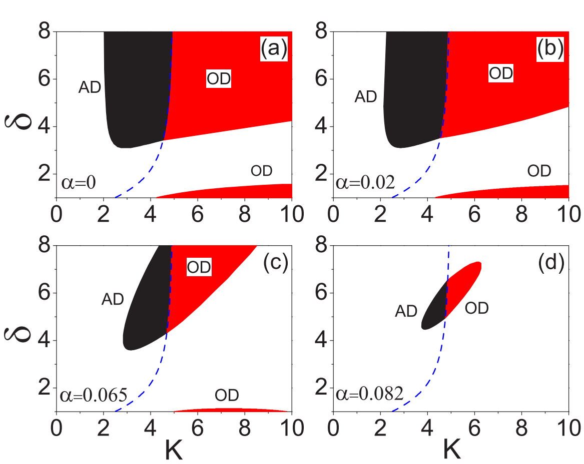

FIG. 5. Stability diagrams of AD and OD of the coupled system (7) in the parameter space of (K,δ) for $\alpha = 0$ (a), 0.02 (b), 0.065 (c), and 0.082 (d). The black (dark gray) and red (light gray) regions denote the stable HSS (AD) and IHSS (OD), respectively. The dashed blue line represents the critical coupling strength K_c for the birth of IHSS. $w_1 = 2$ and $w_2 = \delta w_1$.

islands of AD and OD survive for $\alpha = 0.082$ [Fig. 5(d)]; both will completely disappear for $\alpha > \alpha_c = 0.087$. Therefore, the local LPF with $\alpha > 0$ in the coupling can revoke not only AD, but also OD and the AD to OD transition by switching the stability of stable HSS and IHSS, which corroborates the generic and robust nature of the local LPF in revoking death.

In conclusion, we have systematically analyzed the dynamical influence of a bandwidth limitation of a communication channel on the emergence of AD and OD in coupled nonlinear oscillators. It is exclusively demonstrated that implementing a LPF in the self-feedback term of the coupling serves as a very simple but highly efficient scheme to revoke deaths. The local LPF with a low cutoff frequency is capable of annihilating not only AD, but also OD and the AD to OD transition by switching the stability of stable HSS and IHSS under diverse scenarios. Its generality and robustness have been further confirmed in an arbitrary network of coupled oscillators with distributed propagation delays. These findings could deepen our general understanding of the underlying mechanisms of sustaining stable oscillations against deaths. Additionally, we have taken an important step toward exploring the role of LPFs in shaping the collective dynamics of coupled oscillator networks, which may initiate numerous further investigations and invoke wide interest in the field of nonlinear dynamics.

We have corroborated our results by employing the Stuart-Landau limit-cycle oscillator, which is a paradigmatic model experiencing a supercritical Hopf bifurcation. Our scheme is applicable to a wide class of real-world systems that are reported to experience AD or OD, such as coupled chemical oscillators, synthetic genetic networks, neuronal systems, etc. The local LPF diminishes the suppression capacity of the coupling, which gives rise to a rich repertoire of oscillatory behavior. Our findings may provide a possible recipe for engineers to design more robust coupled systems with better functional performances in practical applications, such as engineered systems of electronics, communication systems, chaos-based cryptography, and biological networks with synthetic circuits. The local LPF has great merits in awakening oscillations from deaths, as it only has finite inherent degrees of freedom described by an additional set of ODEs, instead of using time-delayed signals involving an infinite-dimensional phase space. We would like to emphasize that in order to successfully revoke AD or OD, a LPF should be deliberately designed into a communication channel properly. In our study, a LPF is introduced only in the self-feedback term of the coupling to revoke deaths. In contrast, if a LPF is incorporated into the external unit of the coupling, the coupled systems prefer to experience AD or OD easily. We firmly believe that our scheme of a local LPF in revoking deaths is highly feasible in experimental realizations, which could be affirmed directly in pertinent experiments in coupled lasers, nonlinear circuits, and electrochemical reactions.

W.Z. acknowledges Prof. Lei-Han Tang and Prof. Changsong Zhou for valuable discussions. W.Z. was supported by Hong Kong Scholars Program and the Research Grants Council of the HKSAR under Grant No. 12301514. M.Z. was supported by the National Natural Science Foundation of China under Grant No. 11475253.

[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).

[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001).

[3] G. Saxena, A. Prasad, and R. Ramaswamy, Phys. Rep. **521**, 205 (2012).

[4] A. Koseska, E. Volkov, and J. Kurths, Phys. Rep. **531**, 173 (2013).

[5] D. G. Aronson, G. B. Ermentrout, and N. Kopell, Physica D **41**, 403 (1990).

[6] G. B. Ermentrout, Physica D **41**, 219 (1990).

[7] R. E. Mirollo and S. H. Strogatz, J. Stat. Phys. **60**, 245 (1990).

[8] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Phys. Rev. Lett. **80**, 5109 (1998).

[9] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Physica D **129**, 15 (1999).

[10] F. M. Atay, Phys. Rev. Lett. **91**, 094101 (2003).

[11] W. Zou and M. Zhan, Phys. Rev. E **80**, 065204(R) (2009).

[12] K. Konishi, Phys. Rev. E **68**, 067202 (2003).

[13] R. Karnatak, R. Ramaswamy, and A. Prasad, Phys. Rev. E **76**, 035201(R) (2007).

[14] V. Resmi, G. Ambika, and R. E. Amritkar, Phys. Rev. E **84**, 046212 (2011).

[15] D. J. Schwaba, A. Baetica, and P. Mehta, Physica D **241**, 1782 (2012).

[16] W. L. Koukkari and R. B. Sothern, *Introducing Biological Rhythms* (Springer, Netherlands, 2006).

[17] K. Konishi, Phys. Lett. A **341**, 401 (2005).

[18] W. Zou, D. V. Senthilkumar, M. Zhan, and J. Kurths, Phys. Rev. Lett. **111**, 014101 (2013).

[19] W. Zou, D. V. Senthilkumar, R. Nagao, I. Z. Kiss, Y. Tang, A. Koseska, J. Duan, and J. Kurths, Nat. Commun. **6**, 7709 (2015).

[20] D. Ghosh, T. Banerjee, and J. Kurths, Phys. Rev. E **92**, 052908 (2015).

[21] A. Sedra and P. Brackett, *Filter Theory and Design: Active and Passive* (Matrix, Beaverton, OR, 1978).

[22] A. Namajūnas, K. Pyragas, and A. Tamaševičius, Phys. Lett. A **204**, 255 (1995).

[23] N. F. Rulkov, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **50**, 314 (1994).

[24] A. Schenck zu Schweinsberg and U. Dressler, Phys. Rev. E **63**, 056210 (2001).

[25] K. Pyragas, V. Pyragas, I. Z. Kiss, and J. L. Hudson, Phys. Rev. Lett. **89**, 244103 (2002).

[26] K. Pyragas, V. Pyragas, I. Z. Kiss, and J. L. Hudson, Phys. Rev. E **70**, 026215 (2004).

[27] A. Ahlborn and U. Parlitz, Phys. Rev. Lett. **96**, 034102 (2006).

[28] E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, and A. Tamaševičius, Phys. Rev. E **88**, 060901(R) (2013).

[29] T. L. Carroll, Phys. Rev. E **50**, 2580 (1994); IEEE Trans. Circ. Syst. I **42**, 105 (1995); Phys. Rev. E **53**, 3117 (1996).

[30] N. J. Corron and D. W. Hahs, IEEE Trans. Circ. Syst. I **44**, 373 (1997).

[31] G. F. Keyser, J. A. Holloway, and D. D. Prather, J. Interdisc. Cycle Res. **10**, 239 (1979).

[32] M.-Y. Kim, C. Sramek, A. Uchida, and R. Roy, Phys. Rev. E **74**, 016211 (2006).

[33] M. C. Soriano, F. Ruiz-Oliveras, P. Colet, and C. R. Mirasso, Phys. Rev. E **78**, 046218 (2008).

[34] F. R. K. Chung, *Spectral Graph Theory* (American Mathematical Society, Providence, RI, 1997).

[35] A. Koseska, E. Volkov, and J. Kurths, Phys. Rev. Lett. **111**, 024103 (2013).

[36] A. Zakharova, M. Kapeller, and E. Schöll, Phys. Rev. Lett. **112**, 154101 (2014).

[37] The stability of HSS (AD) and IHSS (OD) of coupled system (7) with $\alpha > 0$ can be identified from the characteristic equation

$$\begin{vmatrix} a-\lambda & -w-b & -K & K & 0 & 0 \\ w-b & c-\lambda & 0 & 0 & 0 & \\ \frac{1}{a} & 0 & -\frac{1}{\alpha}-\lambda & 0 & 0 & 0 \\ K & 0 & 0 & \tilde{a}-\lambda & -w-\tilde{b} & -K \\ 0 & 0 & 0 & w-\tilde{b} & \tilde{c}-\lambda & \\ 0 & 0 & 0 & \frac{1}{\alpha} & 0 & -\frac{1}{\alpha}-\lambda \end{vmatrix} = 0,$$

where $a = 1 - 3x_1^{*2} - y_1^{*2}$, $b = 2x_1^* y_1^*$, $c = 1 - x_1^{*2} - 3y_1^{*2}$, $\tilde{a} = 1 - 3x_2^{*2} - y_2^{*2}$, $\tilde{b} = 2x_2^* y_1^*$, and $\tilde{c} = 1 - x_2^{*2} - 3y_2^{*2}$. $(x_1^*, y_1^*, x_2^*, y_2^*)$ is a steady-state solution of coupled system (7), which can either be the homogeneous origin or the inhomogeneous fixed points newly created by the coupling [35].