Einstein relation and hydrodynamics of nonequilibrium mass transport processes

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We derive hydrodynamics of paradigmatic conserved-mass transport processes on a ring. The systems, governed by chipping, diffusion, and coalescence of masses, eventually reach a nonequilibrium steady state, having nontrivial correlations, with steady-state measures in most cases not known. In these processes, we analytically calculate two transport coefficients, bulk-diffusion coefficient and conductivity. Remarkably, the two transport coefficients obey an equilibrium-like Einstein relation even when the microscopic dynamics violates detailed balance and systems are far from equilibrium. Moreover, we show, using a macroscopic fluctuation theory, that the probability of large deviation in density, obtained from the above hydrodynamics, is in complete agreement with the same derived earlier by Das *et al.* [Phys. Rev. E **93**, 062135 (2016)] using an additivity property.

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I. INTRODUCTION

The Einstein relation (ER) [1], also known as the Einstein-Smoluchowski relation, is a celebrated equality in equilibrium physics. It connects, quite unexpectedly, two seemingly unrelated transport coefficients, bulk-diffusion coefficient $D(\rho)$ and conductivity $\chi(\rho)$, as $D(\rho) = \chi(\rho)/\sigma^2_{\rm eq}(\rho)$, where $\sigma^2_{\rm eq}(\rho) = \lim_{v\to\infty}(\langle n_v^2\rangle_{\rm eq} - \langle n_v\rangle^2_{\rm eq})/v$ is scaled variance of particle number n_v in a subvolume v (subvolume still much smaller than system volume), and ρ is local number density; angular bracket $\langle\cdot\rangle_{\rm eq}$ denotes equilibrium average. Here the diffusion coefficient $D(\rho)$ is defined from Fourier's law for diffusive current $J_D = -D(\rho)\partial\rho/\partial x$, where $\partial\rho/\partial x$ is spatial density gradient in a particular direction, say along the x axis. The conductivity $\chi(\rho)$ is defined from Ohm's law for drift current $J_d = \chi(\rho)F/k_BT$, due to a small external biasing force F also along the x axis, with k_B and T being the Boltzmann constant and temperature, respectively.

For systems in equilibrium, where detailed balance is obeyed, the ER is universal, irrespective of the details of interparticle interactions or whether the systems are liquids or gases, etc. Indeed, the ER is one of the earliest known forms of a more general class of equilibrium fluctuation relations, collectively called fluctuation-dissipation theorems (FDTs); the FDTs can be proved using linear-response theory around equilibrium state having the Boltzmann-Gibbs distribution [2].

However, systems having a nonequilibrium steady state (NESS), which is arguably the closest counterpart to equilibrium, generally do not have such relations. Because, unlike in equilibrium, they violate detailed balance and usually cannot be described by the Boltzmann-Gibbs distribution. In fact, in most cases, microscopic probability weights in the steady state are *not* known. Quite interestingly, recent studies [3–15] have indicated that, even in NESSs, there can be fluctuation relations analogous to the FDTs in equilibrium. In particular, the ER has been found, mostly numerically, in several model systems [16–18] having a NESS.

The ER involves two bulk transport coefficients, $D(\rho)$ and $\chi(\rho)$, defined on a macroscopic level from the two phenomenological laws of transport—Fourier's and Ohm's law. One way to understand such macroscopic phenomenological fluctuation relations is to derive, from microscopic dynamics, a hydrodynamic description of the systems on a large space and time scales. However, such a task, for classical deterministic (or quantum) dynamics, is quite difficult. On the other hand, for systems governed by stochastic dynamics, the problem of deriving hydrodynamics is comparatively easier and, recently, there has been considerable progress made in this direction [19–21]. However, for stochastic systems having a NESS, the steady-state probability weights are not always known and tackling the problem analytically in such systems, especially when there are nonzero finite spatial correlations, remains to be a challenging one [22]. Perhaps not surprisingly, so far there are not many nonequilibrium interacting-particle systems for which exact hydrodynamic descriptions, presumably the first step toward exploring fluctuation relations such as the ER, have been derived. In fact, the difficulty arises primarily because fluctuation, diffusion coefficient, and conductivity, which would appear in ER (if any) in such systems, must be calculated in a steady state far from equilibrium, not in or around an equilibrium state.

Here, we study a broad class of nonequilibrium conserved-mass transport processes on a ring. These processes are governed by chipping, diffusion, and coalescence of neighboring masses, with total mass in the system being conserved, and have become paradigm in nonequilibrium statistical physics of driven many-particle systems [23,24]. Indeed, throughout the past couple of decades, they have been explored intensively to model a huge variety of natural phenomena, such as formation of clouds [25] and gels [26,27], force fluctuation in packs of granular beads [28,29], transport of energy in solids [30], dynamics of interacting particles on a ring [31], self-assembly of molecules in organic and inorganic materials [32,33], and distribution of wealth in a society [34], etc.

In this paper, we derive hydrodynamics of the above-mentioned one-dimensional conserved-mass transport processes, which have nontrivial spatial correlations (nonzero and finite), with their steady-state weights in most cases not known. For these processes, we explicitly calculate the two transport coefficients as a function of local mass density ρ, the bulk-diffusion coefficient $D(\rho)$, and the conductivity $\chi(\rho)$, which characterize the hydrodynamics. Remarkably, we found that, for this class of models, the two transport coefficients satisfy an equilibrium-like Einstein relation,

$$D(\rho) = \frac{\chi(\rho)}{\sigma^2(\rho)}, \tag{1}$$

where

$$\sigma^2(\rho) = \lim_{v\to\infty} \frac{\langle m^2\rangle - \langle m\rangle^2}{v}, \tag{2}$$

is scaled variance of mass m in a large subsystem (much smaller than the system) of volume v with $\rho = \langle m\rangle/v$ is average local mass density. The diffusion coefficient $D(\rho)$ and the conductivity $\chi(\rho)$ are suitably defined on a hydrodynamic level from diffusive current $J_D = -D(\rho)\partial\rho/\partial x$ and drift current $J_d = \chi(\rho)F$, respectively, where $\partial\rho/\partial x$ is gradient in local mass density and F is the magnitude of a small biasing force coupled locally to conserved mass variable and applied in a particular direction. For all the processes considered in this paper, we find bulk diffusion coefficient $D(\rho) =$ const. and conductivity $\chi(\rho) \propto \rho^2$, indicating that the processes, on hydrodynamic level, belong to the class of Kipnis-Marchioro-Presutti (KMP) processes on a ring [30]. Moreover, we use the two transport coefficients to find probabilities of large deviations of mass in a subsystem in the framework of recently developed macroscopic fluctuation theory (MFT) [5,15]. The mass large-deviation functions (LDFs) completely agree with that in Refs. [12,14], which were derived earlier using an additivity property.

The paper is organized as follows. In Sec. II, we discuss general aspects of conserved-mass transport processes. In Sec. III, we present a linear-response analysis around a nonequilibrium steady state, which is implemented to calculate the transport coefficients in the model-systems discussed later. We introduce, in Sec. IV (symmetric versions) and Sec. VI (asymmetric versions), a broad class of conserved-mass transport processes (called models I, II, and III) and derive hydrodynamics of these systems in terms of two transport coefficients—the diffusion coefficient and the conductivity. In Sec. V and VI, we discuss how the density large deviation functions in all these models can be calculated using a macroscopic fluctuation theory. In Sec. VII, we summarize with some concluding remarks.

II. GENERAL CONSIDERATIONS AND MOTIVATIONS

Let us first discuss some general aspects of fluctuations in steady states and their connection to hydrodynamics in the context of recently obtained results in conserved-mass transport processes [12–14]. The conserved-mass transport processes are defined on a one-dimensional periodic lattice of L sites, with a continuous mass variable $m_i \geqslant 0$ at site $i \in \{1,2,\dots L\}$ [27,31,35,36]. They are governed by dynamical rules, such as chipping or fragmentation, diffusion, and coalescence of neighboring masses, which eventually lead to a nonequilibrium steady state. Under these dynamical rules, total mass $M = \sum_{i=1}^{L} m_i$ in the system remains conserved. Though these processes are governed by simple dynamical rules, they usually have nontrivial spatial correlations in the steady states. That is why, even in one dimension (which is the case considered here), the *exact* steady-state probability weights for the microscopic configurations, except for a few special cases [27,29,31,35], are not yet known.

In this paper, we study several generalized versions of the above-mentioned mass transport processes [14], which we call Model I, Model II, and Model III. In the symmetric versions of the models (see Sec. IV), mass transfers take place, without any preference, to the right or (and) to the left nearest neighbor(s); consequently, net mass currents are zero in the nonequilibrium steady states. However, as shown later, the systems with the symmetric transfers still remain far from equilibrium as the dynamics in the configuration space violates Kolmogorov criterion and thus also detailed balance [37]. For asymmetric mass transfers (see Sec. VI), the violation is quite evident as there would be nonzero mass current in the systems. Kolmogorov criterion, which provides a necessary and sufficient condition for detailed balance to hold in a system, says the following. If, for each and every possible loop generated by the dynamics in the configuration space, the probability of a forward path and that of the corresponding reverse path are equal, detailed balance is satisfied, and vice versa. As a consequence, if a reverse path corresponding to a forward path in a particular transition in the configuration space does not exist, it suffices to say that Kolmogorov criterion, and therefore detailed balance, is violated. Indeed, in the absence of the knowledge of exact steady-state measures in these mass transport processes, Kolmogorov criterion helps one to check whether detailed balance is satisfied or not.

At a coarse-grained level where one divides such a system of volume V into $\nu = V/v$ subsystems, each of volume $v \ll V$, one could however have a simpler description. Provided that the subsystem sizes are large compared to the microscopic spatial correlation length but much smaller than the size of the full system, one expects that the system would possess an additivity property [8,10,11], which states that large subsystems are statistically almost independent. That is, the *steady-state* joint subsystem mass distribution $\mathcal{P}[\{M_1,M_2,\dots,M_\nu\}]$, with M_k being mass in kth subsystem, can be approximately written in a product form, except for a constraint of global mass conservation. In other words, the joint subsystem mass distribution can be expressed in terms of subsystem weight factor $W_v(M_k)$,

$$\mathcal{P}[\{M_k\}] \simeq \frac{\prod_k W_v(M_k)}{Z(V,M)}\delta\left(\sum_k M_k - M\right), \tag{3}$$

where Z is the normalization constant. For large subsystem size, the weight factor $W_v(M_k)$ can be characterized by a large deviation "density" function $f(\rho_k)$ (or "rate" function; also sometimes called "nonequilibrium free energy" density) as $W_v(M_k) \simeq \exp[-vf(\rho_k)]$, where $\rho_k = M_k/v$ is fluctuating subsystem mass density [13]. The immediate consequence of additivity is that the function $f(\rho)$ is related to the scaled variance $\sigma^2(\rho)$ [as defined in Eq. (2)] of subsystem mass through a fluctuation-response relation (FR) [8,10–14], analogous to equilibrium fluctuation-dissipation theorems,

$$f''(\rho) = \frac{d\mu}{d\rho} = \frac{1}{\sigma^2(\rho)}, \tag{4}$$

where $\mu(\rho) = f'(\rho)$ is defined to be a chemical potential and $\rho = \langle \rho_k \rangle$ is local mass density. Now, instead of subsystem mass variables $\{M_k\}$, additivity property [Eq. (3)] can be written in terms of subsystem density variables $\{\rho_k = M_k/v\}$, or equivalently, in terms of coarse-grained fluctuating density profile $\{\rho(x)\}$ in the system. Then, one can write the joint subsystem density distribution, or large-deviation probability of a given density profile $\{\rho(x)\}$, as

$$\mathcal{P}[\rho(x)] \simeq e^{-\mathcal{F}[\rho(x)]},$$

where $\mathcal{F}[\rho(x)]$ is called large deviation function (LDF). In the mass-transport processes considered here, as the functional form of the scaled variance $\sigma^2(\rho) = \rho^2/\eta$ with η being a model-dependent parameter [e.g., see Eq. (23)], the LDFs can be calculated by using additivity and the FR [Eqs. (3) and (4)] [12,14]. In fact, the LDFs have been previously shown to have the following form:

$$\mathcal{F}[\rho(x)] = \int_V dx\{f(\rho) - f(\rho_0) - \mu(\rho_0)(\rho - \rho_0)\}, \quad (5)$$

where

$$f(\rho) = -\eta \ln \rho, \quad (6)$$

$$\mu(\rho) = f'(\rho) = -\frac{\eta}{\rho}, \quad (7)$$

with $\mu(\rho)$ an equilibrium-like chemical potential and $\rho_0 = M/V$ the global mass density [12,14]. The FR in Eq. (4) can be verified from Eq. (5). Moreover, in this case, the probability distribution function $P_v(m)$ of mass m in a subsystem of volume v can be obtained as

$$P_v(m) \propto m^{v\eta-1} e^{-\eta m/\rho}, \quad (8)$$

which is gamma distribution [12,14].

Thus, additivity property helps one to construct a statistical mechanical framework in these conserved-mass transport processes, through a free energy density $f(\rho)$ and a chemical potential $\mu(\rho)$, which however describes only the static properties of *steady-state* mass fluctuations. At this stage, one could ask whether the above LDFs can be derived in a dynamical setting. To address this issue, here we formulate, within recently developed macroscopic fluctuation theory (MFT) [5,15], a statistical mechanical description of fluctuations for these processes. The formulation provides a dynamical description of mass fluctuations at a coarse-grained level, i.e., a fluctuating hydrodynamics valid in large length and time scales [see Eq. (45)].

Since mass remains conserved locally under the microscopic evolution, one must keep the mass conservation valid also at the hydrodynamic scales. Therefore, the hydrodynamic equation must be written in the form of a continuity equation,

$$\partial_\tau \rho(x,\tau) + \partial_x J(\rho(x,\tau)) = 0, \quad (9)$$

which governs the time evolution of density field $\rho(x,\tau)$ with x and τ being suitably rescaled position and time, respectively. Since the class of processes we consider here are of "gradient type" (i.e., local diffusive current can be expressed as a gradient in local observables) [3] with respect to their microscopic evolutions, one would expect a nonlinear hydrodynamics in the diffusive scaling limit, where the current $J(\rho(x,\tau))$ is the sum of two parts $J = J_D + J_d$. The first part $J_D = -D(\rho)\partial_x \rho$ is the diffusive current with $D(\rho)$ being the diffusion coefficient and the second part $J_d(\rho,\tau) = \chi(\rho)F$ is the drift current due to a small slowly varying biasing field $F(x)$ (conjugate to conserved mass variable) with $\chi(\rho)$ being the conductivity.

According to the hydrodynamic Eq. (9), along with a constitutive relation for the current $J(\rho) = -D(\rho)\partial_x\rho + \chi(\rho)F$, the density field $\rho(x,\tau)$ evolves deterministically in time. However, to study any dynamical aspects of fluctuations, one requires to add a suitable noise term. Clearly, as the noise in this case should maintain the local mass conservation, one must add a noise term ζ to the deterministic part of the current $J(x,\tau) \to J(x,\tau) + \zeta(x,\tau)$, making the total current now a fluctuating one. But the question here is what properties the noise ζ would have. As we see later within MFT [see Eq. (45)], the fluctuating part ζ of the total current can be represented in terms of a weak multiplicative Gaussian white noise, whose strength explicitly depends on the conductivity $\chi(\rho)$. So the problem of formulating a theory of mass fluctuations in these processes essentially boils down to finding the functional dependence of the diffusion coefficient $D(\rho)$ and the conductivity $\chi(\rho)$ on density ρ.

In the following section, we explicitly calculate the two transport coefficients, $D(\rho)$ and $\chi(\rho)$, in a broad class of conserved-mass transport processes. Remarkably, in all cases studied here, we find that the two transport coefficients obey an Einstein relation Eq. (1). We present below the details of computations for different models separately.

III. THEORY: LINEAR RESPONSE AROUND NONEQUILIBRIUM STEADY STATES

Before proceeding to the calculations of the transport coefficients in the nonequilibrium mass transport processes mentioned in the previous section, we first present a proof of the Einstein relation (ER), which is valid in or, strictly speaking, around equilibrium state of a system, in the limit of an external force vanishingly small. In equilibrium, an external force field $\vec{F}$ (here taken to be constant, for simplicity), or equivalently an external potential, can be directly related to chemical potential of the system. For example, consider a one-dimensional system whose two halves are kept at two different external potentials, say, first half at potential V_1 and second half at potential V_2 where $V_2 - V_1 = \Delta V = -\int F dx$ with the force field $\vec{F} = F\hat{x}$. The fact that effective chemical potentials of the two halves equalize implies

$$\mu(\rho_1) + V_1 = \mu(\rho_2) + V_2,$$

where ρ_1 and ρ_2 are densities of the first and second halves, respectively, $\mu(\rho) = df/d\rho$ is chemical potential (canonical) and $f(\rho)$ free-energy density (canonical) in the absence of any external potential. In other words, across a spatial interval Δx, we have the following relation $\Delta\mu/\Delta x = -\Delta V/\Delta x = F$, or

$$\frac{d\mu}{dx} = F, \quad (10)$$

in the limit of $\Delta x \to 0$. Now, in the limit of small force $F \to 0$, drift current $J_d = \chi(\rho)F$ due to the force F and the diffusion current $J_D = -D(\rho)d\rho/dx$ must balance each other so that there is no net current in the system. That is, we must have $J_d + J_D = 0$, which, along with the equality $F = d\mu/dx = (d\mu/d\rho)(d\rho/dx)$ [from Eq. (10)] and the equilibrium fluctuation-response relation between compressibility and fluctuation $d\rho/d\mu = \sigma^2(\rho)$ [Eq. (4)], immediately leads to the ER.

On the other hand, in nonequilibrium, though detailed balance is violated on a microscopic level, the macroscopic mass current in the steady state could still be zero, e.g., in the case of the mass-transport processes with symmetric mass transfer rules. In that case, one would perhaps expect, for a suitably chosen biasing force, an ER even in nonequilibrium. Interestingly, we see later that an ER holds in the cases of both symmetric and asymmetric mass transfers. The issue essentially revolves around the crucial question whether Eq. (10) would hold in such cases, which could be addressed by checking if there is an ER. In fact, provided it holds, an ER would then imply a LDF of the form as in Eq. (5), where $f''(\rho) = D(\rho)/\chi(\rho)$ (see Sec. V for a more rigorous discussion).

To explore the issue further, we perform a linear-response analysis of the conserved-mass transport processes in the presence of a small constant biasing force field $\vec{F} = F\hat{x}$, which is now applied in the system, with $\hat{x}$ being a unit vector along $+ve\, x$ axis. The force field $\vec{F}$, somewhat like a gravitational one, is conjugate to the conserved mass variables (external force is coupled to local masses at the individual sites) and is chosen as follows. The biasing force $\vec{F}$ modifies the original mass transfer rates $c_{i\to j}$, from site i to j, to biased rates $c^F_{i\to j}$ (which are now effectively asymmetric) [15],

$$c^F_{i\to j} = c_{i\to j}\Phi(\Delta e_i), \tag{11}$$

where $\Phi(\Delta e_i) > 0$ is nonnegative function of

$$\Delta e_i = \Delta m_{i\to j}(\vec{F}.\delta\vec{x}_{ij}). \tag{12}$$

The quantity Δe can be physically interpreted as extra energy cost (due to the biasing force $\vec{F}$), for transferring or displacing mass $\Delta m_{i\to j}$ from site i to j in a particular direction with the mass displacement vector $\delta\vec{x}_{ij} = (j-i)a\hat{x}$ and a being the lattice constant. We explicitly write the lattice constant, which would be required later for taking diffusive scaling limit. Clearly, $\Phi|_{F=0} = 1$ as $c^{F=0}_{i\to j} = c_{i\to j}$.

In the case of only nearest-neighbor mass transfer (more generalized version is described below), the mass displacement vector $\delta\vec{x}_{ij}$ can take, depending on the direction of the mass transfer, one of the two values $\delta\vec{x} = \pm a\hat{x}$. Consequently, the form of rates in Eq. (11) makes the modified forward and backward mass-transfer rates across a bond asymmetric and therefore induces a small net current in the system.

To check the ER, we consider, somewhat analogous to equilibrium, the function Φ to have a form $\Phi(\Delta e) = \exp(\Delta e/2)$ [15]. However, note that, in the following linear analysis for small force F where we require only the leading order term $O(F)$ [or $O(\Delta e)$], the whole analysis goes through even for a general functional form of Φ. We expand Φ in $O(F)$,

$$\Phi(\Delta e_i) \simeq 1 + \left[\frac{d\Phi}{d(\Delta e)}\right]_{\Delta e=0}\Delta e_i = 1 + \frac{1}{2}\Delta m_{i\to j}(\vec{F}.\delta\vec{x}_{ij}). \tag{13}$$

For example, see the biased mass-transfer rates $c^F_{i\to j}$ as in Eqs. (31) and (32). In the above equation, without any loss of generality, we put $2[d\Phi/d\Delta e]_{\Delta e=0} = 1$, which essentially implies a rescaling of the applied force $F \to [2d\Phi/d(\Delta e)]_{\Delta e=0} \times F$.

It is possible that several fractions $\Delta m_{i_n\to j_{n'}}$, where $n = 1,2,\dots,K$ and $n' = 1,2,\dots,K'$, of masses from K number of sites $\{i_n\} \equiv i_1,i_2,\dots i_K$ are transferred, at the same instant of time, to K' number of sites $\{j'_n\} \equiv \{j_1,j_2,\dots j_{K'}\}$. For example, see the modified rates for Model I in Eq. (17), where $K = 1$ and $K' = 2$, and in Eq. (27), where $K = K' = L$. The original rate $c_{\{i_n\}\to\{j_{n'}\}}$ for mass transfer from sites $\{i_n\}$ to $\{j_{n'}\}$ and the corresponding modified biased rate $c^F_{\{i_n\}\to\{j_{n'}\}}$ are related as

$$c^F_{\{i_n\}\to\{j_{n'}\}} = c_{\{i_n\}\to\{j_{n'}\}}\Phi(\Delta e), \tag{14}$$

where the total extra energy cost, due to the biasing, can be written by summing over all individual energy costs corresponding to each and every pair $\langle n,n'\rangle$ of departure site n and destination site n' as

$$\Delta e = \sum_{\langle n,n'\rangle} \Delta m_{i_n\to j_{n'}}\left(\vec{F}.\delta\vec{x}_{i_n j_{n'}}\right). \tag{15}$$

In the next, we use this modified biased rate $c^F_{\{i_n\}\to\{j_{n'}\}}$ [as in Eq. (14)] along with Eqs. (13) and (15) for the three models (I, II, III) to derive a hydrodynamic equation like in Eq. (9) and hence, in turn, we compute the diffusivity $D(\rho)$ and the conductivity $\chi(\rho)$.

IV. MODELS AND RESULTS: SYMMETRIC MASS TRANSFERS

In this section, we define the symmetric versions of the models, first in the absence of any biasing force, where masses are transferred symmetrically, without any preferential direction, to the nearest neighbors. Consequently, there is no net mass current in the systems. However, it is important to note that, even in that case, detailed balance condition is still not satisfied. In fact, it would be quite instructive to explicitly show that, for generic values of parameters in the models, Kolmogorov criterion and therefore detailed balance is strongly violated, in the sense that, for a transition (say, forward) from one configuration to another while mass being transferred from a site to its neighbor, the corresponding reverse path of transition may not exist.

Therefore, even in the absence of any biasing force, the system eventually reaches a steady state, which is inherently far from equilibrium, and cannot be described by the equilibrium Boltzmann-Gibbs distribution. To calculate conductivity in such a nonequilibrium steady state, we need to apply a biasing (constant, for simplicity) force field, which would essentially modify the original mass-transfer rates in the systems, inducing a mass current, and then we calculate the current in the limit of biasing force being small.

A. Model I

This particular class of models has been introduced to study mass transport processes accounting for stickiness of masses while fragmenting and diffusing [36]. These processes are variants of various previously studied mass transport processes, such as random average processes (RAP), etc. [24,27,31].

1. Random sequential update

In Model I with random sequential update (RSU), three sites are updated simultaneously where two random fractions of the chipped-off mass from site i are shared randomly with the nearest neighbor sites $i-1$ and $i+1$. The stochastic time evolution of mass $m_i(t)$ at time t after an infinitesimal time dt can be written as

$$m_i(t+dt) = \begin{cases} \lambda m_i(t) & \text{prob. } dt \\ m_i(t) + \tilde{\lambda} r_{i-1} m_{i-1}(t) & \text{prob.} dt \\ m_i(t) + \tilde{\lambda} \tilde{r}_{i+1} m_{i+1}(t) & \text{prob. } dt \\ m_i(t) & \text{prob. } (1-3dt) \end{cases}, \tag{16}$$

where $r_j \in (0,1)$s are independent and identically distributed (i.i.d.) random variables, having a probability density $\phi(r)$ and $\tilde{\lambda} = 1-\lambda$ and $\tilde{r}_{i+1} = 1 - r_{i+1}$. Throughout the paper, we denote the first and the second moments of $\phi(r)$ as

$$\theta_1 = \int_0^1 r\phi(r)dr; \quad \theta_2 = \int_0^1 r^2\phi(r)dr,$$

respectively. Note that if the probability density $\phi(r)$ is not symmetric around $r = 1/2$, it can be shown that, in the hydrodynamic equation for density field, drift dominates diffusion unless the asymmetry is small and comparable to the diffusive contribution. In that case, the analysis would lead to hyperbolic hydrodynamic equations for density field (hydrodynamics of such systems are discussed in Sec. VI). Here we consider the density function $\phi(r)$, which is symmetric around $r = 1/2$, i.e., $\phi(r) = \phi(1-r)$, thus $\theta_1 = 1/2$ is taken throughout; but the probability density $\phi(r)$ is otherwise arbitrary.

Breakdown of Kolmogorov criterion. In this model, with random sequential update, at any instant of time, mass is chipped off from a single departure site and then it arrives at its two nearest-neighbor destination sites. Clearly, the reverse path, where mass would have been simultaneously chipped from two departure sites $i-1$ and $i+1$ and would have arrived at a single destination site i, is not allowed by the actual dynamics as given in Eq. (16). Therefore, Kolmogorov criterion is violated and consequently there is no detailed balance even when there is as such no external biasing force.

Dynamics when $F \neq 0$. Let us now bias the system by applying a small constant biasing force field $\vec{F} = F\hat{x}$, say, along the clockwise direction, which affects the mass transfer rates according to Eq. (14). Since, at every instant of time, two fractions of the chipped-off mass from site i could be simultaneously transferred, to the two neighboring sites $i \pm 1$, the modified biased rate in this case is written as

$$c^F_{i\to\{i+1,i-1\}} = c_{i\to\{i+1,i-1\}}\left[1 + \frac{\Delta e_i}{2}\right], \tag{17}$$

where $c_{i\to\{i+1,i-1\}} = 1$ and $\Delta e_i = Fa(\Delta m_{i\to i+1} - \Delta m_{i\to i-1})$ with $\Delta m_{i\to i+1} = \tilde{\lambda} r_i m_i$ and $\Delta m_{i\to i-1} = \tilde{\lambda}(1-r_i)m_i$. For notational simplicity, we denote the biased rate as $c^F_{i\to\{i+1,i-1\}} \equiv c^F_i$, which can be explicitly written as $c^F_i = 1 + \tilde{\lambda}(2r_i - 1)m_i Fa/2$, with $\tilde{\lambda} = 1-\lambda$. We now write the modified dynamics:

$$m_i(t+dt) = \begin{cases} \lambda m_i(t), & \text{prob.} c^F_i dt \\ m_i(t) + \tilde{\lambda} r_{i-1} m_{i-1}(t), & \text{prob.} c^F_{i-1} dt \\ m_i(t) + \tilde{\lambda}\tilde{r}_{i+1} m_{i+1}(t), & \text{prob.} c^F_{i+1} dt \\ m_i(t), \quad \text{prob.}\left[1 - \left(c^F_i + c^F_{i-1} + c^F_{i+1}\right)dt\right] & \end{cases}. \tag{18}$$

Consequently, the time evolution of the first moment of mass $m_i(t)$ in the infinitesimal time dt can be written as

$$\begin{aligned}\langle m_i(t+dt)\rangle = &\left\langle \lambda m_i(t) c^F_i\right\rangle dt \\ &+ \left\langle [m_i(t) + \tilde{\lambda} r_{i-1} m_{i-1}(t)] c^F_{i-1}\right\rangle dt \\ &+ \left\langle [m_i(t) + \tilde{\lambda}\tilde{r}_{i+1} m_{i+1}(t)] c^F_{i+1}\right\rangle dt \\ &+ \left\langle m_i(t)\left[1 - \left(c^F_i + c^F_{i-1} + c^F_{i+1}\right)dt\right]\right\rangle.\end{aligned}$$

After simplifying the above expression, the time evolution of average mass, or mass density, $\langle m_i\rangle \equiv \rho_i$ at site i, can be rewritten as

$$\begin{aligned}\frac{d\rho_i}{dt} &= \tilde{\lambda}\left\langle r_{i-1} m_{i-1} c^F_{i-1} + (1-r_{i+1}) m_{i+1} c^F_{i+1} - m_i c^F_i\right\rangle \\ &= \frac{\tilde{\lambda}}{2}(\rho_{i-1} + \rho_{i+1} - 2\rho_i) \\ &\quad + \frac{\tilde{\lambda}^2}{2}(2\theta_2 - 1/2)\left[\left\langle m^2_{i-1}\right\rangle Fa - \left\langle m^2_{i+1}\right\rangle Fa\right].\end{aligned} \tag{19}$$

Note that the time evolution of the first moment of local mass, i.e., the density $\rho_i = \langle m_i\rangle$, depends on the second moments $\langle m^2_{i\pm 1}\rangle$ of neighboring masses, and so on. Thus, the hierarchy between the local density and the local fluctuation does not close.

Hydrodynamics. However, we are interested in the hydrodynamic description of the density field at large space and time scales, called diffusive scaling limit as described below. Importantly, on the large spatiotemporal scales, local observables are expected to be slowly varying functions of space and time. Therefore, we could safely assume that a local steady state is achieved throughout the system such that average of any local observable $g(m_i)$ could be replaced by its exact local steady-state average $\langle g(m_i)\rangle_{\mathrm{st}}$, which in that case would be a function of the local density ρ_i only. In other words, we assume $\langle g(m_i)\rangle \approx \langle g(m_i)\rangle_{\mathrm{st}}$. Thus, for the average of the quantity $g(m_i) = m_i^2$, i.e., the second moment of local mass, we have replaced the average by the its local steady-state average,

$$\left\langle m_i^2\right\rangle \approx \left\langle m_i^2\right\rangle_{\mathrm{st}} = \frac{1}{1 - 2\tilde{\lambda}\theta_2}\rho_i^2. \tag{20}$$

The above steady-state expression of the second moment has been exactly calculated before in Ref. [14]. Now substituting Eq. (20) in Eq. (19) and then taking the diffusive scaling limit of Eq. (19), $i \to x = i/L$, $t \to \tau = t/L^2$, and $a \to 1/L$, we obtain the hydrodynamic equation for the density field, $\partial_\tau \rho(x,\tau) + \partial_x J = 0$, where current $J(\rho(x,\tau))$ is given by

$$J = \frac{\tilde{\lambda}^2}{2}\frac{4\theta_2 - 1}{1 - 2\tilde{\lambda}\theta_2}\rho^2 F - \frac{\tilde{\lambda}}{2}\frac{\partial \rho}{\partial x}. \tag{21}$$

In the above equation, we break the current $J = J_d + J_D$ into two parts, drift current $J_d = [\tilde{\lambda}^2(4\theta_2 - 1)/2(1 - 2\tilde{\lambda}\theta_2)]\rho^2 F$ and diffusive current $J_D = -(\tilde{\lambda}/2)(\partial\rho/\partial x)$, to identify the conductivity and the diffusion coefficient, respectively, as

$$\chi(\rho) = \frac{\tilde{\lambda}^2}{2}\frac{(4\theta_2 - 1)}{(1 - 2\tilde{\lambda}\theta_2)}\rho^2, \quad D(\rho) = \frac{\tilde{\lambda}}{2}. \tag{22}$$

Now the scaled variance $\sigma^2(\rho)$ of subsystem mass [as defined in Eq. (2)] can be calculated by summing over the microscopic correlation function $c(n) = \langle m_i m_{i+n}\rangle - \rho^2$,

$$\sigma^2(\rho) = \sum_{n=-\infty}^{\infty} c(n) = \frac{\tilde{\lambda}(4\theta_2 - 1)}{(1 - 2\tilde{\lambda}\theta_2)}\rho^2 \equiv \frac{\rho^2}{\eta}, \tag{23}$$

where $c(n)$ has been exactly calculated in Ref. [14],

$$\begin{aligned} c(n) &= \frac{2\tilde{\lambda}\theta_2}{(1 - 2\tilde{\lambda}\theta_2)}\rho^2 && \text{for } n = 0 \\ &= -\frac{\tilde{\lambda}}{2}\frac{(1 - 2\theta_2)}{(1 - 2\tilde{\lambda}\theta_2)}\rho^2 && \text{for } n = 1 \\ &= 0 && \text{for } n \geqslant 2, \end{aligned}$$

and $\eta = (1 - 2\tilde{\lambda}\theta_2)/\tilde{\lambda}(4\theta_2 - 1)$. Using Eqs. (22) and (23), one can readily verify that the ER as in Eq. (1) is indeed satisfied. We emphasize that the nearest-neighbor spatial correlations here (also in the other models discussed later) are actually finite and our hydrodynamic analysis takes into account the effects of the finite microscopic spatial correlations.

2. Parallel update

In Model I with parallel update (PU), fractions of masses to be transferred to the two nearest neighbors are the same as in the case of random sequential update. However, at a discreet time t, the mass variables at all sites are updated simultaneously according to the following rule:

$$m_i(t+1) = \lambda m_i(t) + \tilde{\lambda} r_{i-1} m_{i-1}(t) + \tilde{\lambda}\tilde{r}_{i+1} m_{i+1}(t), \tag{24}$$

where $\tilde{\lambda} = 1 - \lambda$, $\tilde{r}_i = 1 - r_i$, and $r_i \in (0,1)$ is a symmetrically distributed random variable, having a probability density $\phi(r_i)$. The time evolution equation in the configuration space $\{m_i\} \equiv \{m_1, m_2, \ldots, m_L\}$ can be written as

$$\mathcal{P}[\{m_i\}, t+1] = \left[\prod_j \int dm_j\right] \times \Gamma[\{m_j\} \to \{m_i\}]\mathcal{P}[\{m_j\}, t], \tag{25}$$

where $\mathcal{P}[\{m_i\}, t]$ is the probability of a configuration $\{m_i\}$ at time t and

$$\Gamma[\{m_j\} \to \{m_i\}] = \prod_i \phi(r_i)$$

is the transition probability, per unit time, from a configuration $\{m_j\}$ to another configuration $\{m_i\}$.

Breakdown of Kolmogorov criterion. In the case of parallel update, the breakdown of Kolmogorov criterion, though not quite obvious, can be straightforwardly shown for generic parameter values $\lambda \neq 0$. For example, consider a configuration having two sites $i-1$ and i, with masses m_{i-1} finite and m_i infinitesimal (say, $m_i = 0$, just for the sake of argument), respectively. Then, a chunk of mass is transferred from site $(i-1)$ to site i so that $m_{i-1} \to m'_{i-1} > 0$ and $m_i \to m'_i > 0$. In the next time step, since at least a λ fraction of mass m'_i must be retained at site i, the reverse path where the whole mass m'_i would have been transferred back to $i-1$ from site i is not possible, implying breakdown of Kolmogorov criterion and thus violation of detailed balance. This simple, though not rigorous, argument can be readily extended to any configuration with sufficiently large difference of masses in any two neighboring sites so that there cannot be a reverse path corresponding to a particular possible path of mass transfer. Note that, in this argument, we consider only the unbiased process ($F = 0$). Let us consider transitions $\{m_i\} \to \{m'_i\}$ and $\{m'_i\} \to \{m''_i\}$ at two consecutive time steps. In the second transition, one must have $m''_i > \lambda m'_i$, i.e., the mass retained at site i must be at least $\lambda m'_i$. Now, if the amount of mass $\lambda m'_i$ is greater than mass m_i, the value of mass at site i at the initial step, clearly the path cannot be reversed. Therefore, the condition for which a process cannot be reversed is simply $\lambda m'_i > m_i$, which, after using Eq. (24) $m'_i = \lambda m_i + \tilde{\lambda} r_{i-1} m_{i-1} + \tilde{\lambda}\tilde{r}_{i+1} m_{i+1}$, leads to the condition

$$r_{i-1} m_{i-1} + \tilde{r}_{i+1} m_{i+1} - \frac{1+\lambda}{\lambda} m_i > 0. \tag{26}$$

Therefore, for $\lambda \neq 0$, indeed there are configurations (a finite set in the configuration space) that satisfy the above inequality. This implies breakdown of Kolmogorov criterion and that the steady state is far from equilibrium even in the absence of any biasing force ($F = 0$). Analysis for $\lambda = 0$ requires more effort and is omitted here.

Dynamics when $F \neq 0$. Let us now consider the process in the presence of an externally applied biasing force, $F \neq 0$. Once the random fractions $[\tilde{\lambda} r_i m_i$ and $\tilde{\lambda}(1 - r_i) m_i]$ of mass m_i at a site i are chosen at time t they are transferred, at the next discrete time step, to the nearest neighbor sites $i+1$ and $i-1$, respectively, with probability 1 and this is done simultaneously for all sites. That is, in this case, the mass transfer rate, or the transition probability per unit time, can be written as $c_{\{i_n\}\to\{j_{n'}\}} = 1$, which we modify, in the presence of biasing force, as $c^F_{\{i_n\}\to\{j_{n'}\}} = c_{\{i_n\}\to\{j_{n'}\}} \prod_i \exp(\Delta e_i/2)$, according to Eq. (14). Here we put $\Phi(\Delta e) = \exp(\Delta e/2)$ with $\Delta e = \sum_i \Delta e_i$ and $\Delta e_i = Fa(\Delta m_{i\to i+1} - \Delta m_{i\to i-1}) = \tilde{\lambda}(2r_i - 1)m_i Fa$. The time evolution Eq. (25) can now be written by replacing the original transition probability Γ with the modified one Γ^F,

$$\Gamma^F[\{m_j\} \to \{m_i\}] = \prod_j \left[\frac{\phi(r_j)e^{\Delta e_j/2}}{\gamma(m_j, F)}\right], \tag{27}$$

where $\gamma(m_j, F)$ is a normalization constant, ensuring that the transition probability $\Gamma^F[.]$ is suitably assigned from a normalized probability density function where $(\prod_j \int dr_j)\Gamma^F = 1$. As the probability density $\phi(r)$ is considered to be symmetric about $r = 1/2$, we have the following expansion in powers of F,

$$\begin{aligned} \gamma(m_i, F) &= \int_0^1 \phi(r_i) e^{\tilde{\lambda}(2r_i - 1)m_i Fa/2} dr_i \\ &= 1 + \frac{(\tilde{\lambda} m_i)^2 \theta_2}{8}(Fa)^2 + \ldots, \end{aligned}$$

implying that the leading order term is quadratic $O(F^2)$ in the biasing force F and, therefore, to linear order of F, we can take $\gamma(m_i,F) \approx 1$ in the following analysis (see also Sec. IV B 2).

The expression for the average of mass m_i at site i can now be written as

$$\langle m_i(t+1)\rangle = \left[\prod_j \int dm_j\right] m_i \mathcal{P}[\{m_j\},t+1]$$
$$= \langle[\lambda m_i(t) + \tilde{\lambda} r_{i-1} m_{i-1}(t) + \tilde{\lambda}(1-r_{i+1})m_{i+1}(t)]\rangle,$$

where the angular brackets $\langle . \rangle$ denote average over both random numbers $\{r_j\}$ and the mass variables $\{m_j\}$. Explicitly writing the terms, we get

$$\langle m_i(t+1)\rangle = \left\langle \lambda m_i(t) \int \frac{\phi(r) e^{\tilde{\lambda}(2r-1)m_i Fa/2}}{\gamma(m_i,F)} dr \right\rangle + \left\langle \tilde{\lambda} m_{i-1}(t) \int r \frac{\phi(r) e^{\tilde{\lambda}(2r-1)m_{i-1} Fa/2}}{\gamma(m_{i-1},F)} dr \right\rangle + \left\langle \tilde{\lambda} m_{i+1}(t) \int (1-r) \frac{\phi(r) e^{\tilde{\lambda}(2r-1)m_{i+1} Fa/2}}{\gamma(m_{i+1},F)} dr \right\rangle,$$

which, in leading order in F, leads to

$$\rho_i(t+1) - \rho_i(t) = \frac{\tilde{\lambda}}{2}(\rho_{i-1} + \rho_{i+1} - 2\rho_i) + \frac{\tilde{\lambda}^2}{2}(2\theta_2 - 1/2) \times \left[\langle m_{i-1}^2\rangle Fa - \langle m_{i+1}^2\rangle Fa\right].$$

Hydrodynamics. Now taking the diffusive scaling limit $i \to x = i/L, t \to \tau = t/L^2$, and $a \to 1/L$ and substituting $\langle m_i^2\rangle$ by the following expression of second moment [14], within the assumption of local steady state,

$$\langle m_i^2\rangle_{\rm st} = \frac{1}{\epsilon + (1-\epsilon)\sqrt{\frac{\kappa-1}{\kappa+1}}}\rho_i^2,$$

we obtain the hydrodynamic equation for the density field, $\partial_\tau \rho(x,\tau) + \partial_x(J_d + J_D) = 0$, where the drift current $J_d(\rho(x,\tau))$ and the diffusive current $J_D(\rho(x,\tau))$ are given by

$$J_d = \frac{\tilde{\lambda}^2}{2} \frac{4\theta_2 - 1}{\epsilon + (1-\epsilon)\sqrt{\frac{\kappa-1}{\kappa+1}}}\rho^2 F, \quad J_D = -\frac{\tilde{\lambda}}{2}\frac{\partial \rho}{\partial x}, \tag{28}$$

respectively. Then, the conductivity $\chi(\rho)$ and the diffusion coefficient $D(\rho)$ can be expressed as

$$\chi(\rho) = \frac{\tilde{\lambda}^2}{2} \frac{4\theta_2 - 1}{\epsilon + (1-\epsilon)\sqrt{\frac{\kappa-1}{\kappa+1}}}\rho^2, \quad D(\rho) = \frac{\tilde{\lambda}}{2}, \tag{29}$$

where $\epsilon = 2 - 4\theta_2$ and $\kappa = (1+\lambda)/(1-\lambda)$. The Einstein relation Eq. (1) can be immediately verified using the expression of the scaled variance,

$$\sigma^2(\rho) = \frac{\tilde{\lambda}(4\theta_2 - 1)}{\epsilon + (1-\epsilon)\sqrt{\frac{\kappa-1}{\kappa+1}}}\rho^2,$$

which was exactly calculated earlier in Ref. [14]. We mention here that the microscopic spatial correlations, as in the case of Model I (RSU), are also finite and have been accounted exactly in the above analysis.

B. Model II

The class of models studied in this section is a generalized version of previously known Hammersley process [23] and a variant of random average processes [24]. These models were studied in the past to understand force fluctuations in granular beads [28,29] and dynamics of driven interacting particles on a ring [31,35], etc.

1. Random sequential update

In Model II with random sequential update, two nearest-neighbor sites are updated in an infinitesimal time dt: A random fraction of mass at site i is chipped off and transferred either to site $i-1$ or to site $i+1$, each with probability $(1/2)dt$, i.e., the mass transfer rates $c_{i\to i-1} = 1/2$ and $c_{i\to i+1} = 1/2$.

Breakdown of Kolmogorov criterion. First let us show that the process in the absence of any external bias ($F = 0$) violates Kolmogorov criterion and therefore also detailed balance. Let us consider transitions $\{m_i\} \to \{m_i'\}$ and $\{m_i'\} \to \{m_i''\}$ at two consecutive time steps, where

$$m_i' = (1-\tilde{\lambda} r_i)m_i; m_{i+1}' = m_{i+1} + \tilde{\lambda} r_i m_i,$$
$$m_i'' = (1-\tilde{\lambda} r_i)m_i'; m_{i+1}'' = m_{i+1}' + \tilde{\lambda} r_i m_i'.$$

Now the conditions, $m_i'' = m_i$ and $m_{i+1}'' = m_{i+1}$, for the existence of a reverse path leads to an equality, $r_{i+1}' = r_i m_i/(m_{i+1} + \tilde{\lambda} r_i m_i)$. Or equivalently, an inequality $m_{i+1} \geqslant \lambda r_i m_i$, as $r_{i+1}' \leqslant 1$, must be satisfied for the existence of a reverse path. Said differently, the condition for which a reverse path will not exist can be written as the following inequality on the ratio of neighboring masses,

$$\frac{m_i}{m_{i+1}} > \frac{1}{\lambda r_i}.$$

The above condition is satisfied by a finite set in the configuration space and will then imply the steady state to be far from equilibrium even in the absence of any external biasing force ($F = 0$).

Dynamics when $F \neq 0$. However, in the presence of a biasing force $F \neq 0$, the dynamics is modified as

$$m_i(t+dt) = \begin{cases} \lambda m_i(t) + \tilde{\lambda}(1-r_i)m_i(t) & \text{prob. } dt \\ m_i(t) + \tilde{\lambda} r_{i+1} m_{i+1}(t) & \text{prob. } c_{i+1\to i}^F dt \\ m_i(t) + \tilde{\lambda} r_{i-1} m_{i-1}(t) & \text{prob. } c_{i-1\to i}^F dt \\ m_i(t) & \text{prob. } [1 - (1 + c_{i+1\to i}^F + c_{i-1\to i}^F)dt] \end{cases} \tag{30}$$

where $\tilde{\lambda} = 1 - \lambda$ and the modified mass transfer rates, $c^F_{i\to i\pm 1} = \exp(\pm \Delta m_{i\to i\pm 1} F a/2)$ with transported mass $\Delta m_{i\to i\pm 1} = \tilde{\lambda} r_i m_i(t)$, can be written, in leading order of F, as

$$c_{i-1\to i} = \frac{1}{2} + \frac{\tilde{\lambda}}{4} r_{i-1} m_{i-1} F a, \quad (31)$$

$$c_{i+1\to i} = \frac{1}{2} - \frac{\tilde{\lambda}}{4} r_{i+1} m_{i+1} F a. \quad (32)$$

Clearly, $F = 0$ reproduces the original unbiased dynamics. Now, the time evolution of average mass or density at site i is given by

$$\frac{d\langle m_i\rangle}{dt} = \tilde{\lambda}\langle\big[r_{i-1} m_{i-1} c^F_{i-1\to i} + r_{i+1} m_{i+1} c^F_{i+1\to i} - r_i m_i c^F_i\big]\rangle,$$

which, in leading order of F, can be written as

$$\frac{d\rho_i}{dt} = \frac{\tilde{\lambda}}{2}\theta_1(\rho_{i-1} + \rho_{i+1} - 2\rho_i) + \frac{\tilde{\lambda}^2}{4}\theta_2\big[\langle m^2_{i-1}\rangle F a - \langle m^2_{i+1}\rangle F a\big].$$

Hydrodynamics. Taking the diffusive scaling limit $i \to x = i/L$, $t \to \tau = t/L^2$, and $a \to 1/L$ and using the local steady-state expression for the second moment,

$$\langle m_i^2\rangle = \frac{\theta_1}{\theta_1 - \tilde{\lambda}\theta_2}\rho_i^2,$$

we obtain the hydrodynamic equation governing the density field, $\partial_\tau \rho(x,\tau) + \partial_x(J_d + J_D) = 0$, where the drift current $J_d(\rho(x,\tau))$ and the diffusive current $J_D(\rho(x,\tau))$ are given by

$$J_d = \frac{\tilde{\lambda}^2}{2}\frac{\theta_1\theta_2}{\theta_1 - \tilde{\lambda}\theta_2}\rho^2 F, \quad J_D = -\frac{\tilde{\lambda}}{2}\theta_1\frac{\partial\rho}{\partial x}. \quad (33)$$

Therefore, the conductivity $\chi(\rho)$ and the diffusion coefficient $D(\rho)$ are given by

$$\chi(\rho) = \frac{\tilde{\lambda}^2}{2}\frac{\theta_1\theta_2}{\theta_1 - \tilde{\lambda}\theta_2}\rho^2, \quad D(\rho) = \frac{\tilde{\lambda}}{2}\theta_1. \quad (34)$$

The Einstein relation Eq. (1) can now be verified by using the expression of scaled variance,

$$\sigma^2(\rho) = \frac{\tilde{\lambda}\theta_2}{\theta_1 - \tilde{\lambda}\theta_2}\rho^2,$$

which was obtained earlier in Ref. [14].

2. Parallel update

In Model II with parallel update, at each discrete time step, masses at all sites are updated simultaneously according to the following rule:

$$m_i(t+1) = (1 - \tilde{\lambda} r_i)m_i(t) + \tilde{\lambda} r_{i+1} m_{i+1}(t) + \tilde{\lambda}[s_{i-1} r_{i-1} m_{i-1}(t) - s_{i+1} r_{i+1} m_{i+1}(t)], \quad (35)$$

where $\tilde{\lambda} = 1 - \lambda$. Here we have introduced a set of discrete i.i.d. random variables $\{s_i\}$: When the chipped-off fraction of mass moves to the right, $s_i = 1$, and otherwise, $s_i = 0$. As each of the values $s_i = 0$ and $s_i = 1$ occurs with probability 1/2, we have $\langle s_i^n\rangle = 1/2$ for $n > 0$.

Breakdown of Kolmogorov criterion. In this model, the breakdown of Kolmogorov criterion, for generic parameter values $\lambda \neq 0$, can be shown along the lines of arguments as given in the case of parallel update for Model I in Sec. IV A 2. As before, let us consider transitions $\{m_i\} \to \{m'_i\}$ and $\{m'_i\} \to \{m''_i\}$ at two consecutive time steps. Provided that the mass $(1 - \tilde{\lambda} r'_i)m'_i$, the least amount of mass retained at site i after second transition, is greater than the initial mass m_i, there cannot be a reverse path. Using dynamical rule in Eq. (35), it can be shown that the condition of inequality $(1 - \tilde{\lambda} r'_i)m'_i > m_i$ leads to a condition on the initial masses,

$$s_{i-1} r_{i-1} m_{i-1} + (1 - s_{i+1}) r_{i+1} m_{i+1} - \left(\frac{1}{\lambda} + r_i\right) m_i > 0.$$

The above condition is satisfied for a finite set of configurations in the configuration space and will then imply violation of Kolmogorov criterion, and thus also detailed balance, and that the steady state is far from equilibrium even in the absence of any biasing force ($F = 0$).

Dynamics when $F \neq 0$. In the presence of a biasing force $F \neq 0$, the transition probability $\Gamma[\{m_j\} \to \{m_k\}]$ from a configuration $\{m_j\}$ to another configuration $\{m_k\}$ is modified as

$$\Gamma^F[\{m_j\} \to \{m_k\}] = \prod_i \left[\frac{1}{\gamma(m_i, F)}\phi(r_i) e^{\Delta e_i/2}\right], \quad (36)$$

where $\Delta e_i = [s_i - (1 - s_i)]\tilde{\lambda} r_i m_i F a$ and the normalization factor

$$\gamma(m_i, F) = \Sigma_{s_i} P(s_i) \int_0^1 \phi(r_i) e^{(2s_i - 1)\tilde{\lambda} r_i m_i F a/2} dr_i = 1 + \frac{(\tilde{\lambda} m)^2\theta_2}{8}(F a)^2 + \cdots \approx 1,$$

to the linear order of F. The time evolution of the average mass or density at site i is given by

$$\langle m_i(t+1)\rangle = \langle(1 - \tilde{\lambda} r_i)m_i(t)\rangle + \langle\tilde{\lambda} s_{i-1} r_{i-1} m_{i-1}(t)\rangle + \langle\tilde{\lambda}(1 - s_{i+1}) r_{i+1} m_{i+1}(t)\rangle, \quad (37)$$

where the above angular brackets denote averaging over all three random variables, $\{r_i\}$, $\{s_i\}$, and $\{m_i\}$. Equivalently, we can write

$$\begin{aligned}\langle m_i(t+1)\rangle &= \left\langle(1 - \tilde{\lambda} r_i)m_i(t)\int\frac{\phi(r_i)e^{\Delta e_i/2}}{\gamma(m_i,F)}dr_i\right\rangle \\ &+ \left\langle\tilde{\lambda} s_{i-1} r_{i-1} m_{i-1}(t)\int\frac{\phi(r_{i-1})e^{\Delta e_{i-1}/2}}{\gamma(m_{i-1},F)}dr_{i-1}\right\rangle \\ &+ \left\langle\tilde{\lambda}(1 - s_{i+1}) r_{i+1} m_{i+1}(t)\int\frac{\phi(r_{i+1})e^{\Delta e_{i+1}/2}}{\gamma(m_{i+1},F)}dr_{i+1}\right\rangle, \quad (38)\end{aligned}$$

where, in the second step, we have explicitly written the averaging over the i.i.d. random variables $\{r_i\}$. Next, doing the averaging over the i.i.d. random variables $\{s_i\}$, we obtain, in linear order of F, the time evolution equation for density $\rho_i = \langle m_i\rangle$ at site i,

$$\rho_i(t+1) - \rho_i(t) = \frac{\tilde{\lambda}}{2}\theta_1(\rho_{i-1} + \rho_{i+1} - 2\rho_i) + \frac{\tilde{\lambda}^2}{4}\theta_2\big[\langle m^2_{i-1}\rangle F a - \langle m^2_{i+1}\rangle F a\big]. \quad (39)$$

Hydrodynamics. Now rescaling the space and time by $i \to x = i/L, t \to \tau = t/L^2$, and $a \to 1/L$, and using the expression for second moment of m_i in the local steady state [14],

$$\langle m_i^2 \rangle = \frac{\sqrt{\alpha}}{1-(1-\lambda)\epsilon}\rho_i^2,$$

we obtain the hydrodynamic equation of density field, $\partial_\tau \rho(x,\tau) + \partial_x(J_d + J_D) = 0$, where the drift $J_d(\rho(x,\tau))$ and diffusive currents $J_D(\rho(x,\tau))$, respectively, can be written as

$$J_D = -\frac{\tilde{\lambda}}{2}\theta_1\frac{\partial \rho}{\partial x}, \quad J_d = \frac{\tilde{\lambda}^2}{2}\theta_2\frac{\sqrt{\alpha}}{1-(1-\lambda)\epsilon}\rho^2 F.$$

The above expressions of currents immediately gives the diffusion coefficients and the conductivity as a function of density,

$$\chi(\rho) = \frac{\tilde{\lambda}^2}{2}\theta_2\frac{\sqrt{\alpha}}{1-(1-\lambda)\epsilon}\rho^2, \quad D(\rho) = \frac{\tilde{\lambda}}{2}\theta_1, \tag{40}$$

respectively, with $\epsilon = \frac{\theta_2}{\theta_1}, \alpha = (1+\lambda)/2$. Now, by using the exact expression of scaled variance [14],

$$\sigma^2(\rho) = \frac{\tilde{\lambda}\sqrt{\alpha}\epsilon}{1-\tilde{\lambda}\epsilon},$$

one can verify that the ER as in Eq. (1) is indeed satisfied. Note that, as in the case of Model I, the microscopic spatial correlations are also finite here and have been taken into account in deriving hydrodynamics.

C. Model III

This class of models have been studied intensively in the past to understand distribution of wealth in a population [34,38,39]. In this model, each site keeps a λ fraction (usually called "saving propensity" in the literature) of its own mass, and the remaining mass of two neighboring sites are mixed and are distributed randomly among themselves. Here we study only the random sequential update dynamics, which can be written in an infinitesimal time dt as follows:

$$m_i(t+dt) = \begin{cases} \lambda m_i(t) + \tilde{\lambda} r_i[m_i(t)+m_{i+1}(t)] & \text{prob. } c_{i+1\to i}dt \\ \lambda m_i(t) + \tilde{\lambda}\tilde{r}_{i-1}[m_i(t)+m_{i-1}(t)] & \text{prob. } c_{i-1\to i}dt \\ m_i(t) & \text{prob. } [1-(c_{i+1\to i}+c_{i-1\to i})dt] \end{cases} \tag{41}$$

where $\tilde{\lambda} = 1-\lambda, \tilde{r}_i = 1 - r_i, m_i(t)$ is mass at site i at time $t, r_i \in (0,1)$ is a i.i.d. random variable having a probability density $\phi(r_i)$ (symmetric around 1/2), and the mass transfer rate $c_{i\to j} = 1$ (here $j = i \pm 1$).

Violation of Kolmogorov criterion. Again, let us consider transitions $\{m_i\} \to \{m_i'\}$ and $\{m_i'\} \to \{m_i''\}$ at two consecutive time steps, where, by denoting $\mu_{i,i+1} = m_i + m_{i+1}$,

$$m_i' = \lambda m_i + \tilde{\lambda} r_i \mu_{i,i+1}; m_{i+1}' = \lambda m_{i+1} + \tilde{\lambda}\tilde{r}_i\mu_{i,i+1},$$
$$m_i'' = \lambda m_i' + \tilde{\lambda} r_i \mu_{i,i+1}; m_{i+1}'' = \lambda m_{i+1}' + \tilde{\lambda}\tilde{r}_i\mu_{i,i+1}.$$

The condition, $m_i'' = m_i$ and $m_{i+1}'' = m_{i+1}$, of having a reverse path can be written as an equality $r_i' = (1+\lambda)m_i/\mu_{i,i+1} - \lambda r_i$, or alternatively, as an inequality (as $r_i' \leqslant 1$) on the ratio of neighboring masses $m_i/m_{i+1} \leqslant (1+\lambda r_i)/\lambda\tilde{r}_i$. Said differently, for $m_i/m_{i+1} > (1+\lambda r_i)/\lambda\tilde{r}_i$, Kolmogorov criterion and detailed balance are violated, and thus the steady state is far away from equilibrium even in the absence of any biasing force ($F = 0$).

Dynamics when $F \neq 0$. In the presence of a biasing force, the mass transfer rates are modified as

$$c_{i+1\to i}^F = e^{-\Delta m_{i+1\to i}Fa/2} \approx 1 - \tfrac{1}{2}\Delta m_{i+1\to i}Fa,$$
$$c_{i-1\to i}^F = e^{\Delta m_{i-1\to i}Fa/2} \approx 1 + \tfrac{1}{2}\Delta m_{i-1\to i}Fa,$$

where

$$\Delta m_{i+1\to i} = \tilde{\lambda} r_i m_{i+1}(t) - \tilde{\lambda}(1-r_i)m_i(t) \tag{42}$$

and

$$\Delta m_{i-1\to i} = \tilde{\lambda}(1-r_{i-1})m_{i-1} - \tilde{\lambda} r_{i-1}m_i(t). \tag{43}$$

The time evolution of the first moment of local mass or density $\rho_i = \langle m_i \rangle$ at site i can be written as

$$\begin{aligned}\langle m_i(t+dt)\rangle &= \langle[\lambda m_i(t) + \tilde{\lambda} r_i(m_i(t)+m_{i+1}(t))]c_{i+1\to i}^F dt\rangle \\ &\quad + \langle[\lambda m_i(t) + \tilde{\lambda}(1-r_{i-1})(m_i(t)+m_{i-1}(t))]c_{i-1\to i}^F dt\rangle \\ &\quad + \langle m_i(t)[1-(c_{i-1\to i}^F + c_{i+1\to i}^F)dt]\rangle.\end{aligned}$$

After substituting $\langle m_i \rangle = \rho_i$ and some simplifications, we have the following evolution for density ρ_i,

$$\frac{d\rho_i}{dt} = \frac{\tilde{\lambda}}{2}[\rho_{i+1} - 2\rho_i + \rho_{i-1}] - \frac{1}{2}\langle(\Delta m_{i+1\to i}^2 Fa - \Delta m_{i-1\to i}^2 Fa)\rangle,$$

which leads to

$$\begin{aligned}\frac{d\rho_i}{dt} &= \frac{\tilde{\lambda}}{2}[\rho_{i+1} - 2\rho_i + \rho_{i-1}] \\ &\quad - \frac{\tilde{\lambda}^2}{2}\Big[\theta_2\frac{\lambda + 2\tilde{\lambda}\theta_2}{1-2\tilde{\lambda}\theta_2}(\rho_{i+1}^2 - \rho_{i-1}^2) \\ &\quad - (1-2\theta_2)(\rho_{i+1}^2 - \rho_i^2)\Big]Fa\end{aligned}$$

In the last step, following the assumption of local steady-state, we have used Eqs. (42) and (43) and, subsequently, used the expression of second moment of local mass as well as the expression of nearest-neighbor mass-mass correlation [14],

$$\langle m_i^2 \rangle = \frac{1-\tilde{\lambda}(1-2\theta_2)}{\lambda + \tilde{\lambda}(1-2\theta_2)}\rho_i^2, \quad \langle m_{i-1}m_i\rangle = \rho_i^2.$$

Hydrodynamics. Finally, we take the diffusive limit, by rescaling space and time as $i \to x = i/L, t \to \tau = t/L^2$, and $a \to 1/L$, and obtain the hydrodynamic equation for the density field as $\partial_t \rho(x,\tau) + \partial_x J(\rho(x,\tau)) = 0$, where $J = J_d + J_D$, with

$$J_d(\rho) = \frac{\tilde{\lambda}^2}{2}\frac{4\theta_2 - 1}{1 - 2\tilde{\lambda}\theta_2}\rho^2, \quad J_D(\rho) = -\frac{\tilde{\lambda}}{2}\frac{\partial \rho}{\partial x}.$$

The above functional forms of currents imply that the diffusion coefficient and the conductivity, respectively, have the following expressions:

$$\chi(\rho) = \frac{\tilde{\lambda}^2}{2}\frac{4\theta_2 - 1}{1 - 2\tilde{\lambda}\theta_2}\rho^2, \quad D(\rho) = \frac{\tilde{\lambda}}{2}. \tag{44}$$

The ER as in Eq. (1) can now be verified by using the previously obtained expression of scaled variance [14],

$$\sigma^2(\rho) = \frac{\tilde{\lambda}(4\theta_2 - 1)}{1 - 2\tilde{\lambda}\theta_2}\rho^2.$$

V. DENSITY LARGE DEVIATIONS

The evolution in Eq. (9), in fact, describes the evolution of the average density profile. As mentioned earlier, our microscopic models are, however, stochastic by nature, which gives rise to fluctuations in the density and the associated current fields. According to the macroscopic fluctuation theory (MFT) [15], the fluctuations in these two fields can be introduced by adding a random current field $\zeta(x,\tau)$ to the deterministic one $J(x,\tau)$ as follows. The total current can now be written as $j(x,\tau) = J(x,\tau) + \zeta(x,t)$, where $\zeta(x,\tau)$ is a weak Gaussian multiplicative white noise, whose mean is zero and strength depends on local density through conductivity $\chi(\rho)$,

$$\langle \zeta(x,\tau)\rangle = 0; \langle \zeta(x,\tau)\zeta(x',\tau')\rangle = \frac{1}{L}\chi(\rho)\delta(x - x')\delta(\tau - \tau').$$

Thus, one obtains the following fluctuating-hydrodynamic time-evolution of the density field:

$$\partial_\tau \rho(x,\tau) + \partial_x[-D(\rho)\partial_x \rho(x,\tau) + \chi(\rho)F + \zeta(x,\tau)] = 0. \tag{45}$$

Starting from the stochastic microscopic dynamics, and using the Markov properties of the evolution, one can actually prove the above stochastic hydrodynamic Eq. (45) [15]. Then, using Eq. (45), one can, in principle, find the joint probability of any given time-trajectories of the full density $\rho(x,\tau)$ and current $j(x,\tau)$ profiles, starting from an arbitrary initial condition.

However, here, we are interested in the steady-state probabilities of density large deviations. According to MFT, the probability of an arbitrary density profile $\rho(x)$ in the steady state, which corresponds to Eq. (45) with zero external bias $F = 0$, is given by the following large deviation probability $\mathcal{P}[\rho(x)] \approx e^{-\mathcal{F}[\rho(x)]}$, where the large deviation function $\mathcal{F}[\rho(x)]$ satisfies [15]

$$\int dx \left[\partial_x\left(\frac{\delta\mathcal{F}}{\delta\rho}\right)\chi(\rho)\partial_x\left(\frac{\delta\mathcal{F}}{\delta\rho}\right) - \frac{\delta\mathcal{F}}{\delta\rho}\partial_x J_D(\rho)\right] = 0. \tag{46}$$

After performing a partial integration in the second term, one can readily check that the above equation is satisfied by the LDF $\mathcal{F}[\rho(x)]$, which satisfies the following conditions,

$$\partial_x\left(\frac{\delta\mathcal{F}}{\delta\rho(x)}\right) = \partial_x\{f'[\rho(x)] - f'(\rho_0)\}, \tag{47}$$

$$\frac{1}{f''(\rho)} = \frac{\chi(\rho)}{D(\rho)}. \tag{48}$$

Here, ρ_0 is the average or typical local mass density (which in our case turns out to be the same as the global density since the systems are homogeneous) at which the LDF $\mathcal{F}[\rho]$ has a minimum equal to $\mathcal{F}[\rho(x) = \rho_0] = 0$. Equation (47), together with this minimum condition, gives the following expression of the LDF:

$$\mathcal{F}[\rho(x)] = \int_{-\infty}^{\infty} dx\{f(\rho) - f(\rho_0) - f'(\rho_0)(\rho - \rho_0)\}. \tag{49}$$

Note that the above functional form of the LDF implies the FR as in Eq. (4). Now substituting Eq. (4) in Eq. (48), one immediately obtains the Einstein relation Eq. (1). Moreover, using Eqs. (48) and (4), one can easily see that the LDF in Eq. (49) is exactly the same as in Eq. (5), which was earlier obtained directly from additivity and the FR Eq. (4). Particularly, for the conserved-mass transport processes considered here, one recovers free-energy density $f(\rho)$ and chemical potential $\mu(\rho) = f'(\rho)$, as in Eqs. (6) and (7), by explicitly using the expressions of conductivity $\chi(\rho)$ and diffusion coefficients $D(\rho)$ derived in Secs. IV A, IV B, and IV C.

VI. RESULTS: ASYMMETRIC MASS TRANSFERS

In the asymmetric mass transport processes, masses are transferred preferentially in a particular direction, say, counterclockwise. Consequently, there is, on average, a nonzero mass current and detailed balance is manifestly broken in the system. However, even in the case of such asymmetric mass transfer, we explicitly show below that the bulk-diffusion coefficient $D(\rho)$ and the conductivity $\chi(\rho)$ still satisfy an ER. The conductivity (differential) $\chi(\rho) = [\partial J_d/\partial F]_{F=0}$ here can be defined with respect to a small perturbing biasing force field $\vec{F}$ around the nonzero current-carrying steady state. For simplicity, only the random sequential update rule is considered here; the results can be straightforwardly generalized to the parallel update rules.

To illustrate how one could incorporate asymmetry in transfer of masses, let us now consider a particular model, say, model I where the dynamics is described by Eq. (16) in Sec. IV A. In this case, model I becomes one having asymmetric transfer of masses, provided that the probability density function $\phi(r_i)$ is not symmetric around $r_i = 1/2$. Clearly, the asymmetric mass-transfer gives rise to an inherent bias towards a particular direction. Note that asymmetry can be incorporated in several other ways also, but, for simplicity, we confine our discussions to the cases considered below.

Now, in the above mentioned asymmetric version of model I, the time-evolution of the first moment $\langle m_i(t)\rangle = \rho_i(t)$

of mass at site i is governed by

$$\frac{d\rho_i}{dt} = \tilde{\lambda}\langle r_{i-1}m_{i-1} + (1-r_{i+1})m_{i+1} - m_i\rangle + \frac{\tilde{\lambda}^2}{2}(2\theta_2-\theta_1)a\big[\langle m_{i-1}^2\rangle - \langle m_{i+1}^2\rangle\big]F + \frac{\tilde{\lambda}^2}{2}(2\theta_1-1)a\big[\langle m_{i+1}^2\rangle - \langle m_i^2\rangle\big]F. \tag{50}$$

Let us define strength of asymmetry $\alpha = [1-2\theta_1]$, which in a particular case may depend on system size L through the first moment θ_1 of probability density function $\phi(r_i)$. The parameter α helps us in obtaining concisely the hydrodynamic equation, which can be applicable to both weakly and strongly asymmetric cases, depending on α. We now rescale Eq. (50) by $i \to x = i/L, t \to \tau = t\alpha/L$, and $a \to 1/L$, and using the expression $\langle m_i^2\rangle = \rho_i^2/[\lambda + 2\tilde{\lambda}(\theta_1-\theta_2)]$ [14], we obtain the hydrodynamic equation,

$$\frac{\partial\rho}{\partial\tau} = -\tilde{\lambda}\frac{\partial\rho}{\partial x} + \nu D\frac{\partial^2\rho}{\partial x^2} - \frac{\partial}{\partial x}[\nu\chi(\rho)F], \tag{51}$$

where $\nu = 1/\alpha L$ and

$$\chi(\rho) = \frac{\tilde{\lambda}^2}{2}\frac{1-4(\theta_1-\theta_2)}{\lambda+2\tilde{\lambda}(\theta_1-\theta_2)}\rho^2; \quad D(\rho) = \frac{\tilde{\lambda}}{2}.$$

There is now an additional drift current $\tilde{\lambda}\rho$ appearing in the hydrodynamic equation. However, one can immediately verify that the diffusivity and mobility are indeed connected by the Einstein relation as in Eq. (1). Note that conductivity now depends on the strength of asymmetry α through $\theta_1 = (1-\alpha)/2$.

In the case of weak asymmetry where $\alpha(L) = \text{const.}/L$ is $O(1/L)$, the above rescaling of time $(\tau \sim t/L^2)$ leads to diffusive hydrodynamics with conductivity $\nu\chi(\rho)$ and diffusion coefficient $\nu D(\rho)$ both being finite. Whereas, in the case of strong asymmetry where $\alpha = \text{const.}$ is $O(1)$, the above rescaling of time $(\tau \sim t/L)$ gives hyperbolic hydrodynamics with conductivity $\nu\chi(\rho)$ and diffusion coefficient $\nu D(\rho)$ both being infinitesimally small as $\nu \to 0$ in the hydrodynamic limit. However, the MFT is still expected to describe the density fluctuation in both cases [15] and density field $\rho(x,\tau)$ would then satisfy the following stochastic hydrodynamic equation with a Gaussian multiplicative noise-current $\zeta(x,\tau)$:

$$\frac{\partial\rho}{\partial\tau} = -\partial_x\left[\tilde{\lambda}\rho - \nu D\frac{\partial\rho}{\partial x} + \zeta(x,\tau)\right], \tag{52}$$

where $\langle\zeta(x,\tau)\rangle = 0$ and $\langle\zeta(x,\tau)\zeta(x',\tau')\rangle = [\nu\chi(\rho)/L]\delta(x-x')\delta(\tau-\tau')$. Note that the structure of stochastic hydrodynamics for asymmetric cases remains quite similar to Eq. (45), where J_D is now replaced by $J_D + \tilde{\lambda}\rho$ and D and χ are now replaced by νD and $\nu\chi$, respectively. Consequently, the density large deviation function can be obtained by solving a slightly modified version of Eq. (46),

$$\int dx\left[\partial_x\left(\frac{\delta\mathcal{F}}{\delta\rho}\right)\nu\chi(\rho)\partial_x\left(\frac{\delta\mathcal{F}}{\delta\rho}\right) + \frac{\delta\mathcal{F}}{\delta\rho}\partial_x\nu D(\rho)\partial_x\rho\right] + \tilde{\lambda}\int dx\frac{\delta\mathcal{F}}{\delta\rho}\partial_x\rho = 0. \tag{53}$$

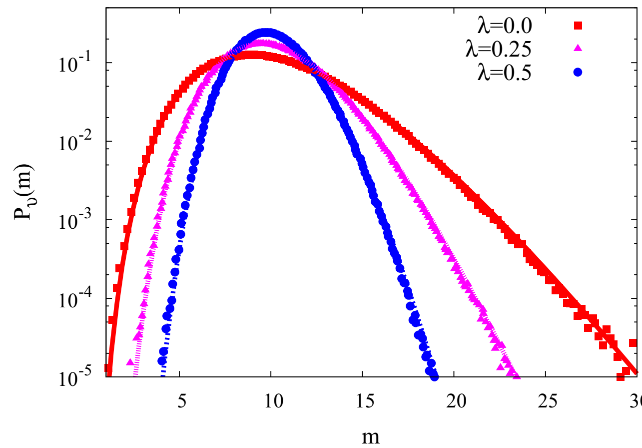

FIG. 1. Weakly asymmetric mass transfers: Model I (random sequential update), steady-state probability distribution $P_v(m)$ is plotted as a function of subsystem mass m for $\lambda = 0, 0.25$ and 0.5 and subsystem volume $v = 10$.

By noting that the last term in the left-hand side of the above equation is identically zero when integration is performed over a periodic boundary, Eq. (49), along with Eq. (48), provides the density LDF, having the same functional form as in Eqs. (5), (6), and (7). One could check that the same LDF can also be recovered by directly using additivity. The only difference in the two cases of symmetric and asymmetric mass transfers is that the exact expressions of free energy $f(\rho)$ may differ as it is directly obtained from the ratio (related to parameter η) of conductivity $\chi(\rho)$ and diffusion coefficient $D(\rho)$ (or, from the mass fluctuation $\sigma^2(\rho)$), and the ratios may be different in these two cases. Indeed, the LDFs in the cases of symmetric and strongly asymmetric mass transfer are different as the conductivity $\chi(\rho)$ is different in these two cases. However, the LDFs are the same in symmetric and weakly asymmetric cases, which is somewhat expected.

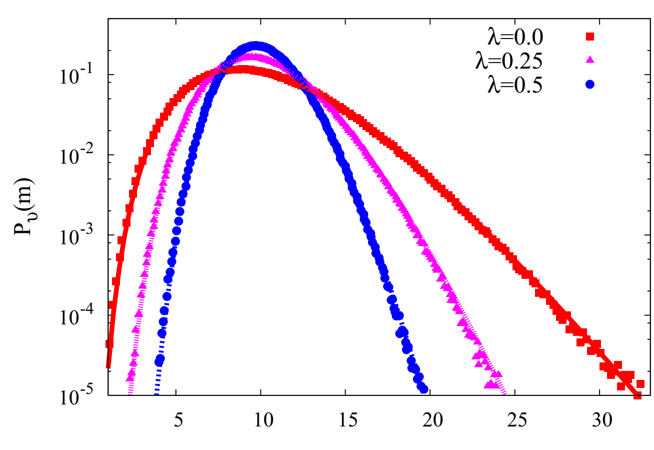

FIG. 2. Strongly asymmetric mass transfers: Model I (random sequential update), steady-state probability distribution $P_v(m)$ is plotted as a function of subsystem mass m for $\lambda = 0, 0.25$, and 0.5 and subsystem volume $v = 10$.

In Figs. 1 and 2, we have plotted steady-state probability distribution $P_v(m)$ [see Eq. (8)] of mass m in a subsystem of volume $v = 10$ as a function of m for $\lambda = 0, 0.25$, and 0.5 and $L = 5000$, which are in excellent agreement with fluctuating hydrodynamics Eq. (52) as well as additivity property in Eq. (3). It should be noted that, for a particular value of λ, the subsystem mass distributions are different for weak and strong asymmetry, depending on the parameters θ_1 (or α, the strength of asymmetry) and θ_2.

We have also considered asymmetric versions of Models II and III, leading to similar conclusions as above (results not presented).

VII. SUMMARY AND CONCLUDING REMARKS

In this paper, we have derived hydrodynamics of paradigmatic conserved-mass transport processes on a one dimensional ring-geometry, which have been intensively studied in the last couple of decades. In these processes, we have calculated two transport coefficients—diffusion coefficient $D(\rho)$ and conductivity $\chi(\rho)$. Remarkably, the two transport coefficients satisfy an equilibrium-like Einstein relation Eq. (1) even when the microscopic dynamics violate detailed balance. In all cases studied here, we find that the diffusion coefficient D is independent of mass density ρ and the conductivity $\chi(\rho) \propto \rho^2$ is proportional to the square of the mass density ρ. Moreover, using these two transport coefficients, a fluctuating hydrodynamic framework for these processes have been set up here, following a recently developed macroscopic fluctuation theory (MFT). The MFT has helped us to calculate density large deviation function (LDF), which is analogous to an equilibrium-like free-energy density function. The LDFs completely agree with that obtained previously in Refs. [12,14] solely using an additivity property Eq. (3).

Interestingly, the analytically obtained functional dependence of the two transport coefficients $D(\rho)$ and $\chi(\rho)$ on density indicates that, on large space and time scales, these mass transport processes belong to the class of Kipnis-Marchioro-Presutti (KMP) processes. However, unlike the KMP processes on a ring, the processes studied in this paper generally have a nontrivial spatial structure in their steady states. That is, they have finite spatial correlations in the steady state. Not surprisingly, the exact probability weights of microscopic configurations in the steady state, except for a few special cases [27,29,31,35], are not yet known. In fact, precisely due to this nontrivial spatial steady-state structure in out-of-equilibrium interacting-particle systems, finding hydrodynamics in such systems poses a great challenge. This is because, in the absence of knowledge of the exact steady-state weights, it is usually difficult to calculate averages of local observables (e.g., moments of local mass variables, which have been actually used here to derive hydrodynamics of these processes).

However, as noted in Ref. [14], there is an important feature in these conserved-mass transport processes (with zero external bias $F = 0$), arising from the fact that the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy involving n-point spatial correlations in the steady states closes. In other words, n-point spatial correlations in the steady state do not depend on $(n+1)$-point or any higher-order spatial correlations. This particular property previously enabled us to exactly calculate the steady-state 2-point spatial correlations and, consequently, the second moment $\langle m_i^2 \rangle$ of local mass at site i [14]. Indeed, the second moment of local mass, which appears in the hydrodynamic equations [e.g., see Eq. (19)], determines the functional dependence of the conductivity $\chi(\rho)$ on density ρ.

Finally, it is worth mentioning that, unlike in equilibrium, microscopic dynamics in the mass transport processes considered here, in general, do not satisfy detailed balance. Even for the processes with symmetric mass transfers, we have explicitly shown that Kolmogorov criterion, and thus detailed balance, is violated (even in the absence of a biasing force) and the microscopic dynamics is not time reversible. That is, for a forward path in the configuration space, there may not exist a reverse path. However, in spite of the lack of any microscopic reversibility in the dynamics of the processes, the observed Einstein relation suggests that these mass transport processes possess a kind of time-reversibility on a coarse-grained macroscopic (hydrodynamic) level. As discussed here, this macroscopic time-reversibility can be understood in the light of a macroscopic fluctuation theory (MFT) [15], which indeed correctly predicts the probabilities of density large-deviations obtained earlier in Refs. [12,14]. From an overall perspective, we believe our study could provide some useful insights in characterizing fluctuations in many other driven many-particle systems, e.g., various driven lattice gases [12,20], where a fluctuating hydrodynamic description is yet to be obtained.

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