Diffusion-controlled formation and collapse of a d-dimensional A-particle island in the B-particle sea

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We consider diffusion-controlled evolution of a d-dimensional A-particle island in the B-particle sea at propagation of the sharp reaction front $A + B \rightarrow 0$ at equal species diffusivities. The A-particle island is formed by a localized (point) A-source with a strength λ that acts for a finite time T. We reveal the conditions under which the island collapse time t_c becomes much longer than the injection period T (long-living island) and demonstrate that regardless of d the evolution of the long-living island radius $r_f(t)$ is described by the universal law $\zeta_f = r_f/r_f^M = \sqrt{e\tau |\ln \tau|}$, where $\tau = t/t_c$ and r_f^M is the maximal island expansion radius at the front turning point $t_M = t_c/e$. We find that in the long-living island regime the ratio t_c/T changes with the increase of the injection period T by the law $\propto (\lambda^2 T^{2-d})^{1/d}$, i.e., increases with the increase of T in the one-dimensional (1D) case, does not change with the increase of T in the 2D case and decreases with the increase of T in the 3D case. We derive the scaling laws for particles death in the long-living island and determine the limits of their applicability. We demonstrate also that these laws describe asymptotically the evolution of the d-dimensional spherical island with a uniform initial particle distribution generalizing the results obtained earlier for the quasi-one-dimensional geometry. As striking results, we present a systematic analysis of the front relative width evolution for fluctuation, logarithmically modified, and mean-field regimes, and we demonstrate that in a wide range of parameters the front remains sharp up to a narrow vicinity of the collapse point.

I. INTRODUCTION

In recent decades, the reaction-diffusion systems $A + B \rightarrow 0$, where unlike species A and B diffuse and irreversibly annihilate in the bulk of a d-dimensional medium, have attracted great interest owing to the remarkable property of effective *dynamical repulsion* of unlike species ([1–4]). In systems with initially spatially separated reactants this property results in the formation and self-similar propagation of a *localized reaction front* which, depending on the interpretation of A and B (chemical reagents, quasiparticles, topological defects, etc.), plays a key role in a wide range of applications from Liesegang patterns formation [5–7] to electron-hole luminescence in quantum wells [8–10]. The simplest model of a planar reaction front, introduced by Galfi and Racz (GR) [11], is the quasi-one-dimensional model,

$$\partial a/\partial t = D_A \nabla^2 a - R, \quad \partial b/\partial t = D_B \nabla^2 b - R, \quad (1)$$

for two initially separated reactants which are uniformly distributed on the left side ($x < 0$) and on the right side ($x > 0$) of the initial boundary. Taking the reaction rate in the mean-field form $R(x,t) = ka(x,t)b(x,t)$ (k being the reaction constant), GR discovered that in the long time limit $kt \gg 1$ the reaction profile $R(x,t)$ acquires a universal scaling form with the width $w \propto (t/k^2)^{1/6}$, so that on the diffusion length scale $L_D \propto t^{1/2}$ the relative width of the front asymptotically contracts unlimitedly $w/L_D \sim (kt)^{-1/3} \rightarrow 0$ as $kt \rightarrow \infty$. Based on this fact a general concept of the front dynamics, the quasistatic approximation (QSA), was developed ([12–16]). The key property of the QSA is that $w(J)$ depends on t only through the time-dependent boundary current, $J_A = |J_B| = J$, the calculation of which is reduced to solving the external diffusion problem with the moving *absorbing boundary* (Stefan problem) $R = J\delta(x - x_f)$. Following this approach, in most subsequent works the use of the QSA was traditionally

restricted by the GR *sea-sea* problem with unlimited number of A and B particles where the stage of monotonous quasistatic front propagation is always reached asymptotically.

In the recent Refs. [17–24] it has been shown that based on the QSA the scope of the $A + B \rightarrow 0$ problems which allow for analytic description can be appreciably broadened including the systems with *finite* number of particles and *nonmonotonous* front propagation where asymptotically the QSA is violated. In Rapid Communication Ref. [17], the problem of the death of an A-particle island in the uniform B-particle sea at equal species diffusivities was considered. It has been established that at sufficiently large initial number of A particles, N_0, and a sufficiently large reaction constant k the death of majority of island particles $N(t)$ proceeds in the universal scaling regime,

$$N = N_0 \mathcal{G}(t/t_c), \quad (2)$$

where $t_c \propto N_0^2$ is the lifetime of the island in the limit $k, N_0 \rightarrow \infty$. This result was obtained under the assumption that the reaction front propagates quasistatically and is sustained to be quite sharp, $w/x_f \ll 1$ until the collapse time $t \approx t_c$, so that the law of the motion of the front center $x_f(t)$ governs the island width evolution. It has been shown that while dying, the island first expands to a certain maximal amplitude $x_f^M \propto N_0$ and then begins to contract by the law

$$x_f = x_f^M \zeta_f(t/t_c), \quad (3)$$

so that on reaching the maximal expansion amplitude x_f^M (the turning point of the front),

$$t_M/t_c = 1/e, \quad N_M/N_0 = 0.19886..., \quad (4)$$

and, therefore, irrespective of the initial particle number, $\approx 4/5$ of the particles die at the stage of the island expansion and the remaining $\approx 1/5$ at the stage of its subsequent contraction.

According to Ref. [17], the rapid island contraction is accompanied by the rapid growth of the front width w and, therefore, in some vicinity of the collapse point $\mathcal{T} = (t_c - t)/t_c \sim \mathcal{T}_Q$ the reaction front becomes "blurred" ($w/x_f \sim 1$) and the QSA is no longer applicable. In Ref. [17], it has been shown that for the mean-field front at small $\mathcal{T}$,

$$w/x_f \propto (\mathcal{T}_Q/\mathcal{T})^{2/3}, \tag{5}$$

where

$$\mathcal{T}_Q \propto 1/N_0\sqrt{k},$$

so that $\mathcal{T}_Q \to 0$ at large $k, N_0 \to \infty$.

It should be emphasized, however, that as well as in the case of GR problem these results have been obtained for quasi-one-dimensional geometry (flat front), whereas in numerous applications the urgent need of their generalization for the islands possessing circular (ring-shaped front) or spherical (spherical front) symmetry arises [25–27]. Moreover, in Ref. [17] the evolution of the formed island with uniform initial distribution of A particles has been regarded, whereas in many applications the A-particle island appears in the uniform d-dimensional B-particle sea from a localized (point) source that forms asymptotically the d-dimensional radially symmetric island with particle distribution dependent on intensity and duration of the source action (electron-hole luminescence in quantum wells with localized laser injection of holes into a uniform electron sea is a prominent illustration of such systems [8–10]). The regularities of the d-dimensional radially symmetric island growth from a continuous in time localized (point) source for the physically most important situation when both A and B particles are mobile were studied in Ref. [18]. In the assumption of sharp reaction front formation (QSA) it was established that in the one-dimensional (1D) case the island grows unlimitedly at any reduced source strength λ, and the dynamics of its growth does not depend asymptotically on the diffusivity of B particles. In the 3D case the island grows only at $\lambda > \lambda_c$, achieving asymptotically a stationary state (static island). In the marginal 2D case the island grows unlimitedly at any λ but at $\lambda < \lambda_*$ the time of its formation becomes exponentially large.

In this paper, alongside with generalization of the results [17] for the d-dimensional sphere, as the main goal we study in the frameworks of the QSA the regularities of evolution and collapse of the d-dimensional A-particle island after switching-off a localized source acting for some finite time T. In the assumption of sharp reaction front formation we reveal the complete picture of evolution of front trajectory and particle distribution in the island at change in intensity and duration of the source action at equal species diffusivities. We focus chiefly on the situation when the island collapse time t_c becomes much longer than the injection period T (long-living island) and demonstrate that, in qualitative contrast to a radical change in the island growth laws [18] at the change in the system dimension, the evolution of the long-living island radius regardless of d is described by the universal law

$$\zeta_f = r_f/r_f^M = \sqrt{e\tau|\ln\tau|},$$

where $\tau = t/t_c$ and r_f^M is the radius of the island maximal expansion at the front turning point $t_M = t_c/e$. We reveal the

conditions of long-living island formation and discover that in the long-living island regime the ratio t_c/T changes at the increase in the injection period T by the law $\propto (\lambda^2 T^{2-d})^{1/d}$; i.e., it increases at the increase of T in the 1D case, does not change at the increase of T in the 2D case, and decreases at the increase of T in the 3D case. We show that in the long-living island regime by the time moment of source switching-off the majority of injected particles survive and derive scaling laws of particle death in the island. We also demonstrate that asymptotically these laws describe the evolution of the d-dimensional spherical island with uniform initial particle distribution. Finally, we analyze self-consistently the regularities of the reaction front relative width evolution for fluctuation, logarithmically modified, and mean-field regimes.

II. EVOLUTION OF THE d-DIMENSIONAL SPHERICAL ISLAND WITH UNIFORM INITIAL PARTICLE DISTRIBUTION

We start with generalization of the Ref. [17] results obtained for quasi-one-dimensional geometry. Let the uniform d-dimensional spherically symmetrical A-particle island with a radius L ($r \in [0,L)$) be "submerged" into the uniform d-dimensional sea of particles B ($r \in (L,\infty)$) with initial concentrations a_0 and b_0, respectively (it is clear that in the 1D case the interval $r \in [0,\infty)$ represents a half of radially symmetric distribution $r = |x|$). Particles A and B diffuse with nonzero diffusion constants $D_{A,B}$ and upon contact annihilate with some nonzero probability, $A + B \to 0$. In the continuum version, this process can be described by the reaction-diffusion Eqs. (1), where $a(r,t)$ and $b(r,t)$ are the mean local concentrations of A and B, which, by symmetry, we assume to be dependent only on the radius, and $R(r,t)$ is the macroscopic reaction rate. We shall assume, as usual, that species diffusivities are equal $D_A = D_B = D$. This important condition, due to local conservation of difference concentration $a - b$, leads to a radical simplification that permits to obtain an analytical solution for arbitrary front trajectory (we shall note that at different species diffusivities $D_A \neq D_B$ an analytical solution of the Stefan problem is possible only for a stationary or a monotonically moving front [22]). Then, by measuring the length, time, and concentration in units of $L, L^2/D$, and b_0, respectively, and defining the ratio $a_0/b_0 = c$, we come from Eq. (1) to the simple diffusion equation for the difference concentration $s(r,t) = a(r,t) - b(r,t)$,

$$\partial s/\partial t = \nabla^2 s, \tag{6}$$

at the initial conditions

$$s_0(r \in [0,1)) = c, \quad s_0(r \in (1,\infty]) = -1, \tag{7}$$

with the boundary conditions

$$\nabla s|_{r=0} = 0, \quad s(\infty,t) = -1. \tag{8}$$

As well as in Ref. [17], we shall assume that the ratio of concentrations island/sea is large enough, $c \gg 1$ (concentrated island). Below it will be shown that in the limit of large $c \gg 1$ the "lifetime" of the island $t_c \propto c^{2/d} \gg 1$, so the majority of the particles die at times $t \gg 1$, when the diffusive length exceeds appreciably the initial island radius. As well as in

Ref. [17], the evolution of the island in such a large-t regime is of principal interest to us here.

Asymptotics of the exact solution of the problem Eqs. (6)–(8) for $d = 1,2,3$ at large t and $r/t \ll 1$ has the form

$$s(r,t) = \frac{\gamma N_0}{(4\pi t)^{d/2}} e^{-r^2/4t}(1 - \chi_d) - 1, \quad (9)$$

where $N_0 = \mu_d c (\mu_1 = 2, \mu_2 = \pi, \mu_3 = 4\pi/3)$ is the initial number of particles in the island in units of $b_0 L^d$, $\gamma = (c + 1)/c$, and

$$\chi_d = \alpha_d(1 - r^2/2dt)/t + \ldots,$$

with $\alpha_1 = 1/12$, $\alpha_2 = 1/8$, and $\alpha_3 = 3/20$. According to the QSA for large $k \to \infty$ at times $t \propto k^{-1} \to 0$, there forms a sharp reaction front $w/r_f \to 0$ so that in neglect of the reaction front width the solution $s(r,t)$ defines the law of its propagation $s(r_f,t) = 0$ and the evolution of particle distributions $a = s(r < r_f)$ and $b = |s|(r > r_f)$. Substituting to Eq. (9) the condition $s(r_f,t) = 0$ and assuming that $\chi_d^f \ll 1$, we find the law of the front motion in the form

$$r_f = 2(1 + \alpha_d/dt)\sqrt{t \ln\left[\frac{\gamma N_0(1 - \alpha_d/t)}{(4\pi t)^{d/2}}\right]}. \quad (10)$$

From Eq. (10), it follows that at any d in the limit of large $t, N_0 \gg 1$ the island first expands reaching some maximal radius r_f^M, and then it contracts disappearing in the collapse point $t_c \propto N_0^{2/d}$. Taking $r_f(t_c) = 0$, we find from Eq. (10)

$$t_c = \frac{(\gamma N_0)^{2/d}[1 - \mathcal{O}(N_0^{-2/d})]}{4\pi}, \quad (11)$$

whence neglecting the terms $\mathcal{O}[\max(1/c, 1/c^{2/d})]$, we obtain

$$t_c = (N_0)^{2/d}/4\pi. \quad (12)$$

Neglecting further the terms α_d/t, we finally find from Eq. (10)

$$r_f = \sqrt{2dt \ln(t_c/t)}, \quad (13)$$

whence it follows immediately that in the front turning point $\dot{r}_f(t_M) = 0$,

$$t_c/t_M = e \quad (14)$$

and

$$r_f^M = \sqrt{2dt_M} = (N_0)^{1/d}\sqrt{d/2\pi e}. \quad (15)$$

Introducing the scaling variables $\zeta = r/r_f^M$ and $\tau = t/t_c$, we come to the result announced above that in the limit of large $t, N_0 \gg 1$ *regardless* of the system dimension, the front trajectory is described by the universal law

$$\zeta_f(\tau) = r_f/r_f^M = \sqrt{e\tau|\ln\tau|}. \quad (16)$$

It should be noted that taking into account Eq. (13) in the limit of large t_c the condition $\chi_d^f \ll 1$ reduces to the more rigid requirement $t \gg \alpha_d \ln(t_c/t)$. Assuming that this condition is fulfilled and neglecting the reaction front width, at the same approximation as above for particle distribution in the island, we find from Eq. (9)

$$a(\zeta,\tau) = s(\zeta,\tau) = (e^{-\zeta^2/e\tau}/\tau)^{d/2} - 1. \quad (17)$$

Calculating further the number of particles in the island $N = g_d(r_f^M)^d \int_0^{\zeta_f} a(\zeta,\tau)\zeta^{d-1}d\zeta$ (here $g_1 = 2, g_2 = 2\pi$ and $g_3 = 4\pi$), we obtain immediately the scaling laws of particles death in the island

$$N = N_0 \mathcal{G}_d(\tau), \quad (18)$$

where

$$\mathcal{G}_1(\tau) = \text{erf}(\sqrt{|\ln\tau|/2}) - \sqrt{2\tau|\ln\tau|/\pi},$$
$$\mathcal{G}_2(\tau) = 1 - \tau(1 + |\ln\tau|),$$
$$\mathcal{G}_3(\tau) = \text{erf}(\sqrt{3|\ln\tau|/2}) - \sqrt{6\tau^3|\ln\tau|/\pi}(1 + |\ln\tau|).$$

From Eqs. (17) and (18) we conclude that in the front turning point $\tau_M = 1/e$, *regardless* of the initial number of A particles, their concentration in the center of the island is

$$a(0,\tau_M) = e^{d/2} - 1, \quad (19)$$

and the fraction of the particles that survived in the process of the island expansion is

$$N_M/N_0 = \begin{cases} 0.19886..., & d = 1, \\ 0.26412..., & d = 2, \\ 0.29986..., & d = 3. \end{cases} \quad (20)$$

Assuming that the front remains sharp enough up to a narrow vicinity of the collapse point $\mathcal{T} = (t_c - t)/t_c \ll 1$, we find from Eqs. (16) and (18) that at small $\mathcal{T} \ll 1$ at the final collapse stage particles death proceeds by the law

$$N/N_0 = c_d\mathcal{T}^{(d+2)/2} = c_d(\zeta_f/\sqrt{e})^{d+2}, \quad (21)$$

where $c_1 = \sqrt{2/\pi}/3$, $c_2 = 1/2$, and $c_3 = 3\sqrt{6/\pi}/5$. In Fig. 1 are shown the dependencies $\zeta_f(\tau)$ and $\mathcal{G}_d(\zeta_f)$ that demonstrate the key features of evolution of 1D, 2D, and 3D islands in the limit of large $N_0 \to \infty$.

One of the central points of our analysis is revealing of applicability limits for the assumption that the formed reaction front remains sharp enough, $\eta = w/r_f \ll 1$, up to a narrow vicinity of the collapse point. The detailed discussion of this problem will be presented in Sec. IV. Completing this section we shall reveal the regularities for the evolution of the boundary current density $J = -\partial a/\partial r|_{r=r_f}$ that according to the QSA determines the evolution of the reaction front width $w(J)$. From Eqs. (16) and (17) we find easily

$$\mathcal{J}(\tau) = J/J_M = \sqrt{\frac{|\ln\tau|}{e\tau}}, \quad (22)$$

where $J_M = d/r_f^M = \sqrt{2\pi ed}/(N_0)^{1/d}$. Thus, we conclude that as well as the front trajectory Eq. (16) the boundary current evolution is described by the universal law Eq. (22) regardless of the system dimension that predetermines the universality of the mean-field front relative width evolution.

III. EVOLUTION OF THE d-DIMENSIONAL ISLAND FORMED BY A LOCALIZED SOURCE

Let us proceed now to the analysis of formation regularities of the d-dimensional A-particle spherical island from a localized (point) A-particle source and the consequent island evolution after switching-off this source. Let particles A be injected at $t \geqslant 0$ with a rate Λ at the point $\mathbf{r} = 0$ of the
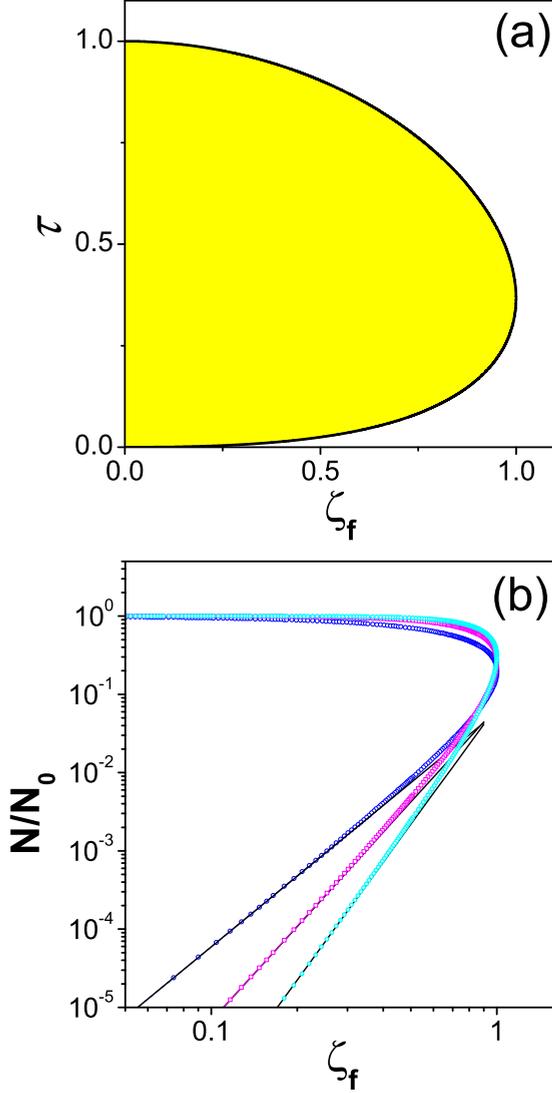
FIG. 1. (a) Universal trajectory of the reduced front radius $\zeta_f(\tau)$ Eq. (16) (line). The island area is colored. (b) Scaling functions $G_d(\zeta_f) = N/N_0$ calculated according to Eqs. (16) and (18) for $d = 1$ (circles), $d = 2$ (squares), and $d = 3$ (hexagons). Straight lines show the small-ζ_f asymptotes according to Eqs. (21).

uniform d-dimensional sea of particles B, distributed with a density ρ. The source is acting for some finite time T, and then it is switched-off. As above, particles A and B diffuse with nonzero diffusion constants $D_{A,B} = D$ and upon contact annihilate with some nonzero probability, $A + B \rightarrow 0$. In the continuum version this process can be described by the reaction-diffusion Eqs. (1) with an additional source term $\Lambda\delta(\mathbf{r})\Theta(T - t)\Theta(t)$ for particles A (Θ denotes here the Heaviside step function) where $a(r,t)$ and $b(r,t)$, by symmetry, we assume to be dependent only on the radius with the initial conditions $a(r,0) = 0, b(r,0) = \rho$, and the boundary condition $b(\infty,t) = \rho$. The initial density of the sea, ρ, defines a natural scale of concentrations and a characteristic length scale of the problem—the average interparticle distance $\ell = \rho^{-1/d}$. So, by measuring the length, time, and concentration in units of $\ell, \ell^2/D$, and ρ, respectively, we introduce the dimensionless source strength $\lambda = \Lambda\ell^2/D$ and the dimensionless reaction

constant $\kappa = k\rho\ell^2/D$. Defining then the difference concentration $s(r,t) = a(r,t) - b(r,t)$, we come to the simple diffusion equation with source

$$\partial s/\partial t = \nabla^2 s + \lambda\delta(\mathbf{r})\Theta(T - t)\Theta(t), \quad (23)$$

at the initial and boundary conditions

$$s(r,0) = s(\infty,t) = -1. \quad (24)$$

According to Eq. (23), in the course of injection in the vicinity of the source there arises a region of A-particle excess, $s(r,t) > 0$, which expands with time. Following Ref. [18], we shall assume that, by analogy with the Galfi-Racz problem, a narrow reaction front has to form at this region boundary, for which the law of motion, $r_f(t)$, according to the QSA, can be derived from the condition $s(r_f,t) = 0$.

The exact solution of the problem Eqs. (23), (24) at the injection stage $0 < t \leqslant T$ has the form

$$s + 1 = \frac{\lambda}{(4\pi)^{d/2}}\int_0^t d\theta e^{-r^2/4(t-\theta)}/(t - \theta)^{d/2}, \quad (25)$$

whereas after source switching-off, at $t > T$, we find

$$s + 1 = \frac{\lambda}{(4\pi)^{d/2}}\int_0^T d\theta e^{-r^2/4(t-\theta)}/(t - \theta)^{d/2}. \quad (26)$$

We shall start with the discussion of the key features of the d-dimensional island formation and growth at the injection stage, reproducing here partially the Ref. [18] results for the completeness.

A. Formation and growth of the d-dimensional A-particle island at the injection stage $0 < t \leqslant T$

Integrating Eq. (25) for $d = 1,2,3$, we find immediately

$$s + 1 = \sqrt{\lambda^2 t}\,\mathrm{ierfc}\left(\frac{r}{2\sqrt{t}}\right), \quad d = 1, \quad (27)$$

$$s + 1 = -(\lambda/4\pi)\mathrm{Ei}\left(-\frac{r^2}{4t}\right), \quad d = 2, \quad (28)$$

$$s + 1 = \left(\frac{\lambda}{4\pi r}\right)\mathrm{erfc}\left(\frac{r}{2\sqrt{t}}\right), \quad d = 3, \quad (29)$$

whence, assuming the front to be formed ($w/r_f \ll 1$) and neglecting its width, from the condition $s(r_f,t) = 0$ we obtain the laws for the island radius (the front center radius) growth $r_f(t)$,

$$\mathrm{ierfc}(r_f/2\sqrt{t}) = 1/\sqrt{\lambda^2 t}, \quad d = 1, \quad (30)$$

$$\mathrm{Ei}\left(-r_f^2/4t\right) = -4\pi/\lambda, \quad d = 2, \quad (31)$$

$$\mathrm{erfc}(r_f/2\sqrt{t}) = 4\pi r_f/\lambda, \quad d = 3, \quad (32)$$

where $\mathrm{ierfc}(u) = \int_u^\infty \mathrm{erfc}(v)dv = e^{-u^2}/\sqrt{\pi} - u\mathrm{erfc}(u)$ and $\mathrm{Ei}(-u^2) = -\int_{u^2}^\infty dv e^{-v}/v$ is the exponential integral. Calculating further the number of A particles surviving in the injection process

$$N = g_d\int_0^{r_f} s(r,t)r^{d-1}dr, \quad (33)$$

we conclude from Eqs.(27)–(33) that at $d \neq 2$ the growth of the island radius $r_f(t)$ and the number of surviving A particles

$N(t)$ are described by the scaling laws

$$r_f = \lambda^{d-2}\mathcal{R}_d(\lambda^{2/(2-d)}t), \tag{34}$$

$$N = \lambda^{d/(d-2)}\mathcal{N}_d(\lambda^{2/(2-d)}t). \tag{35}$$

As a consequence, the fraction of surviving A particles is described by the scaling law

$$q = N/\lambda t = \mathcal{Q}_d(\lambda^{2/(2-d)}t). \tag{36}$$

Following Ref. [18], below we focus on the key features of the d-dimensional island growth for each dimension separately.

1. One-dimensional island

In the 1D case from Eq. (27) it follows that an excess of A particles in the source vicinity forms in a time $t_\star = \pi/\lambda^2$ ($s(0,t_\star) = 0$). It is, however, clear that a hydrodynamic approximation comes into play at times $t \gg \max(1,1/\lambda)$; therefore, at early island formation stages one can distinguish two qualitatively different island growth regimes: (a) $\lambda \ll 1$, when the island formation proceeds under conditions of death of the majority of injected particles, and (b) $\lambda \gg 1$, when a multiparticle "cloud" forms long before the beginning of noticeable annihilation and, therefore, the stage of the developed reaction, $t \gg 1(\gg t_\star)$, is preceded here by a stage of purely diffusive expansion of the cloud, $1/\lambda \ll t \ll 1$ [18]. In the long-time limit $t \gg \max(1,t_\star)$ at any λ we obtain from Eqs. (27), (30), and (33) the exact asymptotics,

$$r_f = \sqrt{2t \ln \Gamma}(1 - \ln \ln \Gamma / \ln \Gamma + \cdots), \tag{37}$$

$$N = \lambda t[1 - \mathcal{O}(\sqrt{\ln \Gamma / \Gamma})], \tag{38}$$

where $\Gamma = t/t_\star$. So, we conclude that forming in qualitatively different regimes from $q \ll 1(\lambda \ll 1)$ to $1 - q \ll 1(\lambda \gg 1)$ the 1D island at any λ crosses over to the universal growth regime Eqs. (37) and (38) with an unlimited decay of the dying particle fraction $1 - q \propto \sqrt{\ln \Gamma / \Gamma} \to 0$ as $\Gamma \to \infty$. Figure 2 shows the dependencies λr_f versus $\lambda^2 t$ calculated according to Eqs. (30) and (37). One can see that asymptotics Eq. (37) gives an exact description of the front trajectory starting with $\lambda^2 t \sim 10^3$.

2. Two-dimensional island

In the 2D case from Eqs. (28), (31), and (33), we find

$$r_f = 2\sqrt{\alpha t}, \tag{39}$$

$$N = \lambda t(1 - e^{-\alpha}), \tag{40}$$

where α is the root of the equation $\text{Ei}(-\alpha) = -\lambda_*/\lambda$, $\lambda_* = 4\pi$ and has the asymptotics $\alpha = e^{-\lambda_*/\lambda}/\gamma(\gamma = 1.781...)$ at $\lambda \ll \lambda_*$ and $\alpha = \ln(\lambda/\lambda_*\alpha)$ at $\lambda \gg \lambda_*$. We conclude that in 2D the island growth rate α and the fraction of surviving A particles q *do not vary in time*: at large $\lambda \gg \lambda_*$ the majority of injected particles survive,

$$1 - q \sim \frac{\ln \lambda}{\lambda} \ll 1, \quad \lambda \gg \lambda_*,$$

whereas at small $\lambda \ll \lambda_*$, the majority of injected particles die,

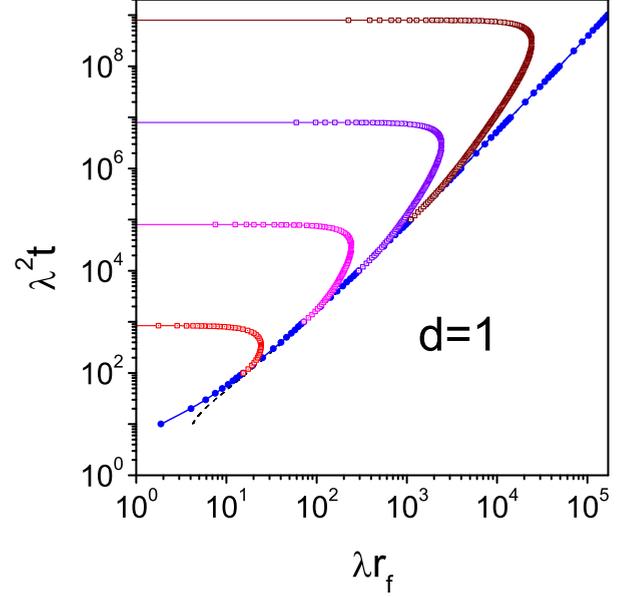$$q \sim e^{-\lambda_*/\lambda} \ll 1, \quad \lambda \ll \lambda_*.$$

FIG. 2. Filled circles show the trajectory of the front radius for the continuous in time point source in the one-dimensional medium calculated according to Eq. (30) in the scaling coordinates λr_f vs $\lambda^2 t$. Dashed line shows long-time asymptotics [Eq. (37)]. Open squares show the trajectories of the front radius after the source switching-off calculated from Eq. (44) for the parameter values $\lambda^2 T = 10^2, 10^3, 10^4$, and 10^5 (from bottom to top).

One of the key consequences of Eqs. (39) and (40) consists in the exponentially strong decrease of the growth rate α and the fraction of surviving particles q in the region $\lambda \ll \lambda_*$. It means that though in the 2D case the island unlimitedly growth at any λ, at small $\lambda \ll \lambda_*$ the 2D island growth is actually *suppressed*.

3. Three-dimensional island

In the 3D case from Eqs. (29) and (32) it follows that at any λ in the long-time limit $t \gg t_s = (\lambda/\lambda_*)^2$ the front radius by the law,

$$r_f = r_s[1 - \mathcal{O}(\sqrt{t_s/t})],$$

reach a stationary value (*stationary island*),

$$r_f(t/t_s \to \infty) = r_s = \lambda/\lambda_*. \tag{41}$$

According to Eqs. (29), (32), and (33), in this limit the number of surviving particles is

$$N = (2\pi/3)r_s^3[1 - \mathcal{O}(\sqrt{t_s/t})],$$

whence is follows that in radical contrast to the 1D case in the 3D case at any injection rate all the injected particles die asymptotically $q = (1/6)(t_s/t) \to 0$ as $t/t_s \to \infty$. In Ref. [18] it is shown that defining a minimal stationary island through the condition $w_s/r_s \sim 1$, we conclude that in the 3D case the island forms only when the injection rate exceeds a
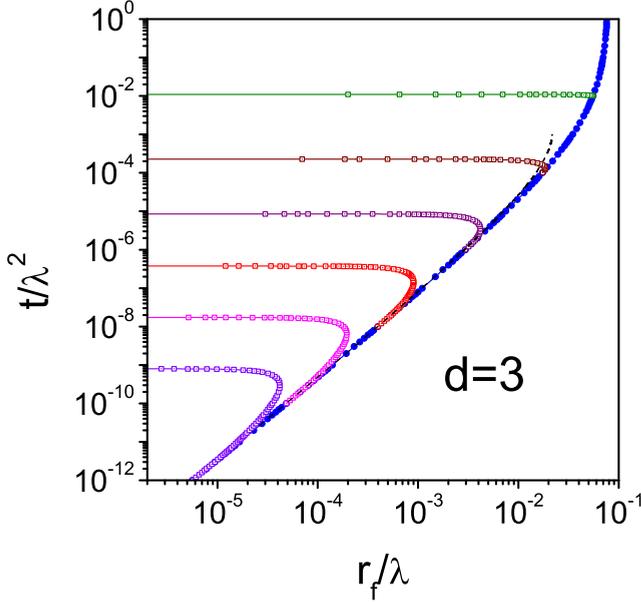
FIG. 3. Filled circles show the trajectory of the front radius for the continuous in time point source in the three-dimensional medium calculated according to Eq. (32) in the scaling coordinates r_f/λ vs t/λ^2. Dashed line shows the intermediate asymptotics [Eq. (42)]. Open squares show the trajectories of the front radius after the source switching-off calculated from Eq. (46) for the parameter values $\lambda^2/T = 10^2, 10^4, 10^6, 10^8, 10^{10}$, and 10^{12} (from top to bottom).

critical value,

$$\lambda_c \sim \lambda_*/\sqrt{\kappa} \gg 1.$$

One of the key consequences of Eq. (32) is that at high injection rates $\lambda \gg \lambda_c$ the stationary state $(t \gg t_s)$ is preceded by an intermediate stage $1 \ll t \ll t_s$, wherein the island grows by the law

$$r_f = \sqrt{2t \ln(t_s/t)}[1 - \ln(\sqrt{\pi}\omega)/\omega + \ldots], \qquad (42)$$

where $\omega = \ln(t_s/t)$. According to Eqs. (29), (32), and (33), at this stage the number of surviving particles is

$$N = \lambda t[1 - \mathcal{O}(\omega^{3/2}\sqrt{t/t_s})], \qquad (43)$$

and therefore the majority of injected particles are *still surviving*,

$$1 - q \sim \omega^{3/2}\sqrt{t/t_s} \ll 1.$$

In Ref. [18] are presented the details of formation of the spherical island Eqs. (42) and (43) from a diffusive cloud which expands in the absence of reaction and it is established that the formed front condition $w/r_f \ll 1$ is realized at $t \gg \ln(t_s/t)/\kappa$. Figure 3 shows the dependencies r_f/λ versus t/λ^2 calculated according to Eqs. (32) and (42). One can see that asymptotics Eq. (42) gives an exact description of the front trajectory up to $t/\lambda^2 \sim 10^{-5}(t/t_s \sim 10^{-3})$.

B. Evolution and collapse of the *d*-dimensional island after source switching-off $t > T$

According to Eq. (26), we find that at $d = 1,2,3$ the evolution of particles distribution $s(r,t)$ after source switching-

off $\Delta t = t - T > 0$ is described by the expressions

$$s + 1 = \lambda\left[\sqrt{t}\,\text{ierfc}\left(\frac{r}{2\sqrt{t}}\right) - \sqrt{\Delta t}\,\text{ierfc}\left(\frac{r}{2\sqrt{\Delta t}}\right)\right], \quad d = 1,$$

$$s + 1 = \frac{\lambda}{4\pi}[\text{Ei}(-r^2/4\Delta t) - \text{Ei}(-r^2/4t)], \quad d = 2,$$

$$s + 1 = \frac{\lambda}{4\pi r}\left[\text{erfc}\left(\frac{r}{2\sqrt{t}}\right) - \text{erfc}\left(\frac{r}{2\sqrt{\Delta t}}\right)\right], \quad d = 3,$$

whence neglecting the front width $w/r_f \ll 1$, we find from the condition $s(r_f,t) = 0$ the island radius trajectory $r_f(t)$,

$$\sqrt{t}\,\text{ierfc}\left(\frac{r_f}{2\sqrt{t}}\right) - \sqrt{\Delta t}\,\text{ierfc}\left(\frac{r_f}{2\sqrt{\Delta t}}\right) = 1/\lambda, \quad d = 1, \quad (44)$$

$$\text{Ei}\left(-r_f^2/4\Delta t\right) - \text{Ei}\left(-r_f^2/4t\right) = \frac{4\pi}{\lambda}, \quad d = 2, \quad (45)$$

$$\text{erfc}\left(\frac{r_f}{2\sqrt{t}}\right) - \text{erfc}\left(\frac{r_f}{2\sqrt{\Delta t}}\right) = \frac{4\pi r_f}{\lambda}, \quad d = 3. \quad (46)$$

Assuming further that the front remains sharp enough up to a narrow vicinity of the collapse point t_c, from the condition $r_f(t_c) = 0$ we obtain

$$t_c = (\lambda T)^2(1 + \pi/\lambda^2 T)^2/4\pi, \quad d = 1, \quad (47)$$

$$t_c = T/(1 - e^{-\lambda_*/\lambda}), \quad d = 2, \quad (48)$$

$$4\pi^{3/2}/\lambda = 1/\sqrt{t_c - T} - 1/\sqrt{t_c}, \quad d = 3, \quad (49)$$

whence for the ratio of the island collapse time t_c to the injection period T it follows immediately,

$$t_c/T = \mathcal{F}_d(\lambda^2 T^{2-d}). \quad (50)$$

One can easily be convinced that the asymptotics of the scaling function $\mathcal{F}_d(z)$ in the limit of large $z \gg 1$ has the form

$$\mathcal{F}_d(z) = \frac{z^{1/d}(1 + 2\pi/z^{1/d} + \ldots)}{4\pi}. \quad (51)$$

Thus, we conclude that (a) at $(\lambda^2 T^{2-d})^{1/d}/4\pi \gg 1$ the island lifetime t_c becomes much longer than the injection period T (the long-living island),

$$t_c/T = (\lambda^2 T^{2-d})^{1/d}/4\pi \gg 1,$$

and (b) in the long-living island regime the ratio t_c/T increases at the increase of T in the 1D case, does not change at the increase of T in the 2D case and decreases at the increase of T in the 3D case. Let us consider the consequences of Eqs. (44)–(49) for each dimension separately.

1. One-dimensional island

According to Eqs. (37) and (38) in the 1D case one of the key conditions for island formation is the requirement $\lambda^2 T \gg 1$ that is why the formed 1D island at *any* λ is long-living. From Eq. (44) it follows that after source switching-off the long-living island continues to expand reaching the maximum,

$$r_f^M \propto \lambda T,$$

whence taking into account Eq. (37), we find

$$r_f^M / r_f^T \propto \sqrt{\frac{\lambda^2 T}{\ln(\lambda^2 T)}} \to \infty,$$

as $\lambda^2 T \to \infty$. The trajectories of the 1D island radius calculated according to Eq. (44) in the scaling coordinates λr_f versus $\lambda^2 t$ are shown in Fig. 2, demonstrating the evolution of these trajectories with the growing parameter $\lambda^2 T$.

2. Two-dimensional island

According to Eq. (45) in the 2D case in the limit of anomalously slow growth $\lambda \ll \lambda_*$, when the majority of injected particles die, we find

$$r_f^M / r_f^T \approx 1,$$

whence taking into account Eq. (48), it follows that regardless of the injection time after source switching-off the formed 2D island begins to contract immediately disappearing for exponentially small (in comparison with the injection period) time interval,

$$(t_c - T)/T \sim e^{-\lambda_*/\lambda} \ll 1.$$

In the opposite limit of the long-living island $\lambda \gg \lambda_*$ after source switching-off the island continues to expand reaching the maximum,

$$r_f^M \propto \sqrt{\lambda T},$$

whence taking into account Eq. (39), we find that regardless of the injection duration,

$$r_f^M / r_f^T \propto \sqrt{\frac{\lambda}{\ln \lambda}} \to \infty,$$

as $\lambda \to \infty$.

3. Three-dimensional island

According to Eq. (46) in the 3D case in the limit of the stationary island $T \gg t_s = (\lambda/\lambda_*)^2$, when the majority of the injected particles die, we find

$$r_f^M / r_s \approx 1,$$

whence, as expected, it follows that in this limit regardless of the injection duration after source switching-off the formed 3D island begins to contract immediately disappearing for the time $t_c - T \propto r_s^2 \ll T$. Indeed, from Eq. (49) in the stationary limit $T \gg t_s$, we find

$$t_c - T = \frac{\lambda^2 (1 - \sqrt{\lambda^2/T}/2\pi^{3/2} + \dots)}{16\pi^3}.$$

In the opposite limit of the long-living island $1 \ll T \ll t_s$ after source switching-off the island continues to expand reaching the maximum,

$$r_f^M \propto (\lambda T)^{1/3},$$

whence taking into account Eq. (42), we find

$$r_f^M / r_f^T \propto \left[\frac{\lambda^2}{T \ln^3(t_s/T)} \right]^{1/6} \to \infty,$$

as $\lambda^2/T \to \infty$. The trajectories of the 3D island radius calculated according to Eq. (46) in the scaling coordinates r_f/λ versus t/λ^2 are shown in Fig. 3, demonstrating the evolution of these trajectories with the growing parameter λ^2/T.

As was stated in the Introduction, and as it is clear from the presented analysis, the evolution regularities in the long-living island regime are of the main interest, which is why below we shall focus namely on this regime. According to Eqs. (44), (45), and (46), the maximal volume of d-dimensional island expansion in the front turning point is proportional to the number of particles injected by the time of source switching-off,

$$\Omega_M = \mu_d (r_f^M)^d \propto N_T = \lambda T.$$

But according to Eqs. (38), (40), and (43), in the long-living island regime the majority of injected particles survives up to the time of source switching-off; i.e., the value of N_T determines the number of A-particles at the moment of source switching-off. Comparing these results and taking into account Eqs. (37), (39), and (42), we come to the important conclusion that in the limit of the long-living island regardless of the medium dimension and the injection duration the ratio of the d-dimensional island maximal volume in the front turning point, Ω_M, to the starting island volume at the moment of source switching-off, Ω_T, is proportional to the island particles mean concentration $<a>_T = N_T/\Omega_T$ at the moment of source switching-off and inversely proportional to the fraction of the particles died by this moment $1 - q_T$,

$$<a>_T \propto \frac{1}{1 - q_T} \propto \frac{\Omega_M}{\Omega_T} = \left(\frac{r_f^M}{r_f^T} \right)^d. \tag{52}$$

From Eq. (52) it follows that the long-living island regime is realized in the limit when at the moment of source switching-off the island particles mean concentration $<a>_T$ becomes large as compared to the initial sea density. The value of $<a>_T$ increases with the increase of T in the 1D case, does not change with the increase of T in the 2D case, and decreases with the increase of T in the 3D case, which leads to the corresponding behavior of the island expansion relative amplitude Ω_M/Ω_T and, as a consequence, to the corresponding behavior of the relative island lifetime t_c/T.

Comparing the long-living island evolution after source switching-off to the evolution of the initially uniform concentrated island, we conclude that in both of the problems the maximal island expansion amplitude Ω_M is determined unambiguously by the "starting" number of particles in the island N_{st} and, as a consequence, in both of the problems the island collapse time is $t_c = (N_{st})^{d/2}/4\pi$, where $N_{st} = N_T$ or N_0, respectively. This fact gives grounds to assert that in the long-time limit ($t \gg T$ and $t \gg 1$, respectively), *regardless of the starting particle distribution the island evolution takes a universal form characterizing island death in the instantaneous source regime. Below we shall demonstrate the validity of this statement and reveal the conditions for universalization of the long-time island evolution.

C. Self-similar evolution of the long-living island

We shall assume that the island lifetime t_c exceeds the injection period T considerably. Then, in the long-time limit $T \ll t < t_c$ taking into account the additional requirement $r^2 T / t^2 \ll 1$, from Eq. (26) we easily find the long-time asymptotics of particle distribution,

$$s + 1 = \frac{\lambda T}{(4\pi t)^{d/2}} e^{-r^2/4t} [1 + (d - r^2/2t)T/4t + \ldots], \quad (53)$$

whence, neglecting the front width, from the condition $s(r_f) = 0$ we obtain the long-time asymptotics of front trajectory,

$$r_f = 2(1 - T/4t)\sqrt{t \ln\left[\frac{\lambda T(1 + dT/4t)}{(4\pi t)^{d/2}}\right]}, \quad (54)$$

and in accordance with Eqs. (50) and (51), for the island collapse point $r_f(t_c) = 0$ we find

$$t_c = \frac{(\lambda T)^{2/d}[1 + 2\pi/(\lambda^2 T^{2-d})^{1/d} + \ldots]}{4\pi}. \quad (55)$$

Neglecting in Eq. (54) the terms $\mathcal{O}(T/t)$, we come exactly to the long-time front trajectory asymptotics of the initially uniform island [Eq. (13)],

$$r_f = \sqrt{2dt \ln(t_c/t)},$$

where now

$$t_c = (N_T)^{2/d}/4\pi,$$

whence it follows that in the island maximal expansion point,

$$t_c/t_M = e,$$

$$r_f^M = \sqrt{2dt_M} = (N_T)^{1/d}\sqrt{d/2\pi e}. \quad (56)$$

Thus, we conclude that as well as in the case of the initially uniform island, i.e., regardless of the initial A-particles distribution, the long-time front trajectory of the long-living island regardless of the system dimension is described by the universal law Eq. (16),

$$\zeta_f = r_f/r_f^M = \sqrt{e\tau|\ln\tau|}, \quad \tau = t/t_c.$$

Satisfying within the island the requirement $r^2 T / t^2 \leqslant r_f^2 T / t^2 \ll 1$, we find the key condition for front trajectory universalization,

$$\tau \gg (T/t_c)\max(1, |\ln\tau|). \quad (57)$$

As an illustration, Figs. 4(a) and 4(b) demonstrate collapse of 1D and 3D islands front trajectories to the universal trajectory Eq. (16) with the growing parameters $\lambda^2 T$ and λ^2/T, respectively.

Assuming that the condition Eq. (57) is fulfilled and neglecting in Eq. (53) the terms $(d - r^2/2t)T/4t + \ldots$, as well as in the case of the initially uniform island, we come to the independent of the starting distribution self-similar evolution of the long-time particles distribution in the island $a(\zeta, \tau)$ [Eq. (17)] and, as a consequence, we obtain the scaling laws of particles death in the form

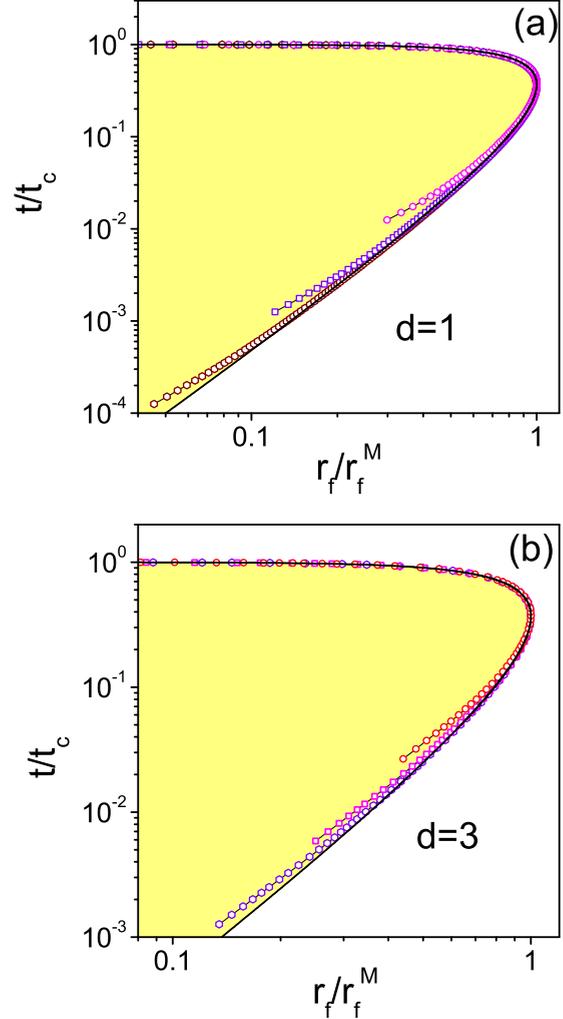$$N/N_T = \mathcal{G}_d(\tau), \quad (58)$$

FIG. 4. Collapse of the front trajectories shown in Figs. 2 and 3 after the source switching-off to the universal trajectory $\zeta_f(\tau)$ [Eq. (16)] (thick line) in the scaling coordinates r_f/r_f^M vs t/t_c with growing parameters $\lambda^2 T$ and λ^2/T, respectively: (a) $d = 1$, $\lambda^2 T = 10^3$ (circles), 10^4 (squares), and 10^5 (hexagons); (b) $d = 3$, $\lambda^2/T = 10^8$ (circles), 10^{10} (squares), and 10^{12} (hexagons).

where scaling functions $\mathcal{G}_d(\tau)$ are determined by Eq. (18). In the limit of large $t_c/T \to \infty$, hence, it follows that in the front turning point τ_M, *regardless of the number of the injected particles* $N_T = \lambda T$, the particles distribution in the island takes the form

$$a(\zeta, \tau_M) = e^{d(1-\zeta^2)/2} - 1,$$

and the fraction of the particles survived at the island expansion stage N_M/N_T is determined by Eq. (20). From Eq. (58), it also follows that at the final island collapse stage the fraction of the surviving particles $N(\mathcal{T})/N_T$ decays according to the power law Eq. (21). Completing the long-living island evolution analysis, it should be emphasized that according to Eqs. (16), (17), and (57), as well as in the case of the initially uniform island, we conclude that regardless of the system dimension the reduced boundary current behavior $\mathcal{J}(\tau) = J/J_M$ is described by the universal law Eq. (22) that predetermines universality of the mean-field front relative width evolution.

IV. EVOLUTION OF THE REACTION FRONT

So far we have assumed formally that the reaction front is sharp enough so that the front relative width $\eta = w/r_f$ remains negligibly small up to a narrow vicinity of the island collapse point. In this section, we shall reveal the conditions for this assumption realization in the long-living island regime. Supposing that $|\zeta - \zeta_f|/\zeta_f \ll 1$, we find from Eqs. (16) and (17),

$$s + 1 = e^{-d|\ln\tau|(\zeta-\zeta_f)/\zeta_f+\cdots},$$

whence it follows that in the front center vicinity $|\zeta - \zeta_f|/\zeta_f \ll \min(1, 1/d|\ln\tau|)$, the quantity s becomes a linear function of ζ,

$$s = -d|\ln\tau|(\zeta - \zeta_f)/\zeta_f + \ldots.$$

It means that the boundary current density Eq. (22) determines the evolution of the quasistatic reaction front width $w(J)$ in fulfilling the requirement

$$\eta = w/r_f \ll \min(1, 1/d|\ln\tau|). \tag{59}$$

It is not difficult to show that, in accordance with Eq. (22), the front quasistaticity condition $t_w/t_J \ll 1$ [where the value $t_J = -(d\ln J/dt)^{-1}$ describes the rate of the boundary current change and $t_w \sim w^2$ is the equilibration time of the reaction front] takes the form

$$t_w/t_J \sim d\eta^2(1 + |\ln\tau|) \ll 1,$$

whence it follows self-consistently that in fulfilling the requirement Eq. (59) the reaction front becomes quasistatic automatically.

In Refs. [13–16] it is established that at $d > d_c = 2$ in the dimensional variables the dependence of the quasistatic front width on the boundary current density is described by the mean-field law,

$$w_{\mathrm{MF}} \sim (D^2/kJ)^{1/3}, \tag{60}$$

whereas in the 1D case in the diffusion-controlled limit the quasistatic front width becomes k-independent and it is determined by the fluctuation law,

$$w_{\mathrm{F}} \sim \sqrt{D/J}. \tag{61}$$

At upper critical dimension $d = d_c = 2$ in the diffusion-controlled limit in the mean-field law Eq. (60) a logarithmic correction appears (logarithmically modified front), $w_{\mathrm{L}} \propto (|\ln J|/J)^{1/3}$ [14,28,29]; its full form will be presented below. At first we shall consider the evolution of the 1D fluctuation front width, then we shall analyze the behavior of the modified two-dimensional front width, and finally we shall reveal the regularities of the mean-field front width evolution for quasi-one-dimensional, quasi-two-dimensional, and three-dimensional geometry.

A. Fluctuation front

According to Eq. (61), in the units that we have accepted, the fluctuation front width reads

$$w_{\mathrm{F}} \sim 1/\sqrt{n_0 J},$$

where $n_0 = b_0 L$ for the initially uniform island and $n_0 = \rho\ell = 1$ for the island formed by the localized source. Substituting here Eq. (22), we find

$$w_{\mathrm{F}} = w_{\mathrm{F}}^M (e\tau/|\ln\tau|)^{1/4}, \tag{62}$$

where $w_{\mathrm{F}}^M \sim \sqrt{r_f^M/n_0}$. From Eqs. (62) and (16) it follows that the fluctuation front relative width $\eta_{\mathrm{F}} = w_{\mathrm{F}}/r_f$ changes by the law

$$\eta_{\mathrm{F}} = \eta_{\mathrm{F}}^M/(e\tau|\ln^3\tau|)^{1/4}, \tag{63}$$

where in accordance with Eqs. (15) and (56) the relative width amplitude in the front turning point is

$$\eta_{\mathrm{F}}^M \sim 1/\sqrt{n_0 r_f^M} \sim 1/\sqrt{\mathcal{N}}, \tag{64}$$

and $\mathcal{N}$ is the initial number of particles in the originally uniform island $\mathcal{N} = n_0 N_0$ or the number of injected particles $\mathcal{N} = N_T$. According to Eq. (63), at the island expansion stage, the value η_{F} is decreasing relatively slowly reaching the minimum $\min(\eta_{\mathrm{F}}) \approx 0.72\eta_{\mathrm{F}}^M$ at $\tau_m = 1/e^3$ and then, at the island contraction stage at $\mathcal{T} = 1 - \tau = (t_c - t)/t_c \ll 1$, it begins to increase fast by the law

$$\eta_{\mathrm{F}} \sim (\mathcal{T}_Q/\mathcal{T})^{3/4},$$

where $\mathcal{T}_Q \sim 1/\mathcal{N}^{2/3} \to 0$ as $\mathcal{N} \to \infty$. Thus, we conclude that at sufficiently large initial number of particles $\mathcal{N}$, the fluctuation front remains sharp enough up to a narrow vicinity of the island collapse point. According to Eqs. (59), (63), and (64) at the island expansion stage far from the collapse point ($\tau \ll 1$) the front becomes sharp $\eta_{\mathrm{F}}|\ln\tau| < \epsilon \ll 1$ under the condition

$$\tau/|\ln\tau| > 1/(\epsilon^2\mathcal{N})^2.$$

It should be emphasized that in contrast to the front width the amplitude η_{F}^M determining characteristic scale of the fluctuation front relative width does not depend on the sea density.

B. Modified front at $d = d_c = 2$

Following Krapivsky's approach [29] along with the arguments of Ref. [12], for the modified two-dimensional front width in the diffusion-controlled limit we find (in dimensional variables)

$$w_{\mathrm{L}} \sim \left[\left(\frac{D}{J}\right)\ln\left(\frac{D}{Jr_a^3}\right)\right]^{1/3},$$

where r_a is the reaction radius and $D/Jr_a^3 \gg 1$ according to the requirement $w_{\mathrm{L}} \gg r_a$. Substituting here Eq. (22), we obtain

$$w_{\mathrm{L}} = w_{\mathrm{L}}^M \frac{[1 + \ln(\sqrt{e\tau/|\ln\tau|})/\ln\phi]^{1/3}}{(|\ln\tau|/e\tau)^{1/6}}, \tag{65}$$

whence in accordance with Eqs. (15), (16), and (56), it follows that the front relative width $\eta_{\mathrm{L}} = w_{\mathrm{L}}/r_f$ changes by the law

$$\eta_{\mathrm{L}} = \eta_{\mathrm{L}}^M \left[\frac{1 + \ln(\sqrt{e\tau/|\ln\tau|})/\ln\phi}{e\tau\ln^2\tau}\right]^{1/3}, \tag{66}$$

where the relative width amplitude in the front turning point is

$$\eta_{\rm L}^M \sim \left[\frac{\ln \phi(\mathcal{N})}{\mathcal{N}} \right]^{1/3},$$

$$\phi(\mathcal{N}) = (\ell/r_a)^3 \sqrt{\mathcal{N}}, \qquad (67)$$

and $\mathcal{N}$ is the initial number of particles in the originally uniform island $\mathcal{N} = b_0 L^2 N_0$ or the number of injected particles $\mathcal{N} = N_T$. According to Eq. (66), at the island expansion stage, the value $\eta_{\rm L}$ is decreasing relatively slowly reaching at large enough ϕ the minimum $\min(\eta_{\rm L}) \approx 0.88(1 - 0.28/\ln \phi + \dots) \eta_{\rm L}^M$ at $\tau_m = e^{-2}(1 - 3/2 \ln \phi + \dots)$, and then at the island contraction stage at $\mathcal{T} \ll 1$, it begins to increase fast by the law

$$\eta_{\rm L} \sim (\mathcal{T}_Q/\mathcal{T})^{2/3},$$

where $\mathcal{T}_Q \sim [\ln(\phi/\sqrt{\mathcal{T}})/\mathcal{N}]^{1/2} \to 0$ as $\mathcal{N} \to \infty$ at fixed $\mathcal{T}$. From Eqs. (59) and (66) it follows that at the island expansion stage far from the collapse point ($\tau \ll 1$) the front becomes sharp $\eta_{\rm L}|\ln \tau| < \epsilon \ll 1$ under the condition

$$\tau/|\ln \tau| > \frac{\ln(\phi \sqrt{e\tau/|\ln \tau|})}{\epsilon^3 \mathcal{N}},$$

with the additional requirement $\tau/|\ln \tau| \gg \phi^{-2}$. Thus, we conclude that at sufficiently large "starting" number of particles $\mathcal{N}$, the modified two-dimensional front becomes sharp at early stages of the island expansion and remains sharp up to a narrow vicinity of the collapse point. Let us note that in contrast to the fluctuation front the relative width amplitude $\eta_{\rm L}^M$ increases logarithmically slowly at the decrease of the sea density. As an illustration, assuming that $r_a \sim 10^{-8}$ cm, $b_0 = 10^{12}$ cm^{-2}, $L = 0.1$ cm, and $c = 10^3$, we find $\mathcal{N} \approx 3 \times 10^{13}$, $\ln \phi \approx 30$, and $\eta_{\rm L}^M \approx 10^{-4}$.

C. Mean-field front

According to Eq. (60), in the units that we have accepted, the mean-field front width reads

$$w_{\rm MF} \sim 1/(\kappa J)^{1/3},$$

where $\kappa = k b_0 L^2/D$ for the originally uniform island and $\kappa = k\rho \ell^2/D$ for the island formed by the localized source. Substituting here Eq. (22), we find

$$w_{\rm MF} = w_{\rm MF}^M (e\tau/|\ln \tau|)^{1/6}, \qquad (68)$$

where $w_{\rm MF}^M \sim (r_f^M/d\kappa)^{1/3}$. From Eqs. (68) and (16) it follows that the relative mean-field front width $\eta_{\rm MF} = w_{\rm MF}/r_f$ changes by the law

$$\eta_{\rm MF} = \eta_{\rm MF}^M/(e\tau \ln^2 \tau)^{1/3}, \qquad (69)$$

where in accordance with Eqs. (15) and (56) the relative width amplitude in the front turning point is

$$\eta_{\rm MF}^M \sim \left(\sqrt{d\kappa} r_f^M \right)^{-2/3} = m \left(\frac{D\ell^{d-2}}{k} \right)^{1/3} \mathcal{N}^{-2/3d}, \quad (70)$$

where $m = (2\pi e/d^2)^{1/3}$ and $\mathcal{N}$ is the initial number of particles in the originally uniform island $\mathcal{N} = b_0 L^d N_0$ or the number of the injected particles $\mathcal{N} = N_T$. According to

Eq. (69) at the island expansion stage the value $\eta_{\rm MF}$ is decreasing relatively slowly reaching the minimum $\min(\eta_{\rm MF}) \approx 0.88 \eta_{\rm MF}^M$ at $\tau_m = 1/e^2$ and then, at the island contraction stage at $\mathcal{T} \ll 1$, it begins to increase fast by the law

$$\eta_{\rm MF} \sim (\mathcal{T}_Q/\mathcal{T})^{2/3},$$

where $\mathcal{T}_Q \sim (\eta_{\rm MF}^M)^{3/2} \propto \mathcal{N}^{-1/d} \to 0$ as $\mathcal{N} \to \infty$. From Eqs. (59) and (69) it follows that at the island expansion stage far from the collapse point ($\tau \ll 1$) the front becomes sharp $\eta_{\rm MF}|\ln \tau| < \epsilon \ll 1$ under the condition

$$\tau/|\ln \tau| > \left(\eta_{\rm MF}^M/\epsilon \right)^3 \propto \epsilon^{-3} \mathcal{N}^{-2/d}.$$

Barkema, Howard, and Cardy have shown analytically and numerically [15] that in the 1D case the fluctuation front is formed under the condition $w_{\rm F}/w_{\rm MF} \gg 1 (k/\sqrt{JD} \gg 1)$, while in the opposite limit $w_{\rm F}/w_{\rm MF} \ll 1 (k/\sqrt{JD} \ll 1)$ the front width is determined by the mean-field law Eq. (60). Comparing Eqs. (63) and (69), we find $\eta_{\rm F}/\eta_{\rm MF} = (\eta_{\rm F}^M/\eta_{\rm MF}^M)(e\tau/\ln \tau)^{1/12}$ whence by virtue of weak dependence on τ it follows that as a characteristic crossover point from the mean-field to the fluctuation regime (MF $\to$ F) it is reasonable to accept the ratio $\eta_{\rm F}^M/\eta_{\rm MF}^M \sim 1$. Substituting here Eqs. (64) and (70), for the fluctuation regime area we find

$$k \gg k_{\rm F} \sim D/\ell \sqrt{\mathcal{N}},$$

whence it follows that with the increase in the initial number of particles in the island and the decrease of the sea density the fluctuation regime domain expands indefinitely ($k_{\rm F} \to 0$ as $\ell \sqrt{\mathcal{N}} \to \infty$). Determining further the lower bound of the sharp mean-field front regime by the condition $\eta_{\rm MF}^M < \varepsilon \ll 1$, we find from Eq. (70)

$$k > k_{\rm MF}^\varepsilon \sim (D/\ell)/\varepsilon^3 \mathcal{N}^2,$$

whence it follows

$$k_{\rm F}/k_{\rm MF}^\varepsilon \sim (\varepsilon^2 \mathcal{N})^{3/2},$$

and we conclude that the area of the island death in the sharp mean-field front regime $k_{\rm MF}^\varepsilon < k \ll k_{\rm F}$ appears under the condition $\mathcal{N} > \varepsilon^{-2}(\eta_{\rm F}^M < \varepsilon)$ and expands fast with the increase of $\mathcal{N}$.

Repeating the presented above argumentation, for the crossover from the mean-field to the logarithmically modified front (MF $\to$ L) at $d = d_c = 2$ we find from Eqs. (67) and (70)

$$k \gg k_{\rm L} \sim D/\ln \phi(\mathcal{N}),$$

whence it follows that with the increase of the starting number of the particles in the island and the decrease of the sea density the LM regime area expands logarithmically slowly. Determining the lower bound of the sharp mean-field front regime by the condition $\eta_{\rm MF}^M < \varepsilon \ll 1$, we find from Eq. (70)

$$k > k_{\rm MF}^\varepsilon \sim D/\mathcal{N}\varepsilon^3,$$

whence it follows

$$k_{\rm L}/k_{\rm MF}^\varepsilon \sim \varepsilon^3 \mathcal{N}/\ln \phi(\mathcal{N}),$$

and we conclude that the area of the island death in the sharp mean-field front regime $k_{\rm MF}^\varepsilon < k \ll k_{\rm L}$ appears under

the condition $\mathcal{N}/\ln\phi > \varepsilon^{-3}(\eta_L^M < \varepsilon)$ and expands fast with the increase of $\mathcal{N}$.

According to Eq. (70), in the 3D case with the growth of the reaction constant the front relative width amplitude decreases $\propto k^{-1/3}$, reaching the minimal value in the diffusion-controlled limit of the perfect reaction $k = k_p = 8\pi D r_a$, where r_a is the reaction radius. Substituting k_p into Eq. (70), we find

$$\eta_{\mathrm{MF}}^M \sim m_p \left(\frac{\ell}{r_a}\right)^{1/3} \mathcal{N}^{-2/9},$$

with $m_p \approx 0.4$, whence taking for illustration $r_a \sim 10^{-8}$ cm, $b_0 = 10^{20}$ cm^{-3}, $L = 0.1$ cm, and $c = 10^3$ we obtain $\eta_{\mathrm{MF}}^M \approx 3 \times 10^{-5}$. Thus, we conclude that the three-dimensional spherical island dies in the sharp front regime in a wide range of parameters (for instance at the decrease of the reaction constant by 9 orders of magnitude the front keeps sharp enough). It should be emphasized that according to Eq. (70) in the region of the sharp mean-field front existence at the decrease of the sea density the front relative width amplitude decreases in the 1D case, does not change in the 2D case, and increases in the 3D case.

D. Quasi-d-dimensional systems

So far we have analyzed the evolution of the d-dimensional spherical island formed by either the point source or the initially uniform spherically symmetric particles distribution. Completing this section we shall consider for completeness the island evolution in the three-dimensional medium for the quasi-one-dimensional (planar front) and the quasi-two-dimensional (cylindrical front) geometry. Let in the uniform three-dimensional B-particle sea acts (a) a planar two-dimensional source of A particles with the injection rate Λ_+ particles in a time unit per a source unit area or (b) a linear one-dimensional source with the injection rate Λ_+ particles in a time unit per a source unit length. Then, by virtue of symmetry, a "planar" island with a width $2x_f(t) = 2r_f(t)$ (wherein the concentration changes only along the normal to the source plane) will be formed around the planar source, and a cylindrical island with a radius $r_f(t)$ (wherein the concentration changes only along the normal to the source axis) will be formed around the linear source. It is not difficult to show that in the units that we have accepted all the results obtained in Sec. III remain valid for the effective dimension d_+ with the only difference that now the reduced source strength takes the form

$$\lambda = \Lambda_+ \ell^{2+\delta}/D,$$

where $\delta = 3 - d_+$ and the effective dimension of the system is $d_+ = 1(\delta = 2)$ for the planar source and $d_+ = 2(\delta = 1)$ for the linear source (nevertheless as before $\ell = \rho^{-1/d} = \rho^{-1/3}$). Besides it is clear that instead of the number of particles in the island N in the quasi-one-dimensional and the quasi-two-dimensional geometry the reduced number of particles appears

$$N_+ \ell^\delta = g_{d_+} \int_0^{r_f} s(r,t) r^{d_+ - 1} dr,$$

where N_+ is the number of particles in the island per unit area ($d_+ = 1$) or per unit length ($d_+ = 2$) of the source, and the

reduced number of injected particles is

$$N_T = \lambda T = \mathcal{N}_+ \ell^\delta,$$

where $\mathcal{N}_+$ is the number of injected particles per unit area ($d_+ = 1$) or per unit length ($d_+ = 2$) of the source. In the long-living island regime, for the island radius amplitude in the front turning point we find from Eq. (56)

$$r_f^M = \sqrt{\frac{d_+}{2\pi e}} (\mathcal{N}_+ \ell^\delta)^{1/d_+},$$

whence after substitution to Eq. (70), for the front relative width amplitude we obtain

$$\eta_{\mathrm{MF}}^M \sim m_+ \left(\frac{D\ell^\sigma}{k}\right)^{1/3} \mathcal{N}_+^{-2/3d_+}, \tag{71}$$

where $\sigma = 3(d_+ - 2)/d_+$ and $m_+ = (2\pi e/d_+^2)^{1/3}$.

Let now the uniform "planar" A-particle island with a width $2L$ or the uniform cylindrical A-particle island with a radius L and the initial concentration $a_0 = cb_0$ ($c \gg 1$) be surrounded by the uniform three-dimensional B-particle sea with the concentration b_0. Then, by virtue of symmetry, as well as in the case with the planar and the linear sources, in the course of the following evolution a "planar" island should remain the "planar" one wherein the concentration changes only along the normal to the front plane ($d_+ = 1$), and a cylindrical island should remain the cylindrical one, wherein the concentration changes only along the normal to the cylinder axis ($d_+ = 2$). It is not difficult to show that in the units that we have accepted all the results obtained in Sec. II remain valid for the effective dimension d_+ with the only difference that instead of the reduced number of particles in the island N evaluated in the units $b_0 L^d$, in the quasi-one-dimensional and the quasi-two-dimensional systems there appears the reduced number of particles in the island N_+ per a front unit area ($d_+ = 1$) or per a cylinder axis unit length ($d_+ = 2$) evaluated in the units $b_0 L^{d_+}$ with the initial number of particles in the island $\mathcal{N}_+ = N_0 b_0 L^{d_+} = \mu_{d_+} c b_0 L^{d_+}$ per a unit of area and length, respectively. In the long-living island regime for the island radius amplitude in the front turning point we find from Eq. (15),

$$r_f^M = \sqrt{\frac{d_+}{2\pi e}} (N_0)^{1/d_+},$$

whence after substitution to Eq. (70) taking into account the relation $\ell = b_0^{-1/3}$ we come to Eq. (71) again for the front relative width amplitude. Substituting further to Eq. (71) the constant of the diffusion-controlled perfect reaction $k = k_p = 8\pi D r_a$, we find

$$\eta_{\mathrm{MF}}^M \sim m_{p+} \left(\frac{\ell^\sigma}{r_a}\right)^{1/3} \mathcal{N}_+^{-2/3d_+},$$

with $m_{p+} = (e/4d_+^2)^{1/3}$ whence taking for illustration the same parameters as for the spherical island $r_a \sim 10^{-8}$ cm, $b_0 = 10^{20}$ cm^3, $L = 0.1$ cm, and $c = 10^3$, we obtain $\mathcal{N}_+ \approx 2 \times 10^{22}$ cm^{-2}, $\eta_{\mathrm{MF}}^M \approx 3 \times 10^{-6}$ for the quasi-one-dimensional island ($d_+ = 1$) and $\mathcal{N}_+ \approx 3 \times 10^{21}$ cm^{-1}, $\eta_{\mathrm{MF}}^M \approx 2 \times 10^{-5}$ for the quasi-two-dimensional island ($d_+ = 2$). Thus, we conclude that at enough large $\mathcal{N}_+$ the quasi-one-dimensional and the

quasi-two-dimensional islands die in the three-dimensional sea in the sharp front regime in a wide interval of parameters. One can easily be convinced that this conclusion remains valid for the quasi-one-dimensional geometry in the two-dimensional medium (linear one-dimensional source in the two-dimensional sea with the logarithmically modified front).

V. CONCLUSION

In this paper, we have presented a systematic analytical study of diffusion-controlled formation and collapse of a d-dimensional A-particle island in the B-particle sea at propagation of the sharp reaction front $A + B \to 0$. Our main purpose was to describe the formation regularities for the d-dimensional spherical island at A particles injection by a point source acting for some finite time T and the following island evolution and collapse after source switching-off. We have focused mainly on the most interesting case when the island collapse time t_c becomes much longer than the injection period T (long-living island) and have revealed the complete picture of the evolution of front trajectory and particle distribution in the island depending on the intensity and duration of source action. Generalizing the results obtained earlier for the quasi-one-dimensional geometry we have also revealed the long-time evolution regularities for the initially uniform d-dimensional spherical A-particle island. The main results can be formulated as follows:

(1) The conditions of the long-living island formation have been found and it was shown that in the long-living island regime the ratio t_c/T changes with the increase of the injection period T and the reduced source strength λ by the law $\propto (\lambda^2 T^{2-d})^{1/d}$, i.e., it increases with the increase of T in the 1D case, does not change with the increase of T in the 2D case and decreases with the increase of T in the 3D case. It has been established that *regardless of the medium dimension and the injection duration*, the ratio of the maximal d-dimensional island volume in the front turning point Ω_M to the initial island volume at the moment of source switching-off Ω_T is proportional to the mean concentration of the island particles at the moment of source switching-off and is inversely proportional to the fraction of the particles died by this moment.

(2) It has been established that *regardless of the number of the injected particles and the system dimension* the long-time

front trajectory of the long-living island is described by the universal law

$$\zeta_f = r_f / r_f^M = \sqrt{e\tau |\ln \tau|},$$

where $\tau = t/t_c$ and $r_f^M = (N_T)^{1/d} \sqrt{d/2\pi e}$ is the radius of the island maximal expansion at the front turning point $t_M = t_c/e$. The scaling laws of evolution of the distribution and the number of particles in the d-dimensional long-living island have been derived, and it has been shown that *regardless of the system dimension* the evolution of the boundary current density J that determines the quasistatic front width is described by the universal law

$$\mathcal{J}(\tau) = J/J_M = \sqrt{\frac{|\ln \tau|}{e\tau}}.$$

(3) It has been shown that regardless of the initial particle distribution, the long-time evolution of the initially uniform concentrated island as well as the long-time evolution of the long-living island converge to the unified universal island death asymptotics in the instantaneous source regime,

$$a(\zeta,\tau) = (e^{-\zeta^2/e\tau}/\tau)^{d/2} - 1.$$

(4) The systematic analysis of the reaction front relative width evolution for the fluctuation, the logarithmically modified, and the mean-field regimes was presented, and it was demonstrated that in a wide range of parameters at a large enough number of injected or initially uniformly distributed particles the front remains sharp up to a narrow vicinity of the island collapse point.

In conclusion, it should be emphasized that as well as in Ref. [17], here the evolution of the island has been considered at equal species diffusivities. Although we believe that the regularities discovered reflect the key features of the island evolution, the study of the much more complicated problem for unequal species diffusivities remains a challenging problem for the future.

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