Simultaneous influence of helicity and compressibility on anomalous scaling of the magnetic field in the Kazantsev-Kraichnan model

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Using the field theoretic renormalization group technique and the operator product expansion, the systematic investigation of the influence of the spatial parity violation on the anomalous scaling behavior of correlation functions of the weak passive magnetic field in the framework of the compressible Kazantsev-Kraichnan model with the presence of a large-scale anisotropy is performed up to the second order of the perturbation theory (two-loop approximation). The renormalization group analysis of the model is done and the two-loop explicit expressions for the anomalous and critical dimensions of the leading composite operators are found as functions of the helicity and compressibility parameters and their anisotropic hierarchies are discussed. It is shown that for arbitrary values of the helicity parameter and for physically acceptable (small enough) values of the compressibility parameter, the main role is played by the composite operators near the isotropic shell in accordance with the Kolmogorov's local isotropy restoration hypothesis. The anomalous dimensions of the relevant composite operators are then compared with the anomalous dimensions of the corresponding leading composite operators in the Kraichnan model of passively advected scalar field. The significant difference between these two sets of anomalous dimensions is discussed. The two-loop inertial-range scaling exponents of the single-time two-point correlation functions of the magnetic field are found and their dependence on the helicity and compressibility parameters is studied in detail. It is shown that while the presence of the helicity leads to more pronounced anomalous scaling for correlation functions of arbitrary order, the compressibility, in general, makes the anomalous scaling more pronounced in comparison to the incompressible case only for low-order correlation functions. The persistence of the anisotropy deep inside the inertial interval is investigated using the appropriate odd ratios of the correlation functions. It is shown that, in general, the persistence of the anisotropy is much more pronounced in the helical systems, while in the compressible turbulent environments this is true only for low-order odd ratios of the correlation functions.

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I. INTRODUCTION

During the few last decades, a great interest has been devoted to the theoretical investigation of the possible deviations from the classical Kolmogorov-Obukhov (KO) theory [1] suggested by both natural and numerical experiments (see, e.g., Refs. [2–5] and references cited therein). It means that the aim of the theory is to verify the validity of the basic principles of the KO phenomenological theory in the framework of a well-defined microscopic model and to identify and understand possible deviations from its predictions. According to the basic conclusions of the KO theory [1–4,6–8], which are standardly formulated in the form of the famous first and second Kolmogorov hypotheses, the statistical properties of various random fields deep inside the inertial interval $l \ll r \ll L$ of a given turbulent environment are independent of the integral scale L (a typical scale of the energy pumping into the system) as well as of the viscous scale l (a typical scale on which the energy begins to dissipate intensively). Under these assumptions, a simple dimensional analysis immediately leads to the prediction of the scaling behavior of correlation functions of the model with exactly defined scaling exponents.

To be more concrete, let us apply the KO theory to the inertial-range behavior of typical correlation functions of the velocity field $\mathbf{v}$ which are usually measured and analyzed in experiments, namely, to the so-called single-time structure functions of the velocity field defined as follows:

$$S_N(r) = \langle [v_r(t,\mathbf{x}) - v_r(t,\mathbf{x}')]^N \rangle, \quad r = |\mathbf{x} - \mathbf{x}'| \tag{1}$$

where v_r denotes the component of the velocity field directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and N is an arbitrary natural number. In this case, the corresponding dimensional analysis together with the assumption of validity of the first and second Kolmogorov hypotheses predict the following scale-invariant inertial-range form of the structure functions (1):

$$S_N(r) = \text{const} \times (\bar{\epsilon} r)^{N/3}, \tag{2}$$

where $\bar{\epsilon}$ is the mean dissipation rate.

However, as was already mentioned, both natural experiments and computational simulations, but also pure theoretical investigations, show the existence of deviations from the predictions of the KO theory which manifest themselves in the significant dependence of the correlation functions on the integral scale L even deep inside the inertial interval in contradiction with the first Kolmogorov hypothesis [2,4,5,9,10]. The existence of such deviations, which are usually referred as anomalous or nondimensional scaling, mean that the corresponding correlation functions must be singular functions of the ratio of the distance r and the integral scale L in the asymptotic inertial-range limit $r/L \to 0$ and, as a consequence, for example the simple scaling representation for the single-time structure functions given in Eq. (2) must be replaced by the following one:

$$S_N(r) = (\bar{\epsilon} r)^{N/3} R_N(r/L), \tag{3}$$

where R_N are unknown scaling functions. The assumption that they have a powerlike asymptotic behavior in the region $r \ll L$

in the form

$$R_N(r/L) \sim (r/L)^{q_N}, \tag{4}$$

with singular dependence on L in the limit $L \to \infty$ and nonlinearity of the exponents q_N as functions of N, is called "anomalous scaling" and is usually explained by the existence of strong developed fluctuations of the dissipative rate known as intermittency (see, e.g., Refs. [2–5,9]).

It is necessary to stress that a complete theoretical understanding of the intermittency and anomalous scaling in the fully developed turbulence on the fundamental microscopic level does not exist yet. At the same time, a significant progress has been achieved in the understanding of the problem of the anomalous scaling in the framework of the investigation of the scaling properties of various single-time correlation functions of passively advected scalar (e.g., the temperature field or an impurity field) or vector (e.g., the magnetic field) fields especially in the models with a Gaussian statistics of the velocity field. There are at least two reasons why such kind of models were and still are so attractive for theoretical study. First of all, it is the fact that the investigation of the scaling properties of correlation functions of passively advected scalar or vector fields (including their anomalous scaling) is usually considerably easier than the genuine problem of the scaling behavior of the velocity field in the framework of the Navier-Stokes turbulence. It is worth mentioning that this is usually true even in the case when the corresponding passive field is advected by the velocity field driven by the stochastic Navier-Stokes equation. The second reason for their intensive investigation is the fact that the anomalous scaling, i.e., the deviations from the predictions of the KO theory, is even more strongly pronounced for various passively advected quantities than for the velocity field itself (see, e.g., Refs. [5,8,9,11–19] and references cited therein). In this respect, the central role was and still is played by the well-known Kraichnan rapid change model of a passive scalar field advected by a self-similar Gaussian δ-correlated in time velocity field [20] as well as by the corresponding Kazantsev-Kraichnan kinematic model of a passive magnetic field in a conductive turbulent environment [21] and by a number of their extensions. Here, it is also worth to mention that even very simplified models of passive advection based on a Gaussian statistics of the velocity field lead to the anomalous scaling behavior of correlation functions which describe many features of the real turbulent advection (see, e.g., Refs. [11,22–27] as well as survey papers [8,9]). Namely, in the framework of the Kraichnan model the systematic theoretical analysis of the anomalous exponents was performed for the first time on the microscopic level by using the so-called zero-mode technique where the anomalous exponents are found from the homogeneous solutions (zero modes) of closed equations for the single-time correlations [22–24] (see also survey [9] and references cited therein).

There exists, however, another powerful method which allows one a systematic investigation of the self-similar scaling behavior, namely, the renormalization group (RG) technique, especially in the field theoretic approach [28–30], which can also be applied for analysis of fully developed turbulence [31,32], of the passive scalar admixture in turbulent environment [33], of the magnetohydrodynamic (MHD) turbulence [34], and many others [7,30].

During the last two decades, the field theoretic RG technique together with the operator product expansion (OPE) was also intensively used for investigation of the problem of anomalous scaling in turbulent systems. In this respect, in Refs. [35,36] the anomalous scaling of a passive scalar field in the framework of the Kraichnan model was investigated in the second and the third orders of approximation of the corresponding perturbative expansion, respectively. Note that, in the field theoretic RG approach, the anomalous scaling is related to the existence of composite operators in the studied model with negative critical dimensions (such operators are usually called "dangerous" operators) in the OPE (see, e.g., Refs. [7,8,30] for details). Thereafter, various (more realistic) generalizations of the simple Kraichnan model with finite time correlations of the Gaussian velocity field [37], compressibility [38], small-scale anisotropy [39], spatial parity violation (helicity) [40,41], and their various combinations [42,43] were investigated in the first-order approximation (the one-loop approximation in the field theory language) as well as in the second-order (two-loop) approximation (except for models with the presence of the small-scale anisotropy where only one-loop calculations exist). The most important general conclusion of all these investigations is the fact that the anomalous scaling behavior which is present in the Kraichnan model remains the basic feature of all generalized models. Note that a few studies were also devoted to the problem of the anomalous scaling of a passive scalar field in the Navier-Stokes turbulence [44].

Thus, it means that the scaling properties of passive scalar fields advected by various turbulent environments are quite well known. On the other hand, the systematic field theoretic RG investigation of the problem of the anomalous scaling of various vector fields (e.g., the weak magnetic field) in developed turbulent systems is still only in beginnings, although some nontrivial results were obtained in the framework of the kinematic Kazantsev-Kraichnan model, an analog of the Kraichnan rapid change model for the passive magnetic field [21]. For a long period, using the field theoretic RG technique and OPE, the scaling properties of the vector fields in various turbulent environments with Gaussian statistics of the velocity field or even in the Navier-Stokes turbulence were investigated only at the lowest order of approximation (the one-loop approximation) [16–18,45,46] which is, however, usually insufficient for complete understanding of the role which is played by the internal tensor structure of the advected field as well as for the understanding of the role of the violation of various symmetries of the corresponding turbulent system for scaling properties of a studied model. In this respect, a typical example is the spatial parity violation (helicity) of turbulent environments, the effects of which on the anomalous scaling of various quantities can be studied starting only from the two-loop approximation [40,41]. Only quite recently a few papers have appeared which are devoted to the systematic two-loop field theoretic RG investigation of the anomalous scaling of the passive vector fields in turbulent environments [47–50]. Although all of them are devoted to the investigation of the anomalous scaling of the passive magnetic field in the framework of the simple kinematic Kazantsev-Kraichnan rapid-change model, the obtained results are, however, very interesting.

First of all, in Refs. [47,48] it was shown that the two-loop corrections to the anomalous scaling in the Kazantsev-Kraichnan model are much more important and lead to the significantly more pronounced anomalous scaling than in the Kraichnan model of a passively advected scalar field. It means that the intermittency and the anomalous scaling of the fluctuations of the magnetic field in the conductive turbulent environments, i.e., in the magnetohydrodynamic (MHD) turbulent environments, are sufficiently more strongly pronounced than in the ordinary fluid turbulence. Note that this pure theoretical result is in agreement with real measurements, numerical experiments, as well as various phenomenological models (see, e.g., Refs. [51] and references cited therein). At the same time, in the subsequent brief studies [49,50], the serious impact of the compressibility [49] and the spatial parity violation [50] of the corresponding turbulent environments on the critical dimensions of the relevant composite operators, which drive the scaling properties of the magnetic field correlation functions, was reported but without a detailed analysis. Maybe the most interesting and, at the same time, important conclusion was obtained in Ref. [50] where it was shown that the conductive turbulent environment with the spatial parity violation leads to the significantly more pronounced anomalous scaling. This result is, on one hand, qualitatively different from the analogous problem in the framework of the Kraichnan model where it was shown that the presence of helicity has no impact on the scaling properties of passively advected scalar field and, on the other hand, it is qualitatively in accordance with conclusions of recent measurements [52].

The aim of this paper is twofold. First, we intend to perform a detail analysis of the separate influence of the compressibility and the spatial parity violation on the anomalous scaling of the correlation functions of the magnetic field in the framework of the Kazantsev-Kraichnan model and, second, our aim is also to investigate the simultaneous nontrivial influence of the compressibility and helicity of the turbulent environment on the asymptotic scaling properties of the passively advected weak magnetic field. It is worth to mention that in this respect, as far as we know, this paper represents the first such field theoretic study at all and, as we shall see, the compressibility together with the helicity exhibit nontrivial impact on the anomalous scaling of the magnetic field as well as on the persistence of the large-scale anisotropy deep inside the inertial interval of the model. In this respect, let us note once more that, as it was shown in Ref. [40] in two-loop approximation, the presence of the spatial parity violation in the turbulent environment has no impact on the anomalous scaling of passively advected scalar field. Finally, let us also note that, although the results of this paper are obtained in the framework of the model with a simple Gaussian statistics of the velocity field, nevertheless, we believe that the obtained results, at least at the qualitative level, are also relevant for the real kinematic MHD turbulence with Navier-Stokes velocity field.

The paper is organized as follows. In Sec. II, the compressible Kazantsev-Kraichnan model with the spatial parity violation of the passively advected weak magnetic field is introduced. The field theoretic formulation of the model is given in Sec. III. The RG analysis of the model is performed in Sec. IV. In Sec. V, the anomalous dimensions of the leading composite operators of the model are found as functions of the compressibility and helicity parameters in the two-loop approximation and, in Sec. VI, they are compared with the corresponding anomalous dimensions of the leading composite operators in the Kraichnan model of passively advected scalar field. In Sec. VII, the anomalous scaling of the single-time two-point correlation functions of the magnetic field in the compressible and helical environment is discussed in detail. The persistence of the large-scale anisotropy deep inside in the inertial interval is investigated in Sec. VIII. Finally, obtained results are briefly reviewed and discussed in Sec. IX.

II. COMPRESSIBLE KAZANTSEV-KRAICHNAN MODEL WITH SPATIAL PARITY VIOLATION

As was discussed in the Introduction, in this paper we shall investigate statistical properties of the solenoidal magnetic field $\mathbf{b} \equiv \mathbf{b}(t,\mathbf{x})$ ($\partial_i b_i = 0$) in the framework of the so-called Kazantsev-Kraichnan model of the kinematic MHD turbulence, where the magnetic field is considered as a vector admixture passively advected by a compressible random velocity field $\mathbf{v}(x) \equiv \mathbf{v}(t,\mathbf{x})$ ($\partial_i v_i \neq 0$) and which is described by the following stochastic advection-diffusion equation (see, e.g., Refs. [16–18] for details):

$$\partial_t b_i = \nu_0 \triangle b_i - \partial_j (v_j b_i) + b_j \partial_j v_i + f_i. \tag{5}$$

Here, the following standard notation is used: $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, $\nu_0 = c^2/(4\pi\sigma_0)$ is the magnetic diffusivity (in what follows, a subscript 0 will denote bare parameters of the unrenormalized theory), c is the speed of light, σ_0 is the electrical conductivity.

The main role of the random force $\mathbf{f}(x) = \mathbf{f}(t,\mathbf{x})$ in Eq. (5), which is also transverse, i.e., we suppose that $\partial_i f_i = 0$, is to maintain the steady state of the system. It represents the source of the fluctuations of the magnetic field and is taken in the Gaussian form with zero mean and with the correlation function

$$D^b_{ij}(x_1;x_2) \equiv \langle f_i(x_1) f_j(x_2)\rangle = \delta(t_1 - t_2) C_{ij}(|\mathbf{x_1} - \mathbf{x_2}|/L). \tag{6}$$

Here, L is an integral scale related to the corresponding stirring, and C_{ij} is a function finite in the limit $L \to \infty$. Its detailed form is unimportant in what follows and the only condition which must be satisfied by C_{ij} is that it must decrease rapidly for $|\mathbf{x_1} - \mathbf{x_2}| \gg L$. In more realistic formulation, the noise f_i in Eq. (5) can be replaced by the term $(B_j \partial_j) v_i$, where $\mathbf{B}$ is a constant large-scale magnetic field, the source of the large-scale anisotropy (see, e.g., Ref. [16] for more details).

In the framework of the Kazantsev-Kraichnan model, the random velocity field $\mathbf{v}(x)$ obeys the Gaussian statistics, δ correlated in time, with zero mean and with the following pair correlation function:

$$D_{ij}(x_1;x_2) \equiv \langle v_i(x_1) v_j(x_2)\rangle = \delta(t_1 - t_2) D_0 \int \frac{d\mathbf{k}}{(2\pi)^d} R_{ij}(\mathbf{k}) k^{-d-\varepsilon} e^{i\mathbf{k}\cdot(\mathbf{x_1}-\mathbf{x_2})}, \tag{7}$$

where d denotes the spatial dimension of the system, $R_{ij}(\mathbf{k})$ describes geometric properties of the velocity correlator (its explicit form will be specified a little bit later), $\mathbf{k}$ is the momentum (wave number), and $D_0 > 0$ is a positive amplitude factor. The interval of possible values of parameter ε is $0 < \varepsilon < 2$. However, usually the interval $0 < \varepsilon \leqslant 1$ is considered in the Kazantsev-Kraichnan model, where the existence of a steady state of the magnetic field (without dynamo effects) is guaranteed (see, e.g., Ref. [12]). The exponent ε can also be considered as a kind of Hölder exponent for measuring of the roughness of the velocity field. For convenience, it is usually appropriate to introduce the coupling constant $g_0 \equiv D_0/\nu_0 \simeq \Lambda^\varepsilon$, where Λ is a characteristic ultraviolet (UV) momentum scale related to the inner turbulent length $l = 1/\Lambda$.

As was already mentioned, the geometric properties of the velocity fluctuations are described by the tensor $R_{ij}(\mathbf{k})$ in correlator (7). The fact that we intend to investigate the influence of the compressible stochastic environment on properties of the model defined in Eqs. (5)–(7) is expressed explicitly in the form of the second-rank tensor $R_{ij}(\mathbf{k})$, namely, it is taken in the form of a linear combination of a transverse projector $T_{ij}(\mathbf{k})$ (its form will be specified below) and the longitudinal projector $L_{ij}(\mathbf{k}) = k_i k_j/k^2$:

$$R_{ij}(\mathbf{k}) = T_{ij}(\mathbf{k}) + \alpha k_i k_j/k^2, \tag{8}$$

where $\alpha \geqslant 0$ is the compressibility parameter which characterizes the deviation of the turbulent environment from the incompressible case. [Strictly speaking, it characterizes the amount of deviation of the tensor $R_{ij}(\mathbf{k})$ from its transversality.] Here, the value $\alpha = 0$ corresponds to the divergence-free (incompressible) velocity field. Note that although the value of the parameter α in Eq. (8) is formally unrestricted, i.e., from pure mathematical point of view it can be arbitrarily large, from consistent physical point of view, however, it must be considered to be close to zero only, i.e., it is necessary to realize that, in fact, parameter α must be restricted by condition $\alpha \ll 1$ or, at least, $\alpha < 1$. This pure physical restriction on the compressibility parameter is related to the fact that our formulation of the model is suitable for investigation of the properties of the system only near the incompressible limit, i.e., in the situation where the system is only slightly deviated from the incompressible state with small density fluctuations of the environment.

On the other hand, our aim is also to study the impact of the spatial parity violation (the presence of the helicity) of the compressible velocity field on the statistical properties of the magnetic field. To this end, the transverse part $T_{ij}(\mathbf{k})$ of the tensor $R_{ij}(\mathbf{k})$ in Eq. (8) is taken in the form of the sum of the ordinary transverse projector $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$, which describes fully symmetric isotropic turbulent environment, and the helical tensor $H_{ij}(\mathbf{k}) = \mathrm{i}\epsilon_{ijl} k_l/|k|$ which responds for the presence of the spatial parity violation in the system:

$$T_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2 + i\rho\epsilon_{ijl} k_l/|k|. \tag{9}$$

Here, ϵ_{ijl} is the Levi-Civita's completely antisymmetric tensor of rank 3 and the real parameter $0 \leqslant |\rho| \leqslant 1$ determines the "amount" of the helicity in the system. Setting $\rho = 0$ means that no violation of the spatial parity is present in the system. On the other hand, $|\rho| = 1$ describes the system with maximal spatial parity violation. Physically, the nonzero helical part expresses the existence of nonzero correlations $\langle \mathbf{v} \cdot \mathrm{rot}\mathbf{v} \rangle$ in the turbulent environment.

Thus, in what follows, the final form of the tensor $R_{ij}(\mathbf{k})$ in correlator (7) is taken to be

$$\begin{aligned} R_{ij}(\mathbf{k}) &= P_{ij}(\mathbf{k}) + \rho H_{ij}(\mathbf{k}) + \alpha L_{ij}(\mathbf{k}) \\ &= \delta_{ij} - k_i k_j/k^2 + \mathrm{i}\rho\epsilon_{ijl} k_l/|k| + \alpha k_i k_j/k^2, \end{aligned} \tag{10}$$

which describes simultaneously the compressible and helical turbulent environment.

Finally, note also that the necessary infrared (IR) regularization in integral (7) is given by the cutoff from below $k = k_{\min} \equiv 1/L$, where L represents the integral turbulent scale. This scale is, in general, different from the stirring scale L introduced in Eq. (6) but, in what follows, this difference is unimportant.

III. FIELD THEORETIC FORMULATION OF THE MODEL

According to the well-known theorem [53], the stochastic problem defined by Eqs. (5)–(7) can be rewritten in the form of the corresponding field theoretic model of the set of three fields $\Phi = \{\mathbf{v},\mathbf{b},\mathbf{b}'\}$ with the following action functional:

$$\begin{aligned} S(\Phi) = &-\frac{1}{2}\int dx_1\, dx_2\, v_i(x_1) D^{-1}_{ij}(x_1;x_2) v_j(x_2) \\ &+\frac{1}{2}\int dx_1\, dx_2\, b'_i(x_1) D^b_{ij}(x_1;x_2) b'_j(x_2) \\ &+\int dx\, b'_i[-\partial_t b_i + \nu_0 \triangle b_i \\ &-\partial_j(v_j b_i) + (b_j\partial_j) v_i], \end{aligned} \tag{11}$$

where $\mathbf{b}'$ is a solenoidal auxiliary field necessary for the field theoretic formulation of the problem, D^b_{ij} and D_{ij} are the correlators (6) and (7), respectively, $dx = dt\, d\mathbf{x}$, and the required summation over dummy indices is assumed. Note that the second and the third lines in Eq. (11) represent the DeDominicis-Janssen action of the studied stochastic problem at fixed velocity field $\mathbf{v}$ and the first line represents the Gaussian averaging over the velocity field $\mathbf{v}$.

The free (i.e., quadratic in fields) part of the action functional (11) defines the set of all nonzero bare propagators of the model in the framework of the Feynman diagrammatic technique of the perturbation theory, namely (in the frequency-momentum representation),

$$\langle b'_i b_j \rangle_0 = \langle b_i b'_j \rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega_k + \nu_0 k^2}, \tag{12}$$

$$\langle b_i b_j \rangle_0 = \frac{C_{ij}(\mathbf{k})}{(-i\omega_k + \nu_0 k^2)(i\omega_k + \nu_0 k^2)}, \tag{13}$$

and the bare propagator $\langle v_i v_j \rangle_0$ for the velocity field is given directly by the correlator $D_{ij}(x;x')$ in Eq. (7). Function $C_{ij}(\mathbf{k})$ in Eq. (13) represents the Fourier transform of the function $C_{ij}(|\mathbf{x_1} - \mathbf{x_2}|/L)$ from Eq. (6). In what follows, only propagators $\langle b'_i b_j \rangle_0 = \langle b_i b'_j \rangle_0^*$ and $\langle v_i v_j \rangle_0$ are important and their graphical representation is given explicitly in Fig. 1. On the other hand, the only interaction vertex of the model has the form $b'_i[-\partial_j(v_j b_i) + b_j \partial_j v_i] = b'_i V_{ijl} b_j v_l$, where the vertex

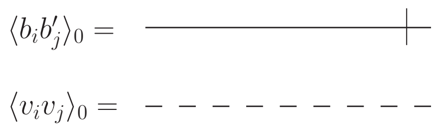

FIG. 1. Graphical representation of the important propagators of the model. The end with a slash in the propagator $\langle b'_i b_j \rangle_0$ corresponds to the field $\mathbf{b}'$ and the end without a slash corresponds to the field $\mathbf{b}$.

factor V_{ijl} in the frequency-momentum representation is given as follows

$$V_{ijl} = i(k_l \delta_{ij} - k_j \delta_{il}). \tag{14}$$

The graphical representation of the interaction vertex is shown explicitly in Fig. 2, where the momentum $\mathbf{k}$ is flowing into the vertex via the auxiliary filed $\mathbf{b}'$.

Note that the formulation of the stochastic problem given in Eqs. (5)–(7) through action functional (11) allows one to use standard field theoretic means, namely, the field theoretic RG technique, to analyze the problem. In the framework of the field theoretic approach, the statistical averages of random quantities in the stochastic problem are replaced with the corresponding functional averages with weight $\exp S(\Phi)$ (see, e.g., Ref. [30] for details).

IV. RENORMALIZATION GROUP ANALYSIS OF THE MODEL

It is not necessary to discuss all details of the RG analysis of the present model because it can be performed in the similar manner as, e.g., in Ref. [16]. At the same time, the general RG analysis of various field theoretic models can be found, e.g., in Ref. [30]. Therefore, here it is enough to summarize only basic facts and results of the corresponding RG analysis which will be important in what follows.

Standardly, the RG analysis of the field theoretic model defined in Eq. (11) is based on the analysis of the UV divergences of the 1-irreducible Green's functions. On the other hand, the identification of the divergent Green's functions is based on the corresponding analysis of the canonical dimensions of the model. The canonical momentum d_Q^k, frequency d_Q^ω, and total d_Q dimensions of all fundamental quantities Q (parameters and fields) of the present two-scale model are summarized in Table I, where also the canonical dimensions of the renormalized parameters are shown. Thus, it is evident that the studied model is logarithmic at $\varepsilon = 0$, for which the coupling constant g_0 is dimensionless and the dimensional analysis shows that the only superficially divergent function of the model is the 1-irreducible Green's function $\langle b'_i b_j \rangle_{1-ir}$.

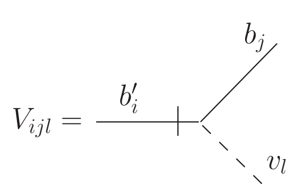

FIG. 2. The graphical representation of the interaction vertex of the model.

TABLE I. Canonical dimensions of the fields and parameters of the model under consideration.

Q	$\mathbf{v}$	$\mathbf{b}$	$\mathbf{b}'$	m,Λ,μ	ν_0,ν	g_0	g,α,ρ
d_Q^k	-1	0	d	1	-2	ε	0
d_Q^ω	1	0	0	0	1	0	0
d_Q	1	0	d	1	0	ε	0

This divergence can be removed multiplicatively by the only counterterm $b'_i \triangle b_j$. At the same time, the renormalization of the model can be explicitly expressed in the form of multiplicative renormalization of the bare parameters g_0 and ν_0, namely (see, e.g., Ref. [16] for all details),

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^\varepsilon Z_g, \quad Z_g = Z_\nu^{-1}, \tag{15}$$

where the dimensionless parameters g and ν are the renormalized counterparts of the corresponding bare ones, μ is the renormalization mass (a scale setting parameter), an artifact of the dimensional regularization, and $Z_\nu(g,\varepsilon,d,\alpha,\rho)$ and $Z_g(g,\varepsilon,d,\alpha,\rho)$ are the so-called renormalization constants of the model, where we have written explicitly all parameters which the renormalization constants can, in principle, depend on.

Here, it is necessary to stress that, strictly speaking, this conclusion is completely true only in the nonhelical case ($\rho = 0$) of the present kinematic Kazantsev-Kraichnan model, which represents an approximation of the problem of real MHD turbulence. It is related to the fact that in the helical case of the present model with $\rho \neq 0$, the linear divergences in the form $\mathbf{b}' \cdot \mathrm{rot}\, \mathbf{b}$ can appear in the 1-irreducible Green's function $\langle b'_i b_j \rangle_{1-ir}$ in the same way as in the field theoretic model of the genuine helical MHD turbulence (see, e.g., Refs. [54] for details). A term of this form, however, is not present in the action (11). In this case, the simplest way to make the model multiplicatively renormalizable is to modify the action functional of the model (11) by introducing the corresponding term and start with the investigation of the obtained enlarged model. However, the presence of such a term in the model leads to the instability that causes the exponential growth in time of the response function $\langle b'_i b_j \rangle$. It means that the direct insertion of such a term into the action functional does not solve the problem. On the other hand, in the framework of the field theoretic approach to the genuine MHD turbulence, the problem of the existence of the linear divergences is solved through the appearance of the homogeneous large-scale magnetic field generated by a kind of the spontaneous symmetry breaking mechanism. However, the proper realization of this mechanism in the framework of the genuine MHD turbulence is possible only due to the presence of the so-called Lorentz force term in the corresponding enlarged form of the Navier-Stokes equation [54]. Therefore, it is immediately evident that this technique cannot be used in the framework of the Kazantsev-Kraichnen model studied in this paper where the statistics of the velocity field is given *a priori* by the explicit form of the correlator (7). This is the reason why we shall leave the problem of the linear divergences untouched in this paper knowing that the existence and stability of the IR scaling regime can be studied

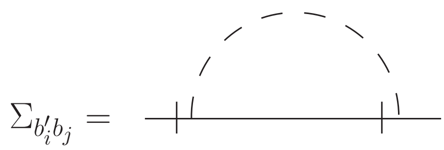

FIG. 3. The only self-energy Feynman diagram which contributes to the UV renormalization of the model.

without considering the linear divergences. At the same time, it is necessary to be aware of the fact that the completely consistent analysis of the problem with the presence of the helicity can be performed only in the framework of the genuine MHD turbulence described by the stochastic MHD equations.

Thus, bearing in mind the above discussion, i.e., forgetting about potential linear divergences, from the third relation in Eq. (15) it is evident that the model is multiplicatively renormalized by only one independent renormalization constant Z_ν which, in fact, does not depend on the helicity parameter ρ and which, in the framework of the minimal subtraction (MS) scheme [29], has the following exact one-loop form [16,18,49]:

$$Z_\nu = 1 - \frac{S_d}{(2\pi)^d}\frac{d-1+\alpha}{2d}\frac{g}{\varepsilon}. \tag{16}$$

Here, $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface of the d-dimensional unit sphere. Note that the expression (16), given by the calculation of the only one-loop self-energy Feynman diagram which is shown explicitly in Fig. 3, is the exact result of the perturbation expansion, i.e., it has no corrections of orders $g^n, n \geqslant 2$. It is related to the fact that all two- and higher-loop diagrams contain at least one close loop of retarded propagators, i.e., all such diagrams vanish (they are identically equal to zero) [30].

Using the explicit form of the renormalization constant Z_ν given in Eq. (16) the RG functions (the β and γ functions) of the model are immediately obtained, namely,

$$\beta_g \equiv \mu\partial_\mu g = g(-\varepsilon + \gamma_\nu), \tag{17}$$

$$\gamma_\nu \equiv \mu\partial_\mu \ln Z_\nu, \tag{18}$$

where the explicit exact expression for function γ_ν is

$$\gamma_\nu = \frac{S_d}{(2\pi)^d}\frac{d-1+\alpha}{2d}g. \tag{19}$$

Then, the inertial-range scaling behavior of various correlations functions of the model is driven by the exact one-loop IR stable fixed point of the RG equations [16,18,49], namely,

$$g_* = \frac{(2\pi)^d}{S_d}\frac{2d\varepsilon}{d-1+\alpha}, \tag{20}$$

which is obtained by the requirement of vanishing of function β_g defined in Eq. (17). This fixed point is IR stable for $\varepsilon > 0$ and corresponds to the so-called kinetic regime in the genuine MHD turbulence.

Note that while the IR fixed point of the model given in Eq. (20) depends explicitly on the parameter which controls the compressibility of the system, namely, the value of g_* decreases as function of parameter α, it is independent of the parameter ρ, i.e., at this stage of the RG analysis the model is absolutely helicity blind (it does not feel the presence of the parity violation in the given turbulent environment).

Besides, let us also note that the fact that the form of the β_g in Eq. (17) is exact means that the value of the anomalous dimension γ_ν at the IR fixed point g_* is also exact, namely,

$$\gamma_\nu^* = \varepsilon. \tag{21}$$

In what follows, we will be interested in the scaling behavior of the single-time two-point correlation functions built solely of the magnetic field (see, e.g., Refs. [16,48]), namely,

$$B_{N-m,m}(r) \equiv \langle b_r^{N-m}(t,\mathbf{x}) b_r^m(t,\mathbf{x}')\rangle, \quad r = |\mathbf{x}-\mathbf{x}'| \tag{22}$$

where b_r denotes the component of the magnetic field directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$.[1] It is an example of a general multiplicatively renormalizable single-time two-point correlation function $G(r)$ which has the following general IR asymptotic scaling form for $r/l \gg 1$ and any fixed r/L:

$$G(r) \simeq \nu_0^{d_G^\omega} l^{-d_G}(r/l)^{-\Delta_G} R(r/L), \tag{23}$$

which is given by the existence of the IR stable fixed point of the RG equations (20). Here, d_G^ω and d_G are the corresponding canonical dimensions of the function G, which can be obtained easily using the basic canonical dimensions of the model given in Table I, $l = 1/\Lambda$, $L = 1/k_{\min}$, and Δ_G is the critical dimension defined as follows:

$$\Delta_G = d_G^k + \Delta_\omega d_G^\omega + \gamma_G^*. \tag{24}$$

Here, γ_G^* is the fixed point value of the anomalous dimension $\gamma_G \equiv \mu\partial_\mu \ln Z_G$, where Z_G is the renormalization constant of the multiplicatively renormalizable quantity G, i.e., $G = Z_G G^R$ [30], and $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of the frequency with γ_ν^* given in Eq. (19) taken at the fixed point, i.e., $\gamma_\nu^* = \varepsilon$ exactly. It means that the critical dimension of frequency is also known exactly, namely, $\Delta_\omega = 2 - \varepsilon$, as well as the critical dimensions of the fields:

$$\Delta_{\mathbf{v}} = 1 - \varepsilon, \quad \Delta_{\mathbf{b}} = 0, \quad \Delta_{\mathbf{b}'} = d. \tag{25}$$

Finally, the function $R(r/L)$ in Eq. (23) is the so-called scaling function, which cannot be determined by the RG equations (see, e.g., Ref. [30] for all details).

[1] For completeness, let us note that from the experimental point of view more convenient are the equal-time structure functions of the magnetic field $S_N(r) = \langle [b_r(t,\mathbf{x}) - b_r(t,\mathbf{x}')]^N\rangle$, which are important in the analysis of inertial-range properties of the MHD turbulence. Nevertheless, from the theoretical point of view it is enough to study much simpler quantities, namely, the equal-time two-point correlation functions $B_{N-m,m}(r)$ defined in Eq. (22). The reason is twofold: first of all, the above defined structure functions are given by linear combinations of the correlation functions (22), hence, the scaling behavior of the structure functions is immediately given by the scaling behavior of the correlation functions (22). At the same time, there is no special need to investigate the structure functions, which are more complex quantities, instead of their building blocks, the correlation functions (22), as a result of the fact that contrary to the passive scalar advection by the velocity field the stochastic equation for the vector field $\mathbf{b}$ is not invariant under the shift $\mathbf{b} \to \mathbf{b} + \mathbf{b_0}$, where $\mathbf{b_0}$ is a constant vector.

Thus, applying the general scaling representation for the equal-time two-point quantity $G(r)$ given in Eq. (23) to the correlation functions (22) one obtains

$$B_{N-m,m}(r) \simeq \nu_0^{-N/2}(r/l)^{-\gamma^*_{N-m}-\gamma^*_m} R_{N,m}(r/L), \quad (26)$$

where γ^*_{N-m} and γ^*_m are the anomalous dimensions of the composite operators b_r^{N-m} and b_r^m taken at the fixed point value g_* and the corresponding scaling functions $R_{N,m}(r/L)$ are unknown in the framework of the standard RG analysis.

On the other hand, the systematic investigation of the asymptotic behavior of the scaling functions $R_{N,m}(r/L)$ deep inside the inertial interval, i.e., in the limit $r/L \to 0$, can be performed by using the OPE technique (see, e.g., Ref. [29]), in the framework of which it is assumed that the scaling functions have the following form:

$$R_{N,m}(r/L) = \sum_i C_{F_i}(r/L)(r/L)^{\Delta_{F_i}}, \quad r/L \to 0. \quad (27)$$

Here, the summation is performed over all possible renormalized composite operators F_i allowed by the symmetry of the problem with critical dimensions Δ_{F_i} and the corresponding coefficient functions $C_{F_i}(r/L)$ are regular in r/L. Note that the very existence of the nontrivial anomalous scaling in the model is related to the existence of the so-called "dangerous" composite operators with negative critical dimensions which give singular contributions to the OPE (27) in the limit $r/L \to 0$ (see, e.g., Ref. [16] for details). The leading contribution to the expansion (27) is given by the composite operators with the smallest critical dimensions and, in our case, is given by the operators constructed solely from the magnetic field $\mathbf{b}(x)$ in the following form [39,46,48]:

$$F_{N,p} = [\mathbf{n}\cdot\mathbf{b}]^p(\mathbf{b}\cdot\mathbf{b})^l, \quad N = 2l + p \quad (28)$$

which is also suitable for taking into account of the uniaxial anisotropy effects represented here by introducing the constant unit vector $\mathbf{n}$ (e.g., $\mathbf{n} = \mathbf{B}/|\mathbf{B}|$, where $\mathbf{B}$ is a large-scale magnetic field discussed in Sec. II).

First of all, consider the simplest case when it is supposed that the system is fully isotropic. In this case, there is no specific direction and the set of all composite operators of the form (28) is reduced to the only one composite operator, namely,

$$F_{N,0} = (\mathbf{b}\cdot\mathbf{b})^{N/2}. \quad (29)$$

Now, using the explicit expression for the critical dimensions given in Eq. (24), one obtains the final asymptotic inertial-range behavior of the correlation functions (22), namely,

$$B_{N-m,m}(r) \simeq \nu_0^{-N/2}(r/l)^{-\gamma^*_{N-m}-\gamma^*_m}(r/L)^{\gamma^*_N} \sim r^{-\gamma^*_{N-m}-\gamma^*_m+\gamma^*_N}, \quad (30)$$

where γ^*_M $(M = N,m,N-m)$ are the anomalous dimensions of the composite operators $F_{M,0}$ given in Eq. (29) which will be discussed in detail in the next section.

Note that in the strictly isotropic case N and m must be even natural numbers and one immediately finds that for $m = 0$ or $m = N$ the correlation functions $B_{N-m,m}(r)$ are reduced to a constant, namely,

$$B_{N,0} \equiv B_{0,N} \simeq \nu_0^{-N/2}. \quad (31)$$

On the other hand, in the anisotropic case, the situation is usually more complicated due to the fact that in this case, the asymptotic inertial-range behavior of the correlation functions (22) is determined by the set of critical dimensions which correspond to the composite operators which are mixed during the renormalization. The leading contribution to the asymptotic behavior of the correlation functions is then given by the corresponding smallest critical dimension. In the next section, we shall investigate this issue in more detail in the case when the large-scale anisotropy is present in the model in the framework of the two-loop approximation. Our aim is to find the explicit form of the critical dimensions for needed composite operators (28) which is necessary for determining the explicit form of the inertial-range behavior of the single-time correlation functions $B_{N-m,m}$ defined in Eq. (22) in the compressible case with the presence of the spatial parity violation in the given turbulent system.

V. CRITICAL DIMENSIONS OF THE COMPOSITE OPERATORS $F_{N,p}$: TWO-LOOP APPROXIMATION

Thus, to be able to discuss the scaling behavior of the single-time two-point correlation functions of the magnetic field defined in Eq. (22) it is necessary to perform the corresponding RG analysis of the composite operators (28) at a given level of the perturbation approximation to obtain needed explicit expressions for their anomalous and critical dimensions. In this paper, we shall find their explicit dependence on the parameters which characterize the compressibility of the stochastic environment as well as its spatial parity violation in the two-loop approximation.

Detailed RG analysis of the composite operators in the framework of the Kazantsev-Kraichnan model can be found, e.g., in Refs. [16,48], therefore, it is not necessary to repeat it here. Instead of that, here we shall discuss only basic conclusions of this analysis which are important in what follows.

Analysis of the composite operators defined in Eq. (28) shows that the composite operators with different values of N are not mixed in the process of renormalization. In addition, in our case, when only the presence of the large-scale anisotropy is considered in the system, the matrix of the renormalization constants for given value of N, $Z_{[N,p][N,p']}$, is in fact triangular, therefore the anomalous dimensions of the basic operators (28) are directly given by the diagonal elements of the matrix of the renormalization constants $Z_{N,p} \equiv Z_{[N,p][N,p]}$, namely,

$$\gamma_{N,p} = \mu\partial_\mu \ln Z_{N,p}. \quad (32)$$

Besides, their critical dimensions defined by the general expression (24) are

$$\Delta_{N,p} = d^k_{F_{N,p}} + \Delta_\omega d^\omega_{F_{N,p}} + \gamma^*_{N,p}. \quad (33)$$

Now, using the fact that $d^k_b = 0$ and $d^\omega_b = 0$ (see Table I) one immediately obtains that

$$\Delta_{N,p} = \gamma^*_{N,p}, \quad (34)$$

i.e., the critical dimensions of the composite operators (28) are equal to their anomalous dimensions taken at the fixed point given in Eq. (20).

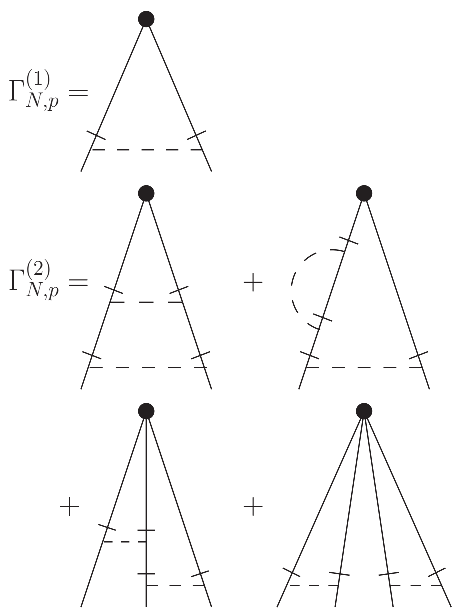

FIG. 4. The Feynman diagrams for the function $\Gamma_{N,p}(x;\mathbf{b})$ in the two-loop approximation. The Feynman rules are the same as in Sec. II. The black circle denotes the vertex of the composite operator $F_{N,p}$.

Thus, to proceed further it is necessary to calculate the renormalization constants $Z_{N,p}$ defined through the following relation between the unrenormalized and the corresponding renormalized composite operators

$$F_{N,p} = Z_{N,p} F^R_{N,p} \tag{35}$$

and, as it is discussed in detail, e.g., in Ref. [16], to find the explicit form of the renormalization constants $Z_{N,p}$ it is necessary to analyze the Nth-order term with respect to the magnetic field $\mathbf{b}$ of the expansion of the generating functional of one-irreducible Green's functions with one composite operator $F_{N,p}$ and any number of fields $\mathbf{b}$. This term has the following form [16]:

$$\Gamma_{N,p}(x;\mathbf{b}) = \frac{1}{n!}\int dx_1 \dots \int dx_n\, b_{i_1}(x_1)\dots b_{i_N}(x_N) \times\langle F_{N,p}(x)\, b_{i_1}(x_1)\dots b_{i_N}(x_N)\rangle_{1-ir} \tag{36}$$

and in the framework of the two-loop approximation it can be written as follows:

$$\Gamma_{N,p} = F_{N,p} + \Gamma^{(1)}_{N,p} + \Gamma^{(2)}_{N,p}, \tag{37}$$

where terms $\Gamma^{(1)}_{N,p}$ and $\Gamma^{(2)}_{N,p}$ represent the corresponding one- and two-loop contributions, respectively. In the framework of the standard Feynman diagrammatic technique they are given graphically by diagrams shown in Fig. 4, where the black circle on the top of each diagram is the vertex related to the composite operator $F_{N,p}$ which is defined as follows:

$$V_{i_1,\dots,i_k}(x;x_1,\dots,x_k) = \frac{\delta^k F_{N,p}}{\delta b_{i_1}(x_1)\dots\delta b_{i_k}(x_k)}, \tag{38}$$

where k denotes the number of attached lines. It represents the only additional Feynman rule needed for analytic investigation of the diagrams in Fig. 4. All the other Feynman rules were already defined in Sec. III.

Finally, the analysis of the UV divergences of the diagrams in Fig. 4 gives the anomalous dimensions $\gamma^*_{N,p}$ [as well as the critical dimensions $\Delta_{N,p}$ given in Eq. (34)] in the the second-order approximation (the two-loop approximation) which can be written as follows (all technical details can be found, e.g., in Ref. [48] and in references cited therein):

$$\gamma^*_{N,p} = \gamma^{*(1)}_{N,p}\varepsilon + \gamma^{*(2)}_{N,p}\varepsilon^2 + O(\varepsilon^3), \tag{39}$$

where $\gamma^{*(1)}_{N,p}$ represents the one-loop contribution to the anomalous dimension $\gamma^*_{N,p}$ and $\gamma^{*(2)}_{N,p}$ is the corresponding two-loop correction.

The one-loop contribution $\gamma^{*(1)}_{N,p}$ does not depend on the helicity (see also Ref. [50]) and its dependence on the parameter of compressibility α has the following explicit form (see also Refs. [17,18] for details):

$$\gamma^{*(1)}_{N,p} = -\{(N-p)(d+N+p-2)(d+1+\alpha) + N(N-1)[d^2\alpha - 2(1+\alpha)]\}/ [2(d+2)(d-1+\alpha)]. \tag{40}$$

Note that in the incompressible case, i.e., for $\alpha = 0$, one comes to the one-loop result obtained, e.g., in Ref. [16], namely,

$$\gamma^{*(1)}_{N,p} = -\frac{(N-p)(d+N+p-2)(d+1) - 2N(N-1)}{2(d+2)(d-1)}. \tag{41}$$

On the other hand, the two-loop contribution $\gamma^{*(2)}_{N,p}$ depends explicitly on the helicity parameter ρ as well as on the compressibility parameter α and has the following form:

$$\gamma^{*(2)}_{N,p} = -\frac{S_{d-1}}{S_d}\frac{d}{(d-1)(d+2)(d-1+\alpha)^2} \times\int_0^1 dx(1-x^2)^{\frac{d-3}{2}}\{\sqrt{1-x^2} \times[(d-2)C_1(W_1Y_1 + 2\rho^2\delta_{3d}Y_3) + C_2W_2Y_1] - 2(C_3W_3 + C_4W_4)Y_2/(d+4)\}, \tag{42}$$

where

$$C_1 = (d+1)(N-p)(d+N+p-2) - 2N(N-1), \tag{43}$$

$$C_2 = -(N-p)(d+N+p-2) + dN(N-1), \tag{44}$$

$$C_3 = (N-2)C_1, \tag{45}$$

$$C_4 = (N-2)[-3(N-p)(d+N+p-2) + (d+2)N(N-1)] \tag{46}$$

and

$$W_1 = 2 + \alpha - \alpha^2, \tag{47}$$

$$W_2 = 2(1-x^2) + \alpha[d(d-3) + 4x^2] - \alpha^2[d(d-1) - 2(1-x^2)], \tag{48}$$

$$W_3 = (1-x^2)(9-5d+4x^2) + \alpha[9(1-2x^2)+x^2(d^2+8x^2)+5d(1-x^2)] - \alpha^2(10-3d-11x^2+4x^4), \quad (49)$$

$$W_4 = -2(1-x^2)^2 + 4\alpha(1-x^2)(d-x^2) + \alpha^2[d^2(d+1-x^2)-2(1-x^2)^2 + d(2x^2-3)]. \quad (50)$$

In addition,

$$Y_1 = x\left[\arctan\left(\frac{1+x}{\sqrt{1-x^2}}\right) - \arctan\left(\frac{1-x}{\sqrt{1-x^2}}\right)\right], \quad (51)$$

$$Y_2 = \frac{x\left[\arctan\left(\frac{2+x}{\sqrt{4-x^2}}\right) - \arctan\left(\frac{2-x}{\sqrt{4-x^2}}\right)\right]}{\sqrt{4-x^2}}, \quad (52)$$

and

$$Y_3 = \pi - \arctan\left(\frac{1+x}{\sqrt{1-x^2}}\right) - \arctan\left(\frac{1-x}{\sqrt{1-x^2}}\right). \quad (53)$$

Note that the presence of the Kronecker delta δ_{3d} in the helical term in Eq. (42), i.e., in the term proportional to ρ^2, means that it plays role (it has sense) only for the spatial dimension $d = 3$.

It is now an easy task to show that in the incompressible and nonhelical case, i.e., when $\alpha = 0$ as well as $\rho = 0$, one has

$$\gamma_{N,p}^{*(2)} = -\frac{S_{d-1}}{S_d}\frac{2d}{(d-1)^3(d+2)}\int_0^1 dx(1-x^2)^{\frac{d-1}{2}} \times\left\{\left[\frac{(d-2)C_1}{\sqrt{1-x^2}} + \sqrt{1-x^2}C_2\right]Y_1 - \frac{[(9-5d+4x^2)C_3 - 2(1-x^2)C_4]Y_2}{d+4}\right\}, \quad (54)$$

which is equivalent to the result obtained in Ref. [48] [see Eq. (86) in Ref. [48]].

Let us also note that in the framework of the so-called zero-mode approach [9], the critical dimensions for $N = 2$, i.e, $\Delta_{2,0}$ and $\Delta_{2,2}$ are known exactly in the incompressible nonhelical case [12–14]. Their expansion up to the second order in ε is

$$\Delta_{2,0} = -\varepsilon - \frac{2(d-2)}{d(d-1)}\varepsilon^2 + O(\varepsilon^3), \quad (55)$$

$$\Delta_{2,2} = \frac{2}{(d-1)(d+2)}\varepsilon + \frac{2(d+4)(d^2-d-4)}{d(d-1)^2(d+2)^3}\varepsilon^2 + O(\varepsilon^3), \quad (56)$$

and for $d = 3$ one has

$$\Delta_{2,0} = -\varepsilon - \frac{\varepsilon^2}{3} + O(\varepsilon^3), \quad (57)$$

$$\Delta_{2,2} = \frac{\varepsilon}{5} + \frac{7\varepsilon^2}{375} + O(\varepsilon^3). \quad (58)$$

It is an easy task to show that these zero-mode results are in full agreement with results given in Eqs. (41) and (54) for $N = 2$ (see also Refs. [47,48]).

Before we shall proceed in the general investigation of the simultaneous influence of the compressibility and helicity on the anomalous scaling in the framework of the Kazantsev-Kraichnan model, let us stress that although, at first sight [see Eq. (42)], it seems that the effects of compressibility and helicity on the anomalous dimensions at the two-loop level of approximation are given by a simple sum of the contributions of the compressibility and of the helicity the dependence is in fact more complicated (nonlinear) due to the existence of the common factor $1/(d-1+\alpha)^2$ in Eq. (42) which is reduced to $1/(2+\alpha)^2$ for $d = 3$ (remind that the spatial parity violation has sense only in three-dimensional space). It means that there exists a nontrivial influence of the helicity contribution by the compressibility, namely, the importance of the contribution of the helicity decreases when compressibility of the system increases.

In addition, in the case when the presence of a large-scale uniaxial anisotropy in the system is assumed, to identify and investigate the leading contribution to the anomalous scaling of phenomenologically interesting and important quantities, e.g., of the single-time two-point correlation functions of the magnetic field defined in Eq. (22), which are the main object of interest in this paper, it is necessary first to identify and analyze various hierarchy relations which are usually valid among the critical dimensions $\Delta_{N,p}$ with different values of N and p. In our case, because $\Delta_{N,p} = \gamma_{N,p}^*$ [see Eq. (34)], the hierarchies among the critical dimensions are directly given by the corresponding hierarchies among the fixed point expressions for the anomalous dimensions. In Refs. [47,48], it was shown that in the incompressible and nonhelical case, i.e., when $\alpha = \rho = 0$, the anomalous dimensions $\gamma_{N,p}^*$ in the two-loop approximation obey the following hierarchies:

$$\gamma_{N,p}^* < \gamma_{N,p'}^*, \quad p < p', \quad (59)$$

$$\gamma_{N,0}^* < \gamma_{N',0}^*, \quad N > N', \quad (60)$$

$$\gamma_{N,1}^* < \gamma_{N',1}^*, \quad N > N', \quad (61)$$

which are valid for arbitrary spatial dimension $d \geqslant 2$ as well as for all values of ε from the interval $0 < \varepsilon < 2$. Note that hierarchy (60) is valid for even values of N and N' and (61) is valid for odd values of N and N', respectively. It means that, in the two-loop approximation and, at least, in the incompressible and nonhelical case, the asymptotic scaling behavior deep inside the inertial interval ($r/L \ll 1$) of various statistical quantities, e.g., of the single-time two-point correlation functions of the magnetic field (22), is given by the anomalous dimensions $\gamma_{N,0}^*$ for even values of N and by $\gamma_{N,1}^*$ for odd values of N. Note also that these properties of the anisotropic anomalous dimensions are in accordance with the well-known Kolmogorov's local isotropy restoration hypothesis.

The question is whether the hierarchy relations (59)–(61) remain valid when the studied system is helical and/or compressible. In this respect, it was shown in Ref. [50] that the assumption of the spatial parity violation in the incompressible

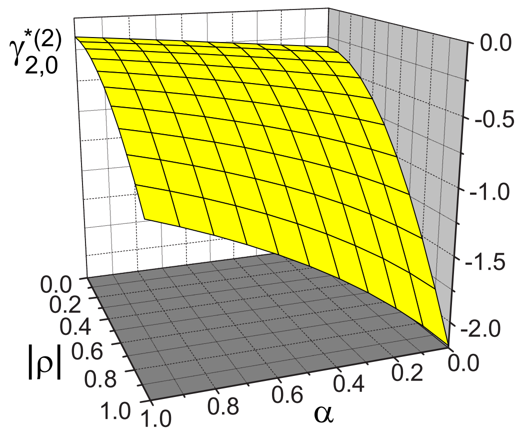

FIG. 5. Dependence of the two-loop correction $\gamma_{2,0}^{*(2)}$ on the parameters of compressibility α and helicity ρ for $d = 3$.

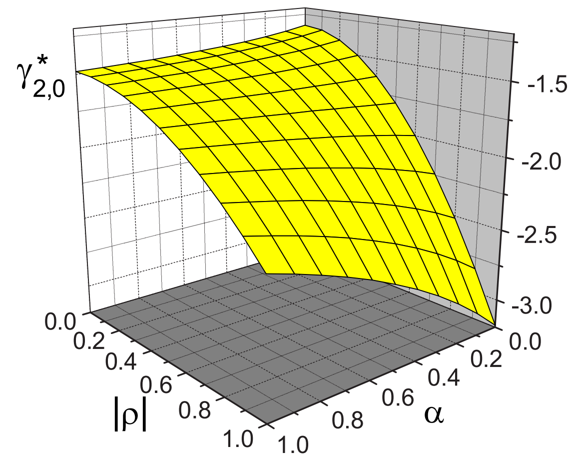

FIG. 6. Dependence of the total two-loop anomalous dimension $\gamma_{2,0}^{*}$ on the parameters of compressibility α and helicity ρ for $d = 3$ and $\varepsilon = 1$.

model does not change the anisotropic hierarchies (59)–(61). It means that even in the helical case the scaling properties of various quantities are again driven by the anomalous dimensions $\gamma_{N,0}^{*}$ for even values of N and by $\gamma_{N,1}^{*}$ for odd values of N. On the other hand, as was shown in Ref. [49], the situation in the case when the compressibility of the model is assumed can be much more complicated. It was shown in Ref. [49] that in the framework of the two-loop approximation, the anisotropic hierarchies (59)–(61) are surely fulfilled only for small enough values of the parameter of compressibility α. However, as it was discussed in Sec. II, the present model correctly describes the properties of the compressible system close enough to its incompressible limit only, i.e., only when one supposes that $\alpha \ll 1$ or at least $\alpha < 1$. In this respect, we can consider the violation of anisotropic hierarchies (59)–(61) in the model with strong enough compressibility as an artifact of the approach in the framework of which the compressibility is introduced into the model. Anyway, in what follows, we shall analyze the properties of the model only in the case with $\alpha < 1$ and relatively small values of N ($N \leqslant 7$) for which the hierarchy relations (59)–(61) are always valid for all values of ε from physically relevant interval $0 < \varepsilon < 2$.

In Figs. 5, 7, 9, 11, 13, and 15, the explicit dependence of the two-loop corrections $\gamma_{N,0}^{*(2)}$ for $N = 2$, 4, and 6 and $\gamma_{N,1}^{*(2)}$ for $N = 3$, 5, and 7 given in Eq. (42) to the corresponding leading total two-loop anomalous dimensions $\gamma_{N,0}^{*}$ and $\gamma_{N,1}^{*}$ of the composite operator $F_{N,0}$ and $F_{N,1}$, respectively, which are defined in Eq. (39), on the parameters of compressibility and helicity is shown for spatial dimension $d = 3$. All these figures show that the two-loop corrections $\gamma_{N,0}^{*(2)}$ (for even values of N) and $\gamma_{N,1}^{*(2)}$ (for odd values of N), which are negative in the incompressible nonhelical case ($\alpha = \rho = 0$), become even more negative, i.e., they decrease, when the system is helical. But, on the other hand, they increase significantly when compressibility of the system is assumed and become even positive for large enough values of α. This behavior of the two-loop corrections $\gamma_{N,0}^{*(2)}$ (for even values of N) and $\gamma_{N,1}^{*(2)}$ (for odd values of N) to the corresponding anomalous dimensions $\gamma_{N,0}^{*}$ and $\gamma_{N,1}^{*}$ with respect to the parameter α is valid regardless of the value of the parameter ρ, i.e, for all $|\rho| \in [0,1]$.

At the same time, in Figs. 6, 8, 10, 12, 14, and 16, the corresponding dependence of the total two-loop anomalous dimensions $\gamma_{N,0}^{*}$ for $N = 2$, 4, and 6 and $\gamma_{N,1}^{*}$ for $N = 3$, 5, and 7 on the parameters of compressibility and helicity is shown for $d = 3$ and $\varepsilon = 1$. As it is evident from all these figures, there is rather strong dependence of the anomalous dimensions on the parameter of helicity ρ, namely, the presence of helicity in the system leads to the significant decreasing of the leading total anomalous dimensions toward negative values and therefore to the more pronounced anomalous scaling of the correlation functions of the magnetic field (see Sec. VII). This situation is radically different from the situation which we have in the framework of the Kraichnan model of passively advected scalar quantity where the anomalous dimensions of the corresponding leading composite operators are helicity blind [41] (see also the next section), i.e., they do not depend on the helicity parameter at all (at least at the two-loop level of approximation) which leads to the fact that the scaling properties of various single-time two-point correlation functions of the passively advected scalar field are

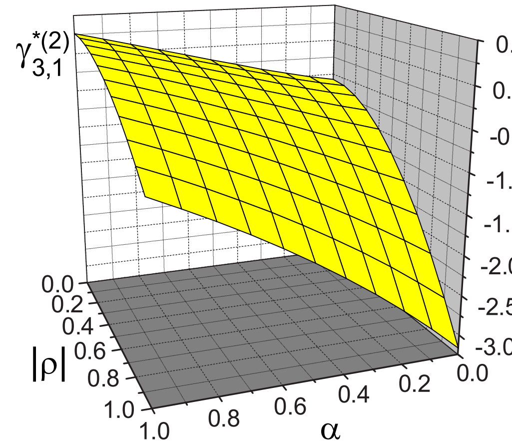

FIG. 7. Dependence of the two-loop correction $\gamma_{3,1}^{*(2)}$ on the parameters of compressibility α and helicity ρ for $d = 3$.

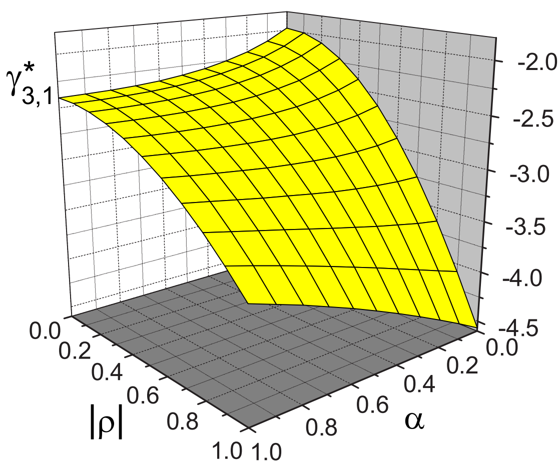

FIG. 8. Dependence of the total two-loop anomalous dimension $\gamma^*_{3,1}$ on the parameters of compressibility α and helicity ρ for $d = 3$ and $\varepsilon = 1$.

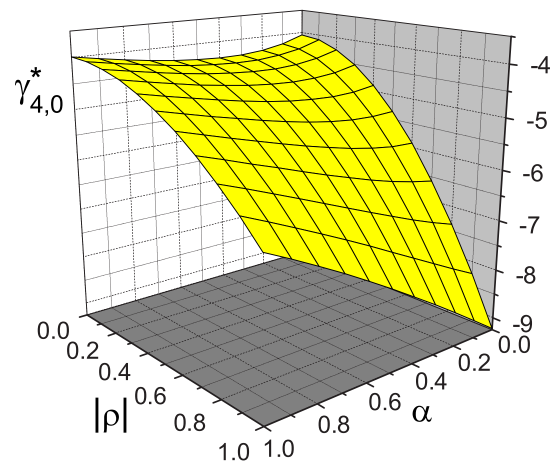

FIG. 10. Dependence of the total two-loop anomalous dimension $\gamma^*_{4,0}$ on the parameters of compressibility α and helicity ρ for $d = 3$ and $\varepsilon = 1$.

also independent of the presence of the spatial parity violation in the turbulent environment [41].

It is also evident from Figs. 6, 8, 10, 12, 14, and 16 that when the small enough compressibility of the turbulent system is assumed, i.e., when $\alpha \ll 1$, then the leading total anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) also decrease when α increases. However, this is true only for small and moderate values of the parameter $|\rho|$. On the other hand, when the spatial parity violation is large enough, i.e., when $|\rho| \sim 1$, the situation is opposite, namely, even small compressibility of the system increases the values of the helical two-loop leading total anomalous dimensions. It is also important to stress that while the small compressibility ($\alpha \ll 1$) of the system leads to the decreasing of the leading two-loop total anomalous dimensions (at least in the nonhelical case $\rho = 0$ as well as in the cases when the spatial parity violation is small enough), the compressibility seriously increases the values of the total two-loop anomalous dimensions independently of the value of the helicity parameter starting from $N = 4$ (see Figs. 10, 12, 14, and 16). As for the leading total two-loop anomalous dimensions $\gamma^*_{2,0}$ and $\gamma^*_{3,1}$, the situation is a little bit different (see Figs. 6 and 8), namely, here the compressibility of the environment decreases the values of these anomalous dimensions for all values of α from the interval $0 \leqslant \alpha \leqslant 1$ for small enough absolute values of the parameter ρ.

The simultaneous dependence of the total two-loop anomalous dimensions $\gamma^*_{N,0}$ (for even values of $N = 2$, 4, and 6) and $\gamma^*_{N,1}$ (for odd values of $N = 3$, 5, and 7) on the parameter of compressibility α and on the spatial dimension $d \geqslant 2$ is shown in Figs. 17–22 for $\rho = 0$ (let us note once more that the helicity of the system has sense only for $d = 3$) and $\varepsilon = 1$. As it follows from these figures, the leading total two-loop anomalous dimensions increase with increasing α for $N > 2$ for spatial dimension $d = 2$ (for $N = 2$ it is a constant, namely, $\gamma^*_{2,0} = -1$ for $d = 2$). As for the three-dimensional case, the situation is more complicated as was already discussed above, namely, for small enough values of the parameter of compressibility α all

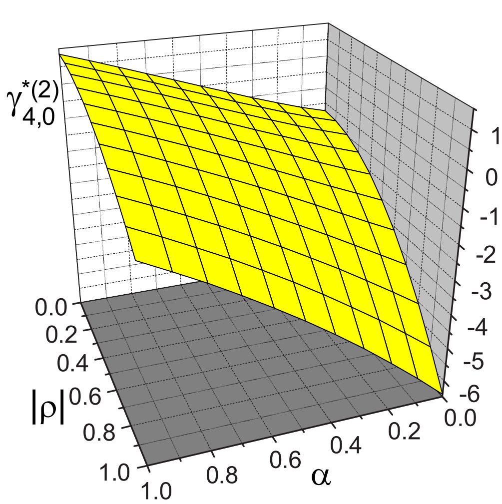

FIG. 9. Dependence of the two-loop correction $\gamma^{*(2)}_{4,0}$ on the parameters of compressibility α and helicity ρ for $d = 3$.

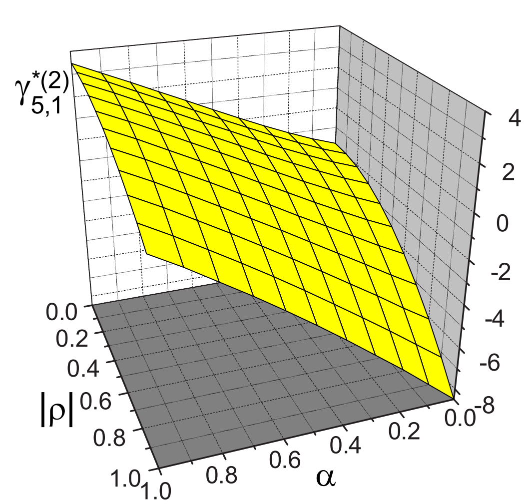

FIG. 11. Dependence of the two-loop correction $\gamma^{*(2)}_{5,1}$ on the parameters of compressibility α and helicity ρ for $d = 3$.

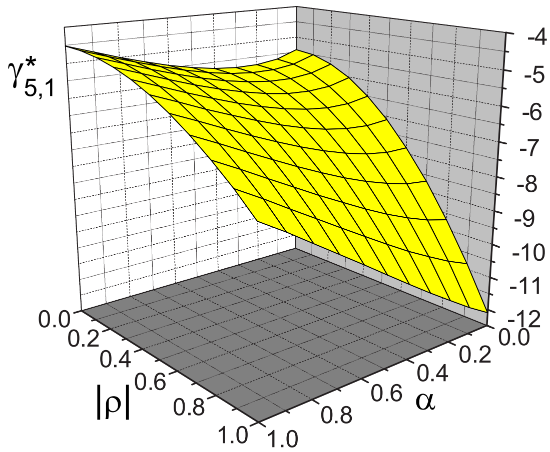

FIG. 12. Dependence of the total two-loop anomalous dimension $\gamma^*_{5,1}$ on the parameters of compressibility α and helicity ρ for $d = 3$ and $\varepsilon = 1$.

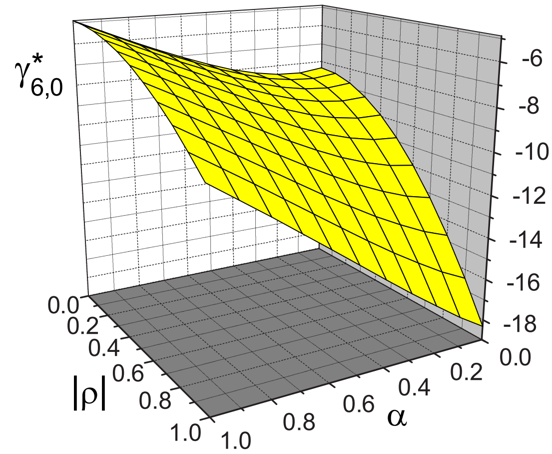

FIG. 14. Dependence of the total two-loop anomalous dimension $\gamma^*_{6,0}$ on the parameters of compressibility α and helicity ρ for $d = 3$ and $\varepsilon = 1$.

leading total two-loop anomalous dimensions decrease when α increases. However, for given value $N \geqslant 4$ there exists the value α from the interval $0 < \alpha < 1$ starting from which the corresponding leading total two-loop anomalous dimension increases. On the other hand, the leading total two-loop anomalous dimensions decrease with increasing $\alpha < 1$ for all spatial dimensions $d \geqslant 4$ (at least up to $N = 7$).

VI. COMPARISON OF THE KAZANTSEV-KRAICHNAN MODEL WITH THE KRAICHNAN MODEL OF PASSIVELY ADVECTED SCALAR FIELD

It is also instructive to investigate the properties of the anomalous dimensions of the composite operators (28) which play the central role in the investigation of the anomalous scaling of the correlation functions of the magnetic field in the framework of the Kazantsev-Kraichnan model studied in the previous section in comparison to the corresponding two-loop anomalous dimensions of the important composite operators in the Kraichnan model of a passively advected scalar quantity [35,36,43].

In contrast to the Kazantsev-Kraichnan model of the passive magnetic field in the framework of which the central role in the investigation of the anomalous scaling properties of the model is played by the composite operators (28), i.e., by the operators built directly from the magnetic field, in the framework of the Kraichnan model of a passively advected scalar field the crucial role in analysis of the anomalous scaling is played by the composite operators built solely of the gradients of the scalar field $\theta(x)$ [35], namely,

$$F'_{N,p} = \partial_{i_1}\theta \dots \partial_{i_p}\theta(\partial\theta \cdot \partial\theta)^l, \quad N = 2l + p. \tag{62}$$

Note that the anomalous dimensions of these operators are known up to the three-loop approximation but only in the incompressible and nonhelical case [36]. It is also known that the anomalous dimensions of the operators (62) in the framework of the Kraichnan model do not depend on the helicity at least up to the two-loop approximation [41]. In

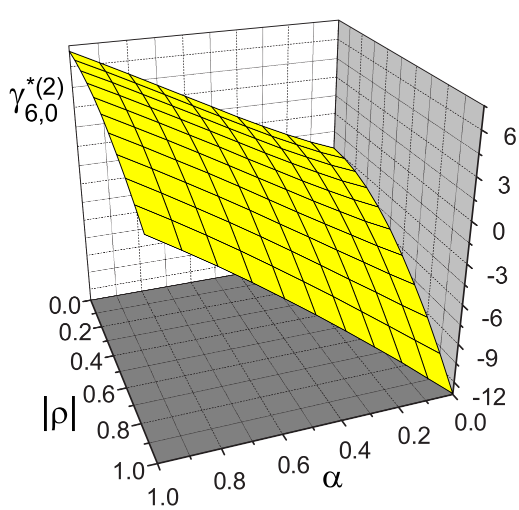

FIG. 13. Dependence of the two-loop correction $\gamma^{*(2)}_{6,0}$ on the parameters of compressibility α and helicity ρ for $d = 3$.

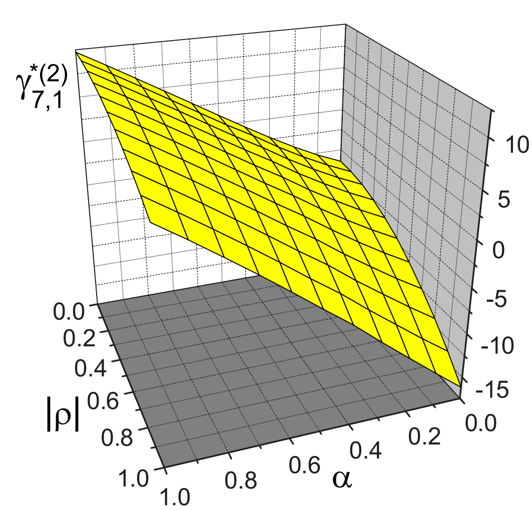

FIG. 15. Dependence of the two-loop correction $\gamma^{*(2)}_{7,1}$ on the parameters of compressibility α and helicity ρ for $d = 3$.

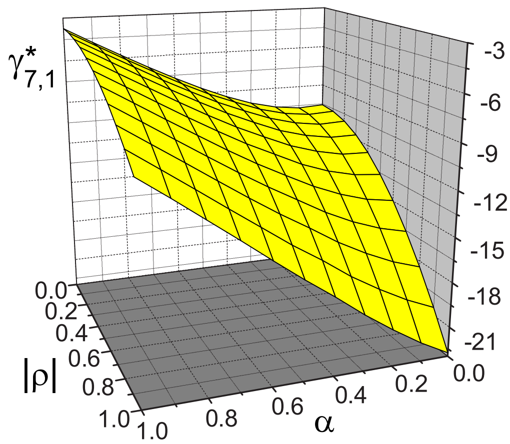

FIG. 16. Dependence of the total two-loop anomalous dimension $\gamma^*_{7,1}$ on the parameters of compressibility α and helicity ρ for $d = 3$ and $\varepsilon = 1$.

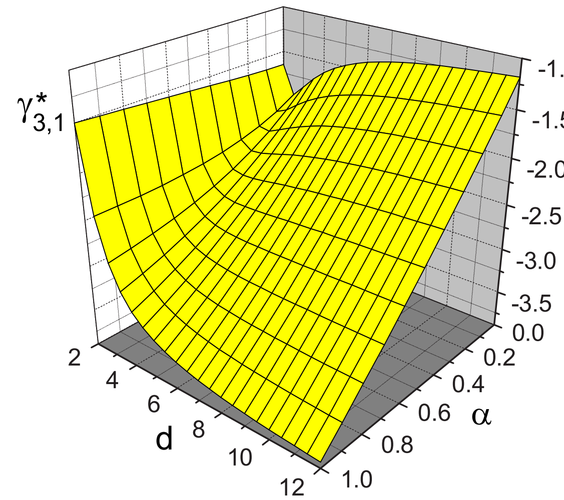

FIG. 18. Dependence of the total two-loop anomalous dimension $\gamma^*_{3,1}$ on the parameter of compressibility α and on the spatial dimension d for $\rho = 0$ and $\varepsilon = 1$.

addition, the dependence of the two-loop anomalous dimensions of the composite operators (62) on the compressibility of the turbulent environment in the framework of the Kraichnan model was studied in Ref. [43].

The general two-loop expressions for the anomalous dimensions of the operators (62) can be written in analogy with Eq. (39), namely,

$$\gamma'^{*}_{N,p} = \gamma'^{*(1)}_{N,p}\varepsilon + \gamma'^{*(2)}_{N,p}\varepsilon^2, \tag{63}$$

where we have used the prime to denote the fact that the expressions are related to the composite operators (62) in the framework of the Kraichnan model describing passive advection of a scalar quantity. Here, the one-loop contribution $\gamma'^{*(1)}_{N,p}$ has the following form (see, e.g., Ref. [43] for details):

$$\begin{aligned}\gamma'^{*(1)}_{N,p} = &-\{(N-p)(d+N+p-2)[(d+1)(d-1+\alpha)\\ &-2\alpha] + 2N(N-1)(d-1)(\alpha-1)\}/\\ &[2(d+2)(d-1)(d-1+\alpha)],\end{aligned} \tag{64}$$

which depends explicitly on the compressibility parameter α. In the incompressible case $\alpha = 0$ one obtains

$$\begin{aligned}\gamma'^{*(1)}_{N,p} = &-\{(N-p)(d+N+p-2)(d+1)\\ &-2N(N-1)\}/[2(d+2)(d-1)].\end{aligned} \tag{65}$$

It is evident that the expression (65) is equal to the corresponding expression for the one-loop contribution $\gamma^{*(1)}_{N,p}$ to the anomalous dimensions $\gamma_{N,p}$ for the composite operators (28) given in Eq. (41), i.e., we have

$$\gamma^{*(1)}_{N,p} = \gamma'^{*(1)}_{N,p} \tag{66}$$

for $\alpha = 0$. However, this equality between the anomalous dimensions of the two sets of completely different composite operators in the Kraichnan and Kazantsev-Kraichnan model, respectively, is valid only at the one-loop level of approximation in the incompressible case. As it was shown in Refs. [47,48], this "universality" of the anomalous dimensions

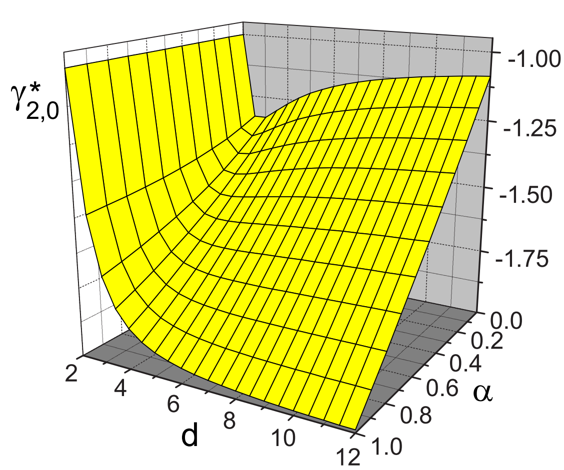

FIG. 17. Dependence of the total two-loop anomalous dimension $\gamma^*_{2,0}$ on the parameter of compressibility α and on the spatial dimension d for $\rho = 0$ and $\varepsilon = 1$.

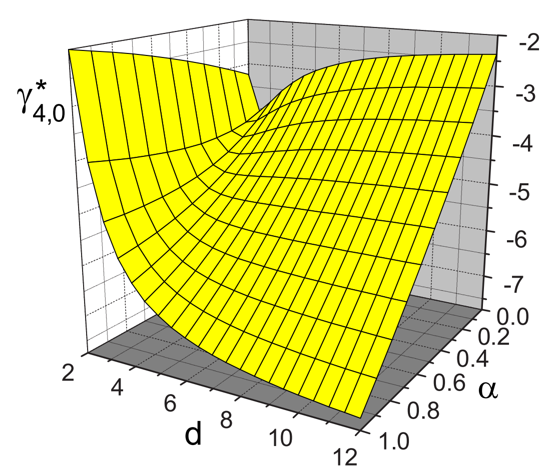

FIG. 19. Dependence of the total two-loop anomalous dimension $\gamma^*_{4,0}$ on the parameter of compressibility α and on the spatial dimension d for $\rho = 0$ and $\varepsilon = 1$.

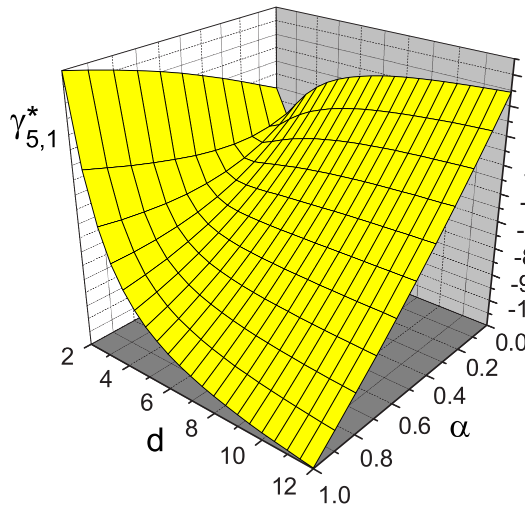

FIG. 20. Dependence of the total two-loop anomalous dimension $\gamma^*_{5,1}$ on the parameter of compressibility α and on the spatial dimension d for $\rho = 0$ and $\varepsilon = 1$.

is destroyed starting from the second-order approximation even in the incompressible case. As was already mentioned, the two-loop corrections $\gamma'^{*(2)}_{N,p}$ to the anomalous dimensions of the composite operators (62) in the framework of the Kraichnan model with compressibility was found in Ref. [43]. Here, we present them in a little bit different representation, namely, in an integral representation, which is suitable for direct comparison with the corresponding two-loop contributions to the anomalous dimensions of the composite operators (28) in the framework of the Kazentsev-Kraichnan model. Thus, in the integral representation we have

$$\gamma'^{*(2)}_{N,p} = -\frac{S_{d-1}}{S_d}\frac{2d}{(d-1)(d+2)(d-1+\alpha)^2} \times \int_0^1 dx(1-x^2)^{\frac{d-3}{2}}\{C_2W'_2Y_1 + (C_3W'_3 + 2C_4W'_4)Y_2/(d+4)\}, \quad (67)$$

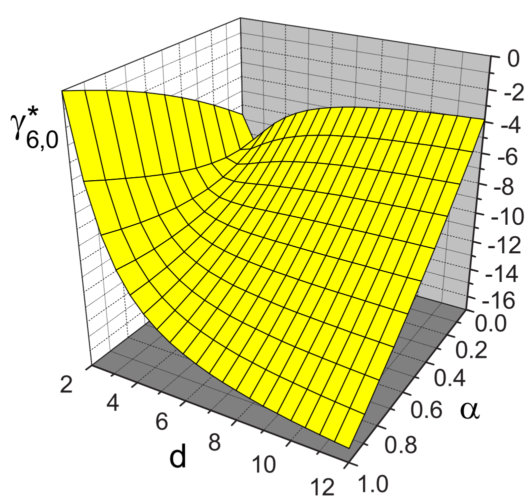

FIG. 21. Dependence of the total two-loop anomalous dimension $\gamma^*_{6,0}$ on the parameter of compressibility α and on the spatial dimension d for $\rho = 0$ and $\varepsilon = 1$.

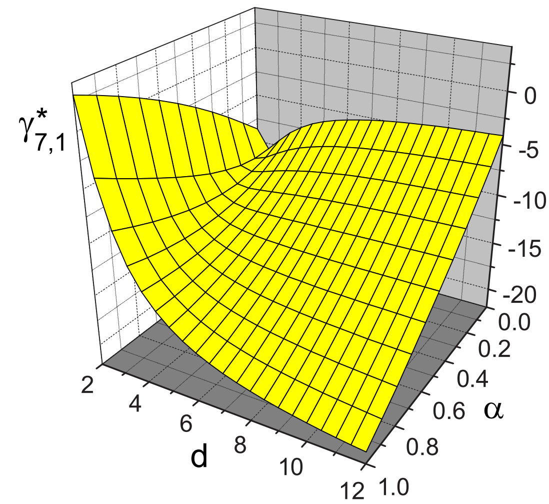

FIG. 22. Dependence of the total two-loop anomalous dimension $\gamma^*_{7,1}$ on the parameter of compressibility α and on the spatial dimension d for $\rho = 0$ and $\varepsilon = 1$.

where C_2, C_3, and C_4 are given in Eqs. (44)–(46), Y_1 and Y_2 are defined in Eqs. (51) and (52), respectively, and

$$W'_2 = \sqrt{1-x^2}[1 - x^2 - \alpha(1-2x^2) - \alpha^2x^2], \quad (68)$$

$$W'_3 = (1-x^2)(7 - 3d - 4x^2) - \alpha[15 - 26x^2 + 8x^4 + d(2+x^2)] - \alpha^2(2 + 11x^2 - 4x^4), \quad (69)$$

and

$$W'_4 = (1-x^2)^2 - 2\alpha(1-x^2)(2-x^2) - \alpha^2(2 + 2x^2 - x^4). \quad (70)$$

Looking at the expressions given in Eqs. (42) and (67), it is clear that at the two-loop level of approximation the anomalous dimensions for composite operators (28) in the framework of the Kazantsev-Kraichnan model of a passively advected vector (magnetic) field and the anomalous dimensions of the composite operators (62) in the framework of the Kraichnan model of a passively advected scalar field are different even in the incompressible and nonhelical case as it was discussed in detail in Ref. [48].

Here, we shall concentrate our attention on the behavior of the one- and the two-loop expressions for the anomalous dimensions $\gamma^*_{N,p}$ and $\gamma'^*_{N,p}$ as functions of the compressibility parameter α. In this respect, in Figs. 23–28 the explicit dependence of the leading one-loop and total two-loop anomalous dimensions, i.e., $\gamma^{*(1)}_{N,0}\varepsilon$, $\gamma'^{*(1)}_{N,0}\varepsilon$, $\gamma^*_{N,0}$, and $\gamma'^*_{N,0}$ for even values of N and $\gamma^{*(1)}_{N,1}\varepsilon$, $\gamma'^{*(1)}_{N,1}\varepsilon$, $\gamma^*_{N,1}$, and $\gamma'^*_{N,1}$ for odd values of N, on the parameter α is shown for $d = 3$, $\rho = 0$, and $\varepsilon = 1$ and for values of N up to $N = 7$. Let us note also that in the case with $\varepsilon = 1$, the one-loop anomalous dimensions $\gamma^{*(1)}_{N,p}\varepsilon$ and $\gamma'^{*(1)}_{N,p}\varepsilon$ are equal directly to $\gamma^{*(1)}_{N,p}$ and $\gamma'^{*(1)}_{N,p}$, respectively, for arbitrary values of N and p. Therefore, for simplicity, in Figs. 23–28 we have written $\gamma^{*(1)}_{N,p}$ and $\gamma'^{*(1)}_{N,p}$ with corresponding N and p instead of more correct expressions $\gamma^{*(1)}_{N,p}\varepsilon$ and $\gamma'^{*(1)}_{N,p}\varepsilon$.

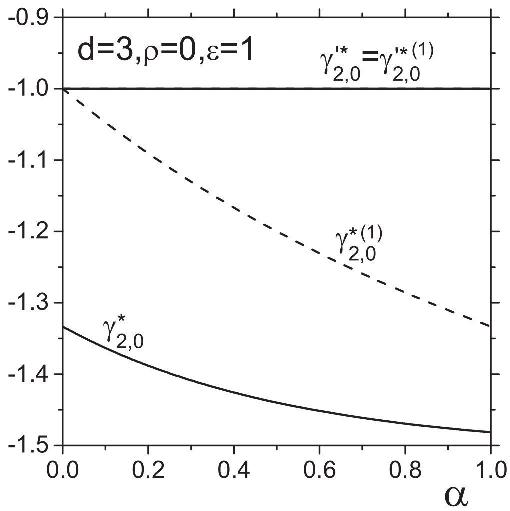

FIG. 23. Dependence of the one-loop anomalous dimensions $\gamma_{2,0}^{*(1)}\varepsilon$ and $\gamma_{2,0}^{\prime*(1)}\varepsilon$ and total two-loop anomalous dimensions $\gamma_{2,0}^{*}$ and $\gamma_{2,0}^{\prime*}$ on the compressibility parameter α for $d=3,\ \rho=0$, and $\varepsilon=1$, i.e., here $\gamma_{2,0}^{*(1)}\varepsilon\equiv\gamma_{2,0}^{*(1)}$ and $\gamma_{2,0}^{\prime*(1)}\varepsilon\equiv\gamma_{2,0}^{\prime*(1)}$.

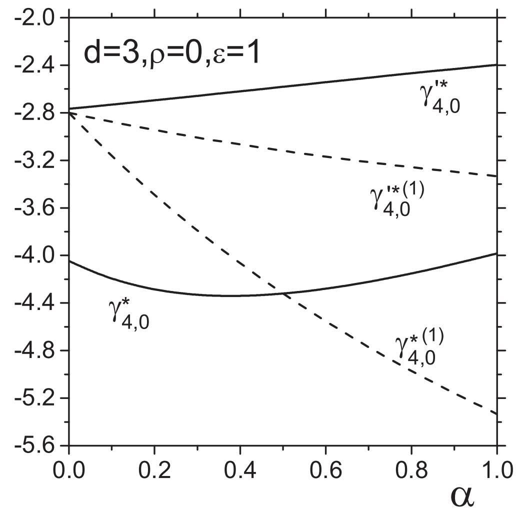

FIG. 25. Dependence of the one-loop anomalous dimensions $\gamma_{4,0}^{*(1)}\varepsilon$ and $\gamma_{4,0}^{\prime*(1)}\varepsilon$ and total two-loop anomalous dimensions $\gamma_{4,0}^{*}$ and $\gamma_{4,0}^{\prime*}$ on the compressibility parameter α for $d=3,\ \rho=0$, and $\varepsilon=1$, i.e., here $\gamma_{4,0}^{*(1)}\varepsilon\equiv\gamma_{4,0}^{*(1)}$ and $\gamma_{4,0}^{\prime*(1)}\varepsilon\equiv\gamma_{4,0}^{\prime*(1)}$.

Now, looking at Figs. 23–28 one can immediately see rather significant difference between behavior of the leading anomalous dimensions in the Kraichnan model and in the Kazantsev-Kraichnan model in the one-loop as well as in the two-loop level of approximation in the most interesting case (at least from the physical point of view) with $d=3$. First of all, as it follows from Eq. (66), the leading one-loop anomalous dimensions $\gamma_{N,0}^{*(1)}\varepsilon$ for even N and $\gamma_{N,1}^{*(1)}\varepsilon$ for odd N are equal to $\gamma_{N,0}^{\prime*(1)}\varepsilon$ and $\gamma_{N,1}^{\prime*(1)}\varepsilon$, respectively, in the incompressible case, i.e., when $\alpha=0$. However, the compressibility of the corresponding turbulent environments destroys this universality of the anomalous dimensions even at the one-loop level of approximation (see the dashed curves in Figs. 23–28) and it is evident that the dependence of the leading one-loop anomalous dimensions $\gamma_{N,0}^{*(1)}\varepsilon$ and $\gamma_{N,1}^{*(1)}\varepsilon$ on the compressibility parameter α in the framework of the Kazantsev-Kraichnan model is much more pronounced than in the case of the anomalous dimensions $\gamma_{N,0}^{\prime*(1)}\varepsilon$ and $\gamma_{N,1}^{\prime*(1)}\varepsilon$ in the framework of the Kraichnan model of a scalar admixture. More precisely, the leading one-loop anomalous dimensions for the Kraichnan model as well as the corresponding leading one-loop anomalous dimensions for the Kazantsev-Kraichnan model decrease (they become more negative) when the parameter of compressibility α increases [except for the anomalous dimension $\gamma_{2,0}^{\prime*(1)}\varepsilon$ which remains constant (see Fig. 23)], however, the decreasing of the anomalous dimensions $\gamma_{N,0}^{*(1)}\varepsilon$ and $\gamma_{N,1}^{*(1)}\varepsilon$ of the vector model

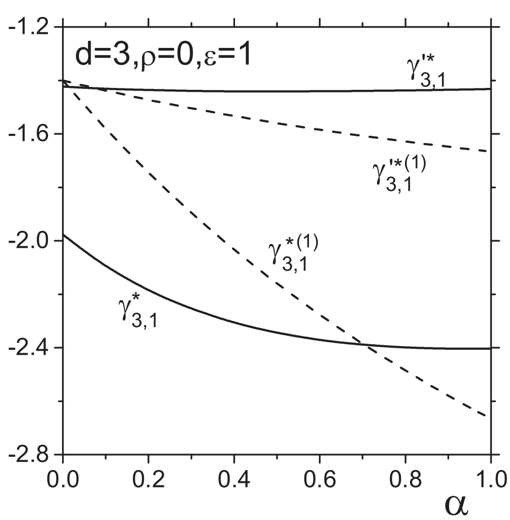

FIG. 24. Dependence of the one-loop anomalous dimensions $\gamma_{3,1}^{*(1)}\varepsilon$ and $\gamma_{3,1}^{\prime*(1)}\varepsilon$ and total two-loop anomalous dimensions $\gamma_{3,1}^{*}$ and $\gamma_{3,1}^{\prime*}$ on the compressibility parameter α for $d=3,\ \rho=0$, and $\varepsilon=1$, i.e., here $\gamma_{3,1}^{*(1)}\varepsilon\equiv\gamma_{3,1}^{*(1)}$ and $\gamma_{3,1}^{\prime*(1)}\varepsilon\equiv\gamma_{3,1}^{\prime*(1)}$.

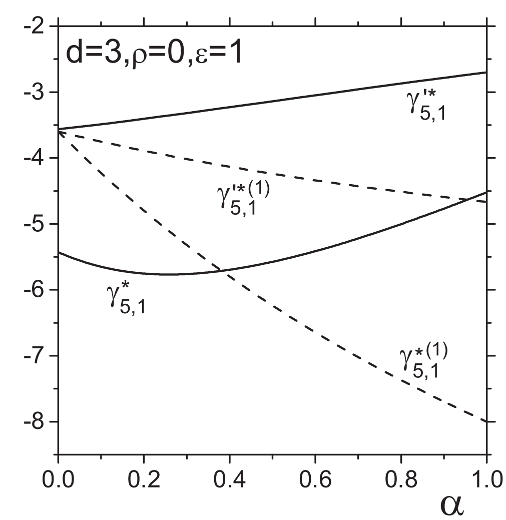

FIG. 26. Dependence of the one-loop anomalous dimensions $\gamma_{5,1}^{*(1)}\varepsilon$ and $\gamma_{5,1}^{\prime*(1)}\varepsilon$ and total two-loop anomalous dimensions $\gamma_{5,1}^{*}$ and $\gamma_{5,1}^{\prime*}$ on the compressibility parameter α for $d=3,\ \rho=0$, and $\varepsilon=1$, i.e., here $\gamma_{5,1}^{*(1)}\varepsilon\equiv\gamma_{5,1}^{*(1)}$ and $\gamma_{5,1}^{\prime*(1)}\varepsilon\equiv\gamma_{5,1}^{\prime*(1)}$.

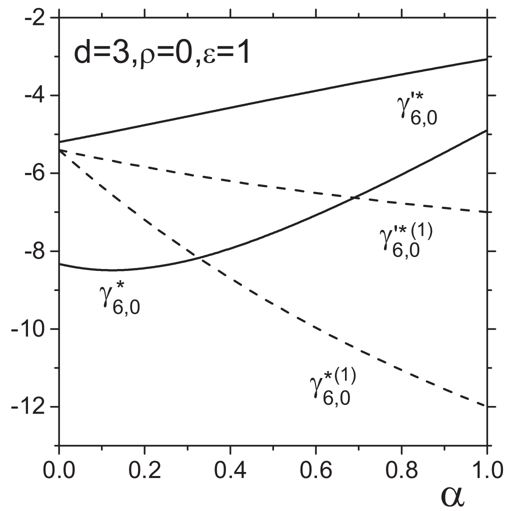

FIG. 27. Dependence of the one-loop anomalous dimensions $\gamma_{6,0}^{*(1)}\varepsilon$ and $\gamma_{6,0}^{\prime*(1)}\varepsilon$ and total two-loop anomalous dimensions $\gamma_{6,0}^{*}$ and $\gamma_{6,0}^{\prime*}$ on the compressibility parameter α for $d=3$, $\rho=0$, and $\varepsilon=1$, i.e., here $\gamma_{6,0}^{*(1)}\varepsilon \equiv \gamma_{6,0}^{*(1)}$ and $\gamma_{6,0}^{\prime*(1)}\varepsilon \equiv \gamma_{6,0}^{\prime*(1)}$.

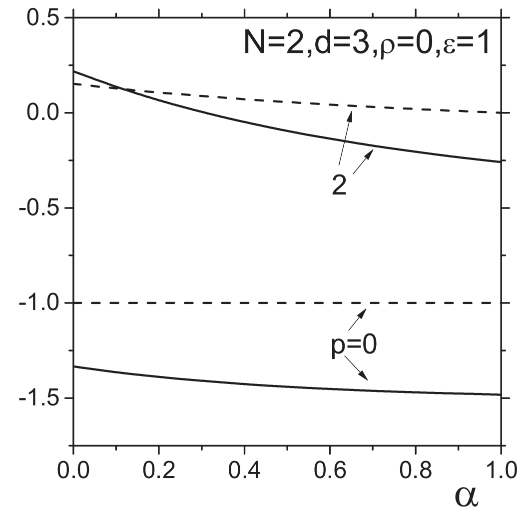

FIG. 29. Dependence of the total two-loop anomalous dimensions $\gamma_{N,p}^{*}$ (solid curves) and $\gamma_{N,p}^{\prime*}$ (dashed curves) for $N=2$ and $p=0,2$ on the compressibility parameter α for $d=3$, $\rho=0$, and $\varepsilon=1$.

as function of α is considerably faster than the decreasing of the anomalous dimensions $\gamma_{N,0}^{\prime*(1)}\varepsilon$ and $\gamma_{N,1}^{\prime*(1)}\varepsilon$ important in the case of the scalar admixture.

However, the fundamental difference between the leading anomalous dimensions of the composite operators (28) and (62) becomes apparent at the two-loop level of approximation. As it follows from Figs. 23–28, there is an essential difference between the leading two-loop anomalous dimensions $\gamma_{N,0}^{*}$ and $\gamma_{N,1}^{*}$, which are crucial for the analysis of the Kazantsev-Kraichnan model, and the corresponding two-loop anomalous dimensions $\gamma_{N,0}^{\prime*}$ and $\gamma_{N,1}^{\prime*}$, which play the central role in the analysis of scaling properties in the framework of the Kraichnan model, even in the incompressible case. At the same time, when the compressibility of the corresponding turbulent systems is considered, while the two-loop anomalous dimensions $\gamma_{N,0}^{\prime*}$ and $\gamma_{N,1}^{\prime*}$ given in Eq. (63) are increasing functions of α [again except for the two-loop anomalous dimension $\gamma_{2,0}^{\prime*}$ which remains constant (see Fig. 23) and except for $\gamma_{3,1}^{\prime*}$ which slightly decreases as the function of α for small enough values of α] the leading two-loop anomalous dimensions $\gamma_{N,0}^{*}$ and $\gamma_{N,1}^{*}$ are decreasing functions of α, at least for small enough values of α ($\alpha \ll 1$), for all values of $N \geqslant 2$. Note that for given value of N and for large enough values of α, the two-loop anomalous dimensions $\gamma_{N,0}^{*}$ and $\gamma_{N,1}^{*}$ become increasing functions of α as in the case of the two-loop anomalous dimensions $\gamma_{N,0}^{\prime*}$ and $\gamma_{N,1}^{\prime*}$. However, this behavior cannot be considered as decisive for us because, as was discussed in Sec. II, the physically relevant region of the compressibility parameter is $\alpha \ll 1$.

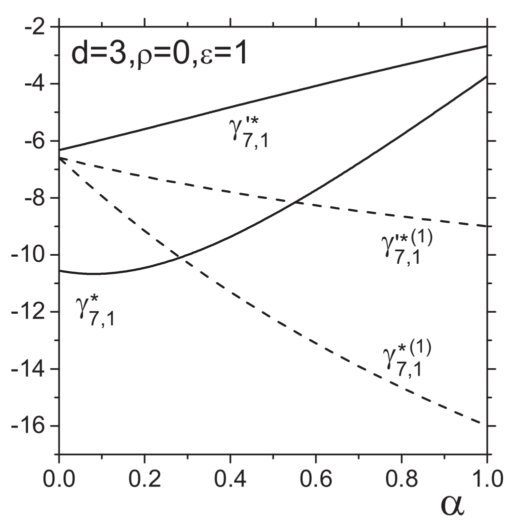

FIG. 28. Dependence of the one-loop anomalous dimensions $\gamma_{7,1}^{*(1)}\varepsilon$ and $\gamma_{7,1}^{\prime*(1)}\varepsilon$ and total two-loop anomalous dimensions $\gamma_{7,1}^{*}$ and $\gamma_{7,1}^{\prime*}$ on the compressibility parameter α for $d=3$, $\rho=0$, and $\varepsilon=1$, i.e., here $\gamma_{7,1}^{*(1)}\varepsilon \equiv \gamma_{7,1}^{*(1)}$ and $\gamma_{7,1}^{\prime*(1)}\varepsilon \equiv \gamma_{7,1}^{\prime*(1)}$.

It is also instructive to demonstrate graphically the validity of the anisotropy hierarchies (59)–(61) among the two-loop anomalous dimensions (39) of the composite operators (28) in the compressible case and compare them with the corresponding hierarchies which are valid among the two-loop anomalous dimensions (63) of the composite operators (62). The validity of the hierarchies is shown explicitly in Figs. 29–34 for various values of N up to $N=7$. From all these figures it is evident that the anisotropy hierarchies (59)–(61) discussed in the previous section are really valid (at least when the parameter of compressibility is not too large, i.e., $\alpha < 1$). It means that the main role in the analysis of the scaling properties of the correlation functions of the magnetic field (see the next section) will be really played by the anomalous dimensions $\gamma_{N,0}^{*}$ for even values of N and by $\gamma_{N,1}^{*}$ for odd values of N in accordance with the incompressible case studied in Refs. [47,48].

Finally, for completeness, let us also show explicitly the simultaneous dependence of the leading two-loop anomalous dimensions $\gamma_{N,0}^{\prime*}$ for even N and $\gamma_{N,1}^{\prime*}$ for odd N of the scalar Kraichnan problem on the spatial dimension d and on the

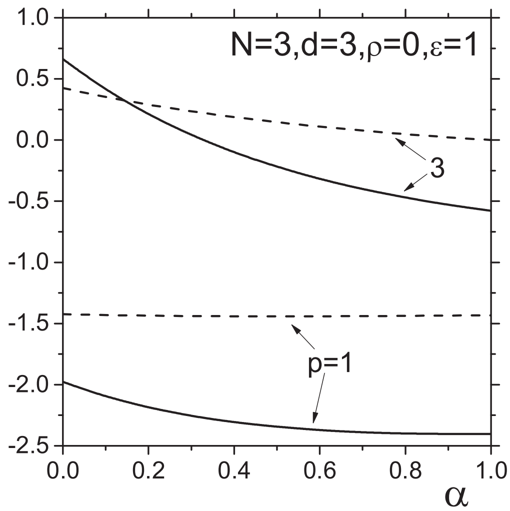

FIG. 30. Dependence of the total two-loop anomalous dimensions $\gamma^*_{N,p}$ (solid curves) and $\gamma'^*_{N,p}$ (dashed curves) for $N = 3$ and $p = 1,3$ on the compressibility parameter α for $d = 3$, $\rho = 0$, and $\varepsilon = 1$.

compressibility parameter α (remind that they do not depend at all on the helicity of the turbulent environment, i.e., on the parameter ρ) to compare them to the corresponding behavior of the two-loop anomalous dimensions $\gamma^*_{N,0}$ and $\gamma^*_{N,1}$ shown in Figs. 17–22 for various N up to $N = 7$. The behavior of the two-loop anomalous dimensions $\gamma'^*_{N,0}$ for even N and $\gamma'^*_{N,1}$ for odd N, respectively, as functions of d and α is shown in Figs. 35–40 for $N = 1,\ldots,7$. Direct comparison of Figs. 35–40 with the corresponding Figs. 17–22 again demonstrates an essential difference in the behavior of the leading two-loop anomalous dimensions $\gamma'^*_{N,0}$ and $\gamma'^*_{N,1}$ of the composite operators (62) in comparison with the leading two-loop anomalous dimensions $\gamma^*_{N,0}$ and $\gamma^*_{N,1}$ of the composite operators (28).

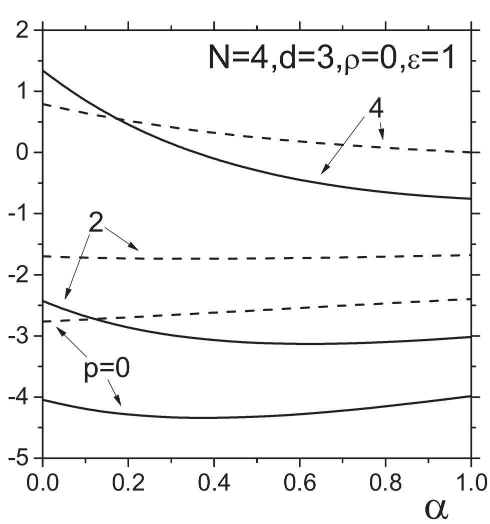

FIG. 31. Dependence of the total two-loop anomalous dimensions $\gamma^*_{N,p}$ (solid curves) and $\gamma'^*_{N,p}$ (dashed curves) for $N = 4$ and $p = 0,2,4$ on the compressibility parameter α for $d = 3$, $\rho = 0$, and $\varepsilon = 1$.

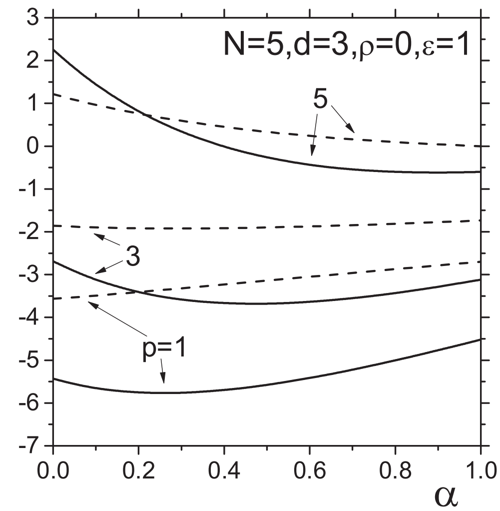

FIG. 32. Dependence of the total two-loop anomalous dimensions $\gamma^*_{N,p}$ (solid curves) and $\gamma'^*_{N,p}$ (dashed curves) for $N = 5$ and $p = 1,3,5$ on the compressibility parameter α for $d = 3$, $\rho = 0$, and $\varepsilon = 1$.

The most visible difference is observed between Figs. 17 and 35 for $N = 2$ and $p = 0$. As it follows from Fig. 35, the two-loop anomalous dimension $\gamma'^*_{2,0}$ does not depend at all on the compressibility as well as on the value of the spatial dimension of the turbulent system studied in the framework of the Kraichnan model of a passively advected scalar quantity. On the other hand, the situation is completely different in the case of the two-loop anomalous dimension $\gamma^*_{2,0}$ which strongly depends on the parameter of compressibility α as well as on the spatial dimension d (see Fig. 17). Let us note, however, that for the spatial dimension $d = 2$ the anomalous dimension $\gamma^*_{2,0}$ is independent of α, i.e., it is a constant, and is equal to $\gamma'^*_{2,0}$. This observation demonstrates the known fact that in two dimensions the magnetic field behaves formally as a scalar

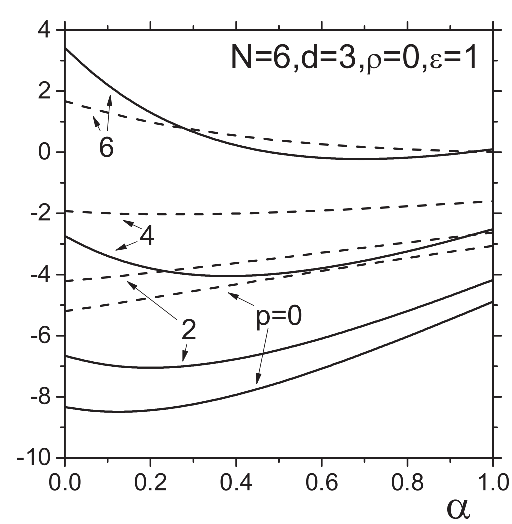

FIG. 33. Dependence of the total two-loop anomalous dimensions $\gamma^*_{N,p}$ (solid curves) and $\gamma'^*_{N,p}$ (dashed curves) for $N = 6$ and $p = 0,2,4,6$ on the compressibility parameter α for $d = 3$, $\rho = 0$, and $\varepsilon = 1$.

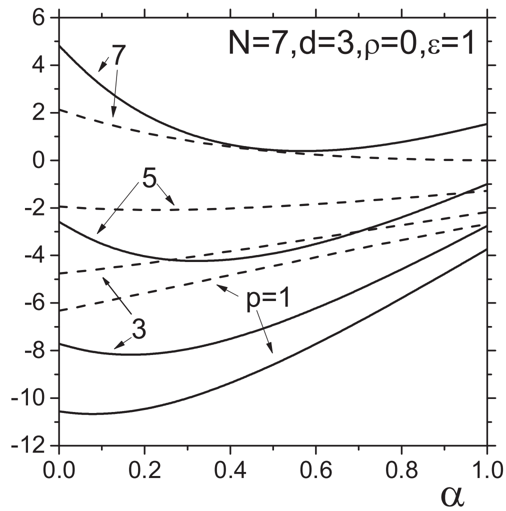

FIG. 34. Dependence of the total two-loop anomalous dimensions $\gamma^*_{N,p}$ (solid curves) and $\gamma'^*_{N,p}$ (dashed curves) for $N = 7$ and $p = 1,3,5,7$ on the compressibility parameter α for $d = 3$, $\rho = 0$, and $\varepsilon = 1$.

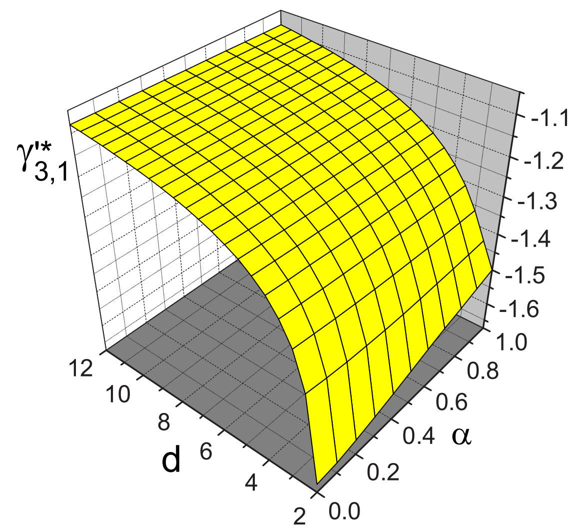

FIG. 36. Dependence of the total two-loop anomalous dimension $\gamma'^*_{3,1}$ on the parameter of compressibility α and on the spatial dimension d for $\varepsilon = 1$.

field, the fact which is represented here by equality of the corresponding anomalous dimensions and therefore is valid for all anomalous dimensions with arbitrary values of N and p, i.e.,

$$\gamma^*_{N,p}(\alpha) = \gamma'^*_{N,p}(\alpha), \quad d = 2. \tag{71}$$

There is also a significant difference in the behavior of the anomalous dimensions $\gamma^*_{3,1}$ (Fig. 18) and $\gamma'^*_{3,1}$ (Fig. 36), namely, while the anomalous dimension $\gamma'^*_{3,1}$ is strictly increasing function as the function of the spatial dimension d for arbitrary value of $0 \leqslant \alpha \leqslant 1$ the behavior of the anomalous dimension $\gamma^*_{3,1}$ is much more complicated (see Fig. 18). Of course, it is easy to check that $\gamma^*_{3,1} = \gamma'^*_{3,1}$ for $d = 2$ in accordance with general relation (71).

Further, starting from $N \geqslant 4$ the behavior of the anomalous dimensions $\gamma'^*_{N,0}$ for even N and $\gamma'^*_{N,1}$ for odd N becomes qualitatively very similar as is demonstrated in Figs. 37–40 for $N = 4$, 5, 6, and 7 but different from the "universal" behavior of the corresponding anomalous dimensions $\gamma^*_{N,0}$ and $\gamma^*_{N,1}$ demonstrated in Figs. 19–22. In this case, maybe the most visible difference between the behavior of the anomalous dimensions $\gamma'^*_{N,0}$ and $\gamma'^*_{N,1}$ and the anomalous dimensions $\gamma^*_{N,0}$ and $\gamma^*_{N,1}$ is the fact that while the anomalous dimensions $\gamma'^*_{N,0}$ and $\gamma'^*_{N,1}$ weakly depend on the parameter of compressibility for large enough values of the spatial dimension (in the limit $d \to \infty$ they become constants), the dependence of the anomalous dimensions $\gamma^*_{N,0}$ and $\gamma^*_{N,1}$ on the parameter α become very strong for large enough values of d, namely, they become linearly decreasing functions of the parameter α

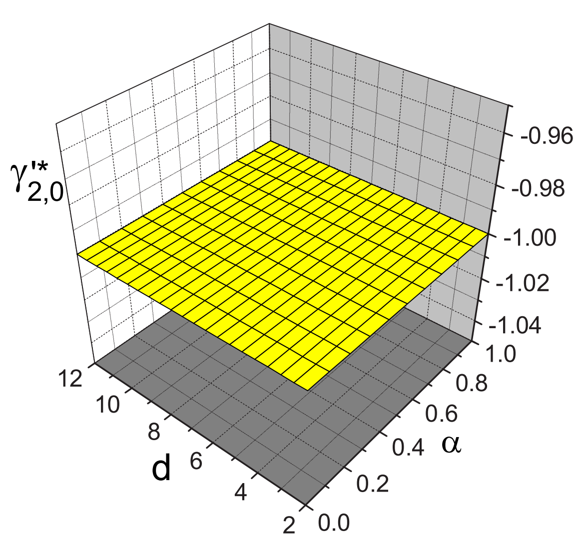

FIG. 35. Dependence of the total two-loop anomalous dimension $\gamma'^*_{2,0}$ on the parameter of compressibility α and on the spatial dimension d for $\varepsilon = 1$.

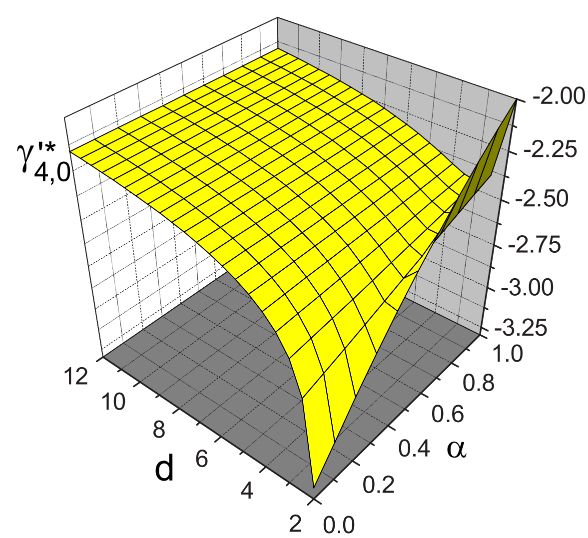

FIG. 37. Dependence of the total two-loop anomalous dimension $\gamma'^*_{4,0}$ on the parameter of compressibility α and on the spatial dimension d for $\varepsilon = 1$.

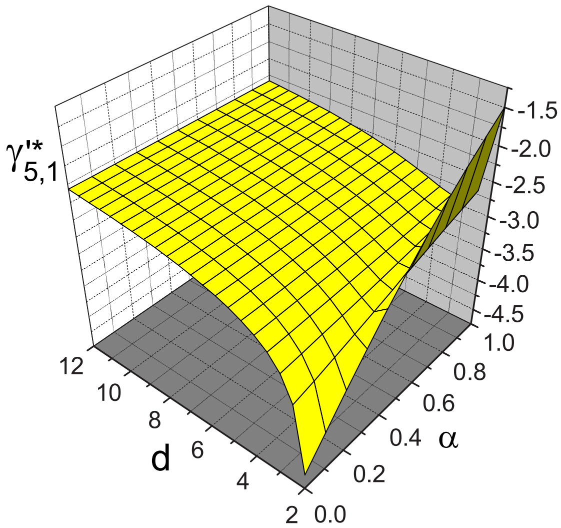

FIG. 38. Dependence of the total two-loop anomalous dimension $\gamma'^{*}_{5,1}$ on the parameter of compressibility α and on the spatial dimension d for $\varepsilon = 1$.

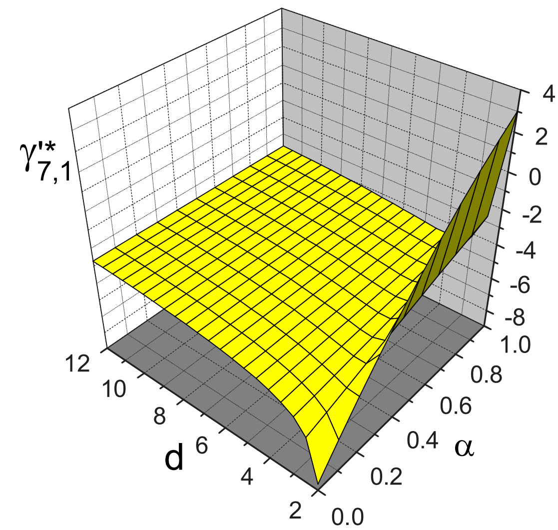

FIG. 40. Dependence of the total two-loop anomalous dimension $\gamma'^{*}_{7,1}$ on the parameter of compressibility α and on the spatial dimension d for $\varepsilon = 1$.

(see Figs. 19–22). Note that this difference is also valid for the anomalous dimensions with $N = 2$ and 3 (compare Figs. 17 and 18 with Figs. 35 and 36). In addition, again, in accordance with the general relation (71), the leading two-loop anomalous dimensions $\gamma^{*}_{N,0}$ and $\gamma^{*}_{N,1}$ of our vector problem are equal to the corresponding anomalous dimensions $\gamma'^{*}_{N,0}$ and $\gamma'^{*}_{N,1}$ of the scalar problem for $d = 2$ regardless of the value of α.

This behavior of the total two-loop anomalous dimensions $\gamma^{*}_{N,0}$, $\gamma'^{*}_{N,0}$ (for even values of N) and $\gamma^{*}_{N,1}$, $\gamma'^{*}_{N,1}$ (for odd values of N) in the limit $d \to \infty$ is related to the fact that in this limit the two-loop corrections $\gamma^{*(2)}_{N,p}$ as well as $\gamma'^{*(2)}_{N,p}$ vanish regardless of the values of N and p as well as regardless of the value of α. It means that, in this case, the total anomalous dimensions $\gamma^{*}_{N,p}$ and $\gamma'^{*}_{N,p}$ of the composite operators (28) and (62) are completely determined by the one-loop corrections

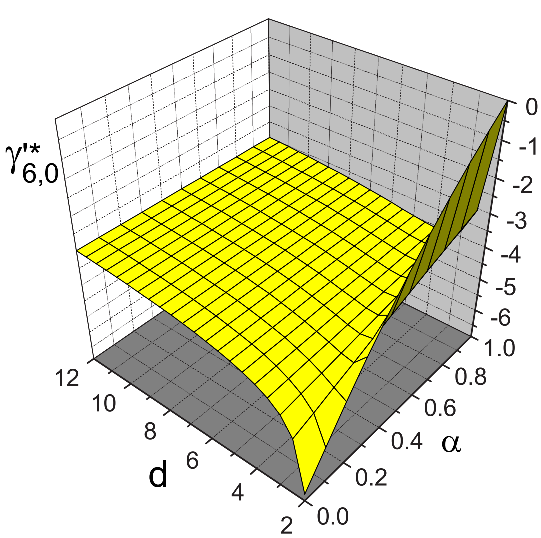

FIG. 39. Dependence of the total two-loop anomalous dimension $\gamma'^{*}_{6,0}$ on the parameter of compressibility α and on the spatial dimension d for $\varepsilon = 1$.

$\gamma^{*(1)}_{N,p}$ and $\gamma'^{*(1)}_{N,p}$ given in Eqs. (40) and (64), namely,

$$\gamma'^{*}_{N,p} = \gamma'^{*(1)}_{N,p}\varepsilon = -\frac{N-p}{2}\varepsilon, \tag{72}$$

which is evidently independent of α, and

$$\gamma^{*}_{N,p} = \gamma^{*(1)}_{N,p}\varepsilon = -\left[\frac{N-p}{2} + \frac{N(N-1)}{2}\alpha\right]\varepsilon, \tag{73}$$

which is an explicit linear function of α.

VII. ANOMALOUS SCALING OF THE SINGLE-TIME CORRELATION FUNCTIONS OF THE MAGNETIC FIELD IN THE COMPRESSIBLE AND HELICAL TURBULENT ENVIRONMENT

As was discussed in Sec. IV, our main aim is to investigate the scaling properties of the single-time two-point correlation functions of the magnetic field defined in Eq. (22) and to find their dependence on the parameters α and ρ which represent compressibility and spatial parity violation of the turbulent system described by the Kraichnan-Kazantsev model defined in Sec. II. In the strictly isotropic case, the asymptotic inertial-range behavior of the correlation functions (22) is given in Eq. (30) and is defined by the anomalous dimensions γ^{*}_{N} of the isotropic composite operators (29). In the two-loop approximation, these isotropic anomalous dimensions are equal to the anomalous dimensions $\gamma^{*}_{N,p}$ for $p = 0$ which are given in Eq. (39) with (40) and (42). Note that in the isotropic case N is always an even number (see the corresponding discussion in Sec. IV).

On the other hand, in the anisotropic case the asymptotic inertial-range behavior of the correlation functions (22) again has the formal form (30) valid in the isotropic case, but now all anomalous dimensions γ^{*}_{N} must be replaced by the anomalous dimensions $\gamma^{*}_{N,p_{\min}}$ with corresponding values of $p_{\min}$ which give the minimal values of $\gamma^{*}_{N,p}$ (see, e.g., Ref. [16] for details).

Note also that in the anisotropic case N can be an even as well as an odd number.

Thus, using the hierarchy relations (59)–(61), which are valid in our model at the one-loop as well as at the two-loop level of approximation at least for not very large values of the compressibility parameter α (see Figs. 29–34 as well as the corresponding discussion in Ref. [49]), one can immediately write the final general asymptotic expressions for the inertial-range behavior of the correlation functions $B_{N-m,m}(r)$ (22):

$$B_{N-m,m}(r) \sim r^{\gamma^*_{[N,0]}-\gamma^*_{[N-m,0]}-\gamma^*_{[m,0]}}, \tag{74}$$

which holds for even values of N and m,

$$B_{N-m,m}(r) \sim r^{\gamma^*_{[N,0]}-\gamma^*_{[N-m,1]}-\gamma^*_{[m,1]}}, \tag{75}$$

which is valid for even value of N and odd value of m, and

$$B_{N-m,m}(r) \sim r^{\gamma^*_{[N,1]}-\gamma^*_{[N-m,0]}-\gamma^*_{[m,1]}}, \tag{76}$$

for odd values of N and m. The remaining (the fourth) possibility with odd value of N and even value of m is already included in the last case, therefore, it is not necessary to write it separately.

Now, using the explicit fixed point expressions for the one- and the two-loop corrections to the anomalous dimensions (39) given in Eqs. (40) and (42), one can write (at the two-loop level of approximation)

$$B_{N-m,m}(r) \sim r^{\zeta_{N,m}} = r^{\zeta^{(1)}_{N,m}\varepsilon+\zeta^{(2)}_{N,m}\varepsilon^2}, \tag{77}$$

where the one-loop corrections $\zeta^{(1)}_{N,m}$ are

$$\zeta^{(1)}_{N,m} = -\frac{m(N-m)(d-1)[1+\alpha(d+1)]}{(d+2)(d-1+\alpha)} \tag{78}$$

for even values of N and m as well as for odd values of N and m, and

$$\zeta^{(1)}_{N,m} = -\frac{(d-1)\{m(N-m)[1+\alpha(d+1)]+d+1+\alpha\}}{(d+2)(d-1+\alpha)}, \tag{79}$$

which is valid for even values of N and odd values of m. On the other hand, the two-loop corrections $\zeta^{(2)}_{N,m}$ in Eq. (77) have the form

$$\zeta^{(2)}_{N,m} = -\frac{S_{d-1}}{S_d}\frac{d}{(d+2)(d-1+\alpha)^2} \times \int_0^1 dx(1-x^2)^{\frac{d-3}{2}}\{\sqrt{1-x^2} \times [(d-2)D_1(W_1Y_1+2\rho^2\delta_{3d}Y_3)+D_2W_2Y_1] - 2(D_3W_3+D_4W_4)Y_2/(d+4)\}, \tag{80}$$

where functions $W_i, i=1,\dots,4$ and Y_i, $i=1,2,3$, are given in Eqs. (47)–(53) and

$$D_1 = 2m(N-m), \tag{81}$$

$$D_2 = 2m(N-m), \tag{82}$$

$$D_3 = m(N-m)(3N+2d-4), \tag{83}$$

$$D_4 = 3m(N-4)(N-m) \tag{84}$$

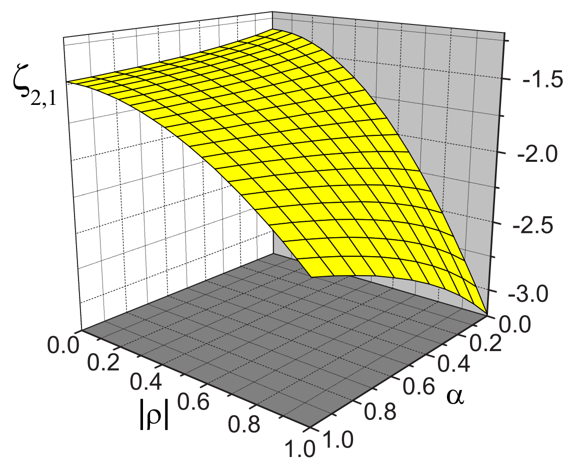

FIG. 41. Dependence of the total two-loop scaling exponent $\zeta_{2,1}$ on the parameters α and ρ for spatial dimension $d=3$ and for $\varepsilon=1$.

for even values of N and m,

$$D_1 = 2[m(N-m)+d+1], \tag{85}$$

$$D_2 = 2[m(N-m)-1], \tag{86}$$

$$D_3 = m(N-m)(3N+2d-4)+(N-4)(d+1), \tag{87}$$

$$D_4 = 3(N-4)[m(N-m)-1] \tag{88}$$

for even N and odd m, and

$$D_1 = 2m(N-m), \tag{89}$$

$$D_2 = 2m(N-m), \tag{90}$$

$$D_3 = (N-m)[m(3N+2d-4)-d-1], \tag{91}$$

$$D_4 = 3(N-m)[m(N-4)+1], \tag{92}$$

which are valid for odd values of N and m.

The behavior of all independent scaling exponents $\zeta_{N,m}$ up to $N=7$ for the single-time two-point correlation functions $B_{N-m,m}$ (22) as functions of the parameters α and ρ is shown explicitly in Figs. 41–52 for $d=3$ and $\varepsilon=1$. Looking at

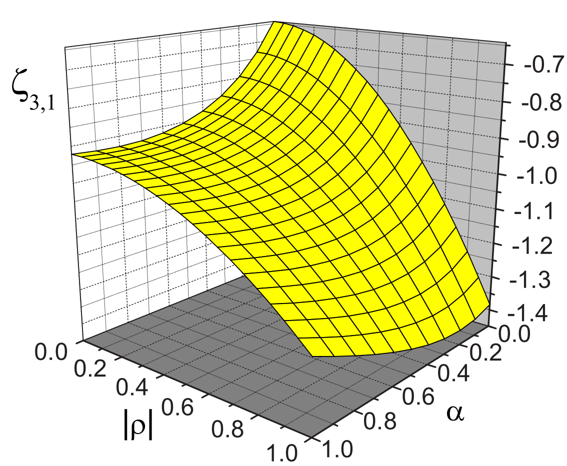

FIG. 42. Dependence of the total two-loop scaling exponent $\zeta_{3,1}$ on the parameters α and ρ for spatial dimension $d=3$ and for $\varepsilon=1$.

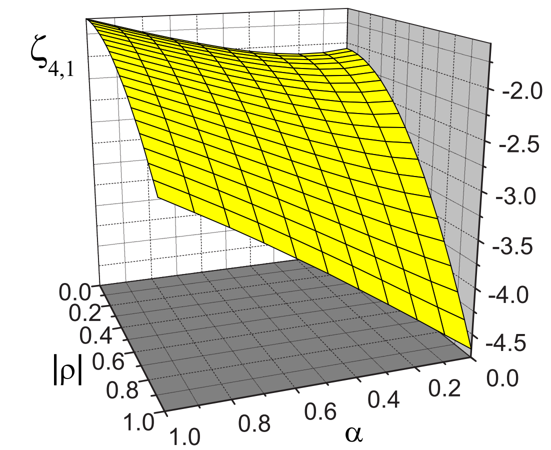

FIG. 43. Dependence of the total two-loop scaling exponent $\zeta_{4,1}$ on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

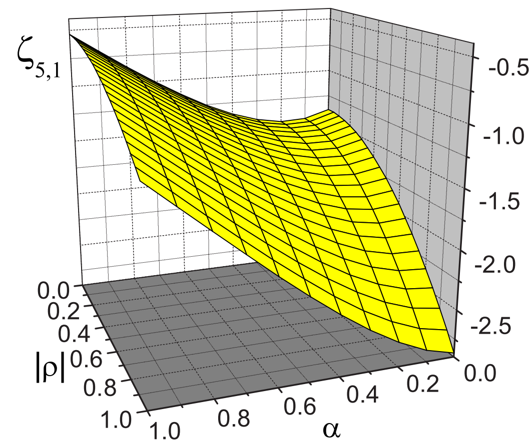

FIG. 45. Dependence of the total two-loop scaling exponent $\zeta_{5,1}$ on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

these figures, one can immediately conclude that the spatial parity violation of the turbulent environment (represented by the parameter ρ) has strong impact on the scaling properties of the correlation functions $B_{N-m,m}$, namely, the corresponding scaling exponents become essentially more negative, i.e., the scaling behavior of the functions $B_{N-m,m}$ become significantly more anomalous. Note that this behavior is in agreement with recent experimental measurements presented in Ref. [52] where it was shown that the intermittent behavior of the magnetic field increases with the injected helicity. On the other hand, when compressibility of the system is considered, the situation seems to be more complicated. It is evident from Figs. 41–52 that there is a significantly different behavior of the scaling exponents $\zeta_{N,m}$ for small values of N, namely, for $N = 2$, 3, 4, and 5, in comparison to the scaling exponents for $N \geqslant 6$. Let us discuss it in more detail.

First, looking at Fig. 41 one can see that for small enough values of the parameter ρ the scaling exponent $\zeta_{2,1}$ is decreasing function of the parameter α in the whole interval $0 \leqslant \alpha \leqslant 1$. However, in the case when the helicity of the turbulent system approaches its maximal absolute value $|\rho| = 1$, the compressibility begins to increase the value of the scaling exponent $\zeta_{2,1}$, i.e., makes the scaling behavior of the function $B_{1,1}$ less anomalous even for small values of α.

A little bit different behavior can be seen in the case of the scaling exponent $\zeta_{3,1}$ which describes the scaling behavior of the correlation function $B_{2,1}$ (see Fig. 42). Here, the scaling exponent $\zeta_{3,1}$ is decreasing function of α for all possible values of ρ if the parameter α is small enough. Only for large enough values of α as well as for large enough value of ρ ($\rho \to 1$), the scaling exponent $\zeta_{3,1}$ increases as the function of α. This behavior is rather specific because it is the only case when a scaling exponent $\zeta_{N,m}$ decreases as the function of α for the system with the maximal violation of the spatial parity ($|\rho| = 1$).

A decreasing of the scaling exponents $\zeta_{N,m}$ as the functions of the compressibility parameter α for small enough absolute values of the helicity parameter ρ can also be seen in the cases with $N = 4$ and 5, i.e., for the correlation functions $B_{3,1}$, $B_{2,2}$, $B_{4,1}$, and $B_{2,3}$ (see Figs. 43–46). As it follows from these figures, this behavior is valid, however, only for small enough values of α. For relatively large values of α, these scaling

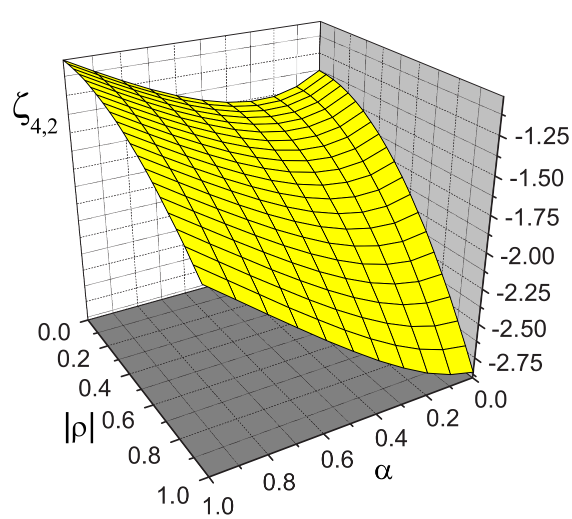

FIG. 44. Dependence of the total two-loop scaling exponent $\zeta_{4,2}$ on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

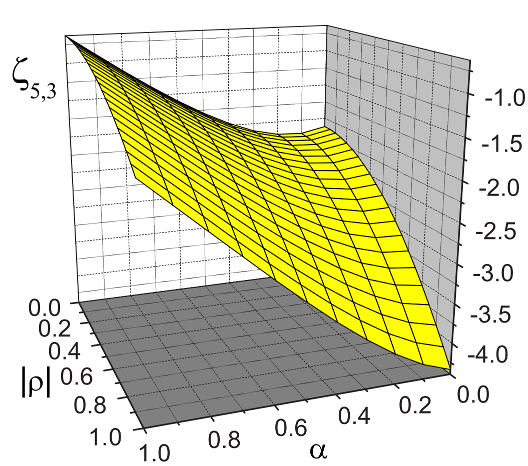

FIG. 46. Dependence of the total two-loop scaling exponent $\zeta_{5,3}$ on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

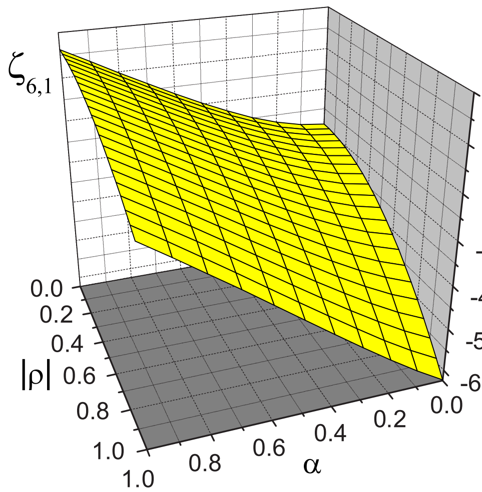

FIG. 47. Dependence of the total two-loop scaling exponent $\zeta_{6,1}$ on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

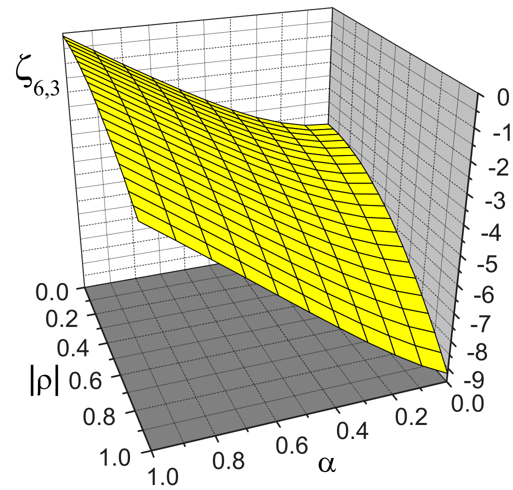

FIG. 49. Dependence of the total two-loop scaling exponent $\zeta_{6,3}$ on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

exponents become increasing functions of α regardless of the absolute value of ρ. At the same time, when $|\rho|$ tends to 1, then these scaling exponents become increasing functions of α for all its values.

Finally, looking at Figs. 47–52 it is evident that the scaling exponents $\zeta_{N,m}$ for $N = 6$ and 7, i.e., for the correlation functions $B_{5,1}$, $B_{4,2}$, $B_{3,3}$, $B_{6,1}$, $B_{4,3}$, and $B_{2,5}$, are always increasing functions of α for all absolute values of the helicity parameter ρ. Note that the similar behavior is also valid for all scaling exponents $\zeta_{N,m}$ with $N \geqslant 8$.

In the end, let us note that the smallest values of the scaling exponents $\zeta_{N,m}$ are always obtained for $\alpha = 0$ and $|\rho| = 1$. There is, however, one exception from this general rule, namely, the scaling exponent $\zeta_{3,1}$ for the correlation function $B_{2,1}$ (see Fig. 42), for which the minimal value is obtained for $|\rho| = 1$ and a nonzero value of the compressibility parameter α.

VIII. PERSISTENCE OF LARGE-SCALE ANISOTROPY AT SMALL SCALES IN COMPRESSIBLE AND HELICAL TURBULENT SYSTEM

To complete our analysis, let us also discuss the problem of isotropy restoration in the presence of an anisotropy of the turbulent system, namely, in our case, the presence of the large-scale anisotropy. As was shown in Ref. [48], the two-loop corrections to the incompressible and nonhelical form of the present model, i.e., in the model with $\alpha = \rho = 0$, lead to more pronounced persistence of the anisotropy in the inertial range in comparison with the corresponding one-loop analysis [16], especially in the phenomenologically most interesting three-dimensional case. Here, our aim is to find the impact of the presence of helicity as well as compressibility of the turbulent environment on the persistence of the large-scale anisotropy deep inside the inertial range and, to this end (see, e.g., Ref. [16] for all details), we shall study the behavior of the dimensionless ratios of the single-time two-point correlation

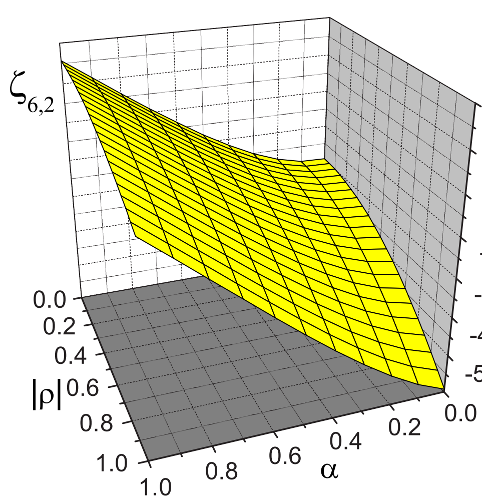

FIG. 48. Dependence of the total two-loop scaling exponent $\zeta_{6,2}$ on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

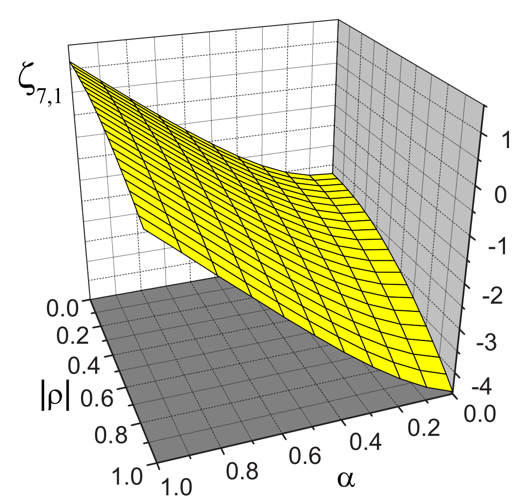

FIG. 50. Dependence of the total two-loop scaling exponent $\zeta_{7,1}$ on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

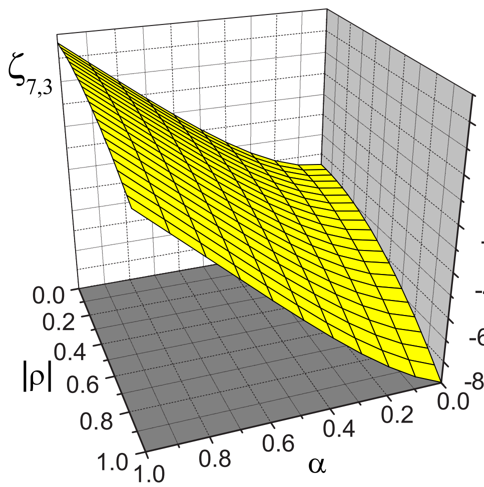

FIG. 51. Dependence of the total two-loop scaling exponent $\zeta_{7,3}$ on the parameters α and ρ for spatial dimension $d=3$ and for $\varepsilon=1$.

functions of the magnetic field $B_{N-m,m}$ (22), namely,

$$R_N \equiv \frac{B_{N-1,1}}{B_{1,1}^{N/2}} = \frac{\langle b_r^{N-1}(t,\mathbf{x}) b_r(t,\mathbf{x}')\rangle}{\langle b_r(t,\mathbf{x}) b_r(t,\mathbf{x}')\rangle^{N/2}}. \tag{93}$$

Now, using the generalization of the asymptotic expression given in Eq. (30) to the anisotropic case together with the hierarchy relations discussed in Sec. V, one can immediately write the explicit dependence of R_N on the ratios r/l and r/L, namely,

$$R_{2n+1} \propto \left(\frac{r}{l}\right)^{-\gamma^*_{2n,0}} \left(\frac{r}{L}\right)^{\gamma^*_{2n+1,1}-(n+1/2)\gamma^*_{2,0}}, \tag{94}$$

which is valid for odd values of $N=2n+1$ and

$$R_{2n+2} \propto \left(\frac{r}{l}\right)^{-\gamma^*_{2n+1,1}} \left(\frac{r}{L}\right)^{\gamma^*_{2n+2,0}-(n+1)\gamma^*_{2,0}}, \tag{95}$$

valid for even values of $N=2n+2$ [16], where, in our case, various $\gamma^*_{x,y}$ represent the corresponding two-loop expressions for the anomalous dimensions given in Eqs. (39), (40), and (42).

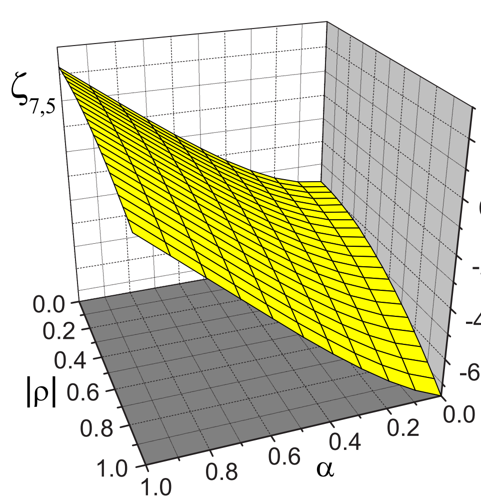

FIG. 52. Dependence of the total two-loop scaling exponent $\zeta_{7,5}$ on the parameters α and ρ for spatial dimension $d=3$ and for $\varepsilon=1$.

To estimate the persistence of the anisotropy deep inside the inertial range, it is convenient to write the dependence of the ratios (94) and (95) on the Péclet number $\mathrm{Pe} \equiv (L/l)^\varepsilon$ which is formally obtained by substituting l instead of r in Eqs. (94) and (95) [24]. One obtains

$$R_{2n+1} \propto \mathrm{Pe}^{[(n+1/2)\gamma^*_{2,0}-\gamma^*_{2n+1,1}]/\varepsilon}, \tag{96}$$

$$R_{2n+2} \propto \mathrm{Pe}^{[(n+1)\gamma^*_{2,0}-\gamma^*_{2n+2,0}]/\varepsilon}, \tag{97}$$

and, in the two-loop approximation, we have

$$R_{2n+1} \propto \mathrm{Pe}^{\xi_{2n+1}} = \mathrm{Pe}^{\xi^{(1)}_{2n+1}+\xi^{(2)}_{2n+1}\varepsilon}, \tag{98}$$

$$R_{2n+2} \propto \mathrm{Pe}^{\xi_{2n+2}} = \mathrm{Pe}^{\xi^{(1)}_{2n+2}+\xi^{(2)}_{2n+2}\varepsilon}, \tag{99}$$

where

$$\xi^{(1)}_{2n+1} \equiv \frac{2n+1}{2}\gamma^{*(1)}_{2,0} - \gamma^{*(1)}_{2n+1,1} = \frac{(d-1)[(4n^2-d-2)(1+\alpha)+4\alpha d n^2]}{2(d+2)(d-1-\alpha)}, \tag{100}$$

$$\xi^{(1)}_{2n+2} \equiv (n+1)\gamma^{*(1)}_{2,0} - \gamma^{*(1)}_{2n+2,0} = \frac{2n(n+1)(d-1)[1+\alpha(d+1)]}{(d+2)(d-1-\alpha)} \tag{101}$$

represent the one-loop expressions which explicitly depend on the compressibility parameter α and which are reduced to the known one-loop incompressible results discussed in Ref. [16] in the limit $\alpha=0$, namely,

$$\xi^{(1)}_{2n+1} = \frac{4n^2-d-2}{2(d+2)}, \tag{102}$$

$$\xi^{(1)}_{2n+2} = \frac{2n(n+1)}{d+2}. \tag{103}$$

On the other hand, the two-loop corrections are given as follows:

$$\xi^{(2)}_N = \frac{S_{d-1}}{S_d}\frac{d}{(d+2)(d-1+\alpha)^2} \times \int_0^1 dx (1-x^2)^{\frac{d-3}{2}} \{\sqrt{1-x^2} \times [(d-2)E_1(W_1Y_1+2\rho^2\delta_{3d}Y_3)+E_2W_2Y_1] - 2(E_3W_3+E_4W_4)Y_2/(d+4)\}, \tag{104}$$

where functions W_i, $i=1,\ldots,4$, and Y_i, $i=1,2,3$, are given in Eqs. (47)–(53) and

$$E_1 = 4n^2-d-2, \tag{105}$$

$$E_2 = 4n^2, \tag{106}$$

$$E_3 = 2n(2n-1)(2n+d+2), \tag{107}$$

$$E_4 = 4n(n-1)(2n-1) \tag{108}$$

for odd values of $N=2n+1$ and

$$E_1 = 4n(n+1), \tag{109}$$

$$E_2 = 4n(n+1), \tag{110}$$

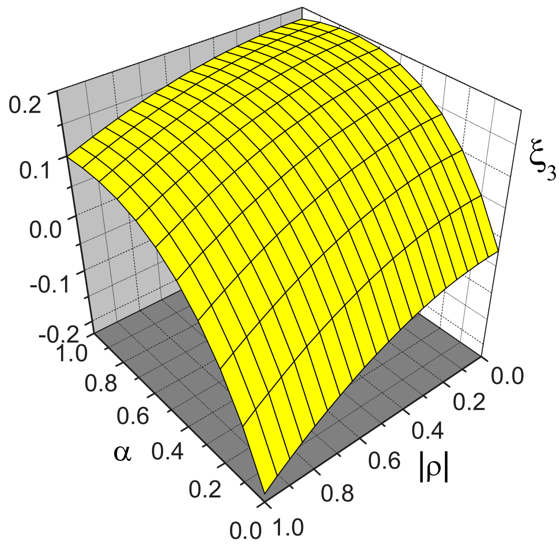

FIG. 53. Dependence of the total two-loop exponent ξ_3 on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

$$E_3 = 4n(n+1)(2n+d+2), \tag{111}$$

$$E_4 = 8n(n^2-1) \tag{112}$$

for even values of $N = 2n+2$. It is an easy task to show that in the limit $\alpha = \rho = 0$ one comes to the two-loop results obtained in Ref. [48] [see Eqs. (115) and (116) in Ref. [48]].

In Figs. 53–57, the behavior of the exponents ξ_N for $N = 3,\ldots,7$ as functions of α and ρ is shown at the two-loop level of approximation for real spatial dimension $d = 3$ and for $\varepsilon = 1$. From the point of view of persistence of the anisotropy in the inertial range, the most interesting is the behavior of the exponents ξ_N for odd values of N, i.e., the exponents for the odd-order quantities (93). Note that these quantities are identically equal to zero by symmetry when the system is isotropic and their behavior in the helical and compressible system with large-scale anisotropy is shown in Figs. 53, 55, and 57 for $N = 3$, 5, and 7, respectively. However, looking at these three figures one can observe immediately that there is

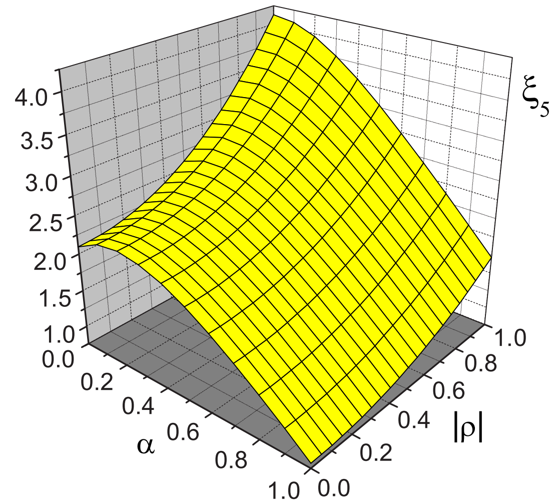

FIG. 55. Dependence of the total two-loop exponent ξ_5 on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

essentially different behavior of the corresponding exponents as functions of α and $|\rho|$ for these three cases. Let us discuss it in detail.

First, let us look at Fig. 53, where the behavior of the exponent ξ_3 is shown explicitly. For $\alpha = \rho = 0$, in two-loop approximation, one obtains $\xi_3 = -0.0235$ (for $\varepsilon = 1$) which was found in Ref. [48]. It means that, in the two-loop approximation, the function R_3 decreases for $\mathrm{Pe} \to \infty$ in the incompressible and nonhelical case but much slower than in the one-loop approximation for which one has $\xi_3^{(1)} = -0.1$ [16] [see also Eq. (102) for $d = 3$ and $n = 1$]. Now, when the spatial parity violation of the system is assumed, as it is evident from Fig. 53, the exponent ξ_3 decreases as the function of the parameter $|\rho|$, i.e., it becomes more negative, and one can conclude that the persistence of the large-scale anisotropy becomes less pronounced in the helical system, at least, if the function R_3 is studied. On the other hand, when the compressibility of the turbulent system is supposed, the situation is opposite, namely, the exponent ξ_3 is rapidly increasing function of α and it acquires positive values already

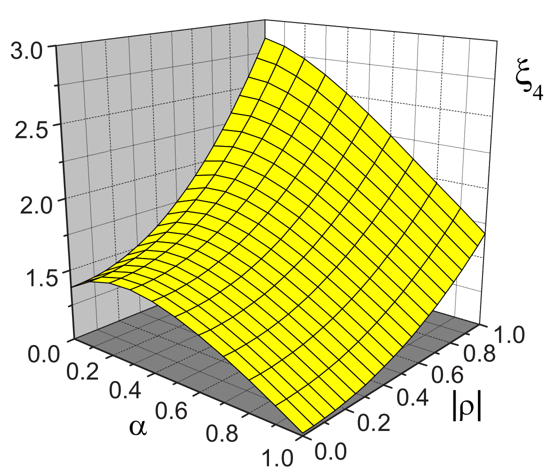

FIG. 54. Dependence of the total two-loop exponent ξ_4 on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

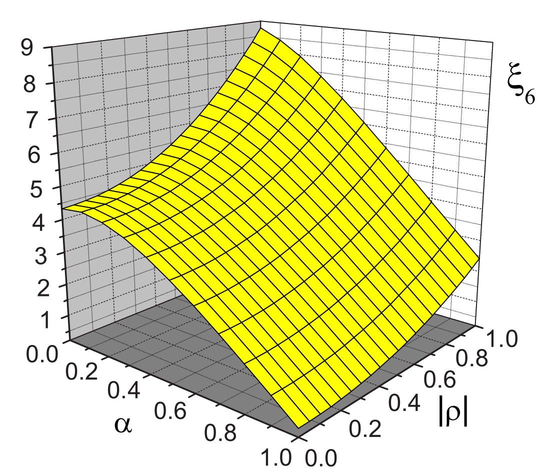

FIG. 56. Dependence of the total two-loop exponent ξ_6 on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

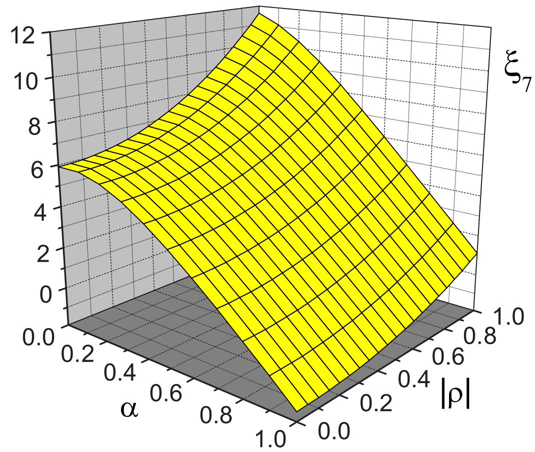

FIG. 57. Dependence of the total two-loop exponent ξ_7 on the parameters α and ρ for spatial dimension $d = 3$ and for $\varepsilon = 1$.

for relatively small values of the parameter α. It means that the persistence of the anisotropy in the inertial range is much more pronounced in the compressible systems than in the incompressible ones, at least, as for the behavior of the function R_3. It is interesting that this behavior of the exponent ξ_3 as the function of α and $|\rho|$ is exceptional because all the other exponents behave completely in a different way.

As it follows from Figs. 55 and 57, the exponents ξ_5 and ξ_7 are increasing functions of the absolute value of ρ. It means that functions R_5 and R_7 as the functions of Pe increase more rapidly when turbulent system is helical in contrast to the corresponding behavior of the function R_3. Note that the same behavior is also valid for all other functions with odd values $N \geqslant 9$. On the other hand, there is little difference between the behavior of the exponents ξ_5 and ξ_7 as for their dependence on the compressibility parameter α. While the exponent ξ_5 is at the beginning an increasing function of α for small enough values of $|\rho|$ and then, starting from some value of α, becomes a decreasing function of α (see Fig. 55) the exponent ξ_7 is always a decreasing function of α regardless of the value of ρ (see Fig. 57). At the same time, the exponent ξ_5 also becomes purely decreasing function of α for large enough absolute values of ρ. Note that the behavior of all the other functions ξ_N with odd values of $N \geqslant 9$ as functions of the parameter α is similar to the behavior of the exponent ξ_7. Finally, looking at Figs. 55 and 57, it is evident that the most pronounced persistence of the large-scale anisotropy is demonstrated in the behavior of the functions R_N with odd values of $N \geqslant 5$ in the incompressible turbulent system ($\alpha = 0$) with maximal spatial parity violation $|\rho| = 1$. However, this is not true for the function R_3 for which the persistence of anisotropy deep inside the inertial range is the most pronounced in the nonhelical system with an appropriate value of the parameter of compressibility. Here, however, it is necessary to bear in mind that our results can be taken seriously only for small enough values of the parameter α (see discussion in Sec. II). Anyway, we can conclude that, depending on the studied function R_N with given odd value of N, one can see that large-scale anisotropy persistence in the inertial range increases in compressible and/or helical turbulent environment.

For completeness, in Figs. 54 and 56 the explicit behavior of the exponents ξ_4 and ξ_6, which drive the asymptotic behavior of the functions R_4 and R_6, respectively, in the limit Pe $\to \infty$, is shown as functions of the parameters α and $|\rho|$. It is evident that the behavior of the exponent ξ_4 is similar to the corresponding behavior of the exponent ξ_5 and the behavior of the exponent ξ_6 matches the corresponding behavior of the exponent ξ_7. However, the exponents ξ_N with even values of N are not crucial for the analysis of the problem of the persistence of the anisotropy in the inertial range of the turbulent system because the corresponding functions R_N are nonzero even in the pure isotropic case.

IX. CONCLUSION

In this paper, we have investigated the scaling properties of the single-time two-point correlations functions of the passive weak magnetic field in the framework of the compressible and helical Kazantsev-Kraichnan model of the kinematic MHD turbulence in the presence of the large-scale anisotropy by using the field theoretic RG technique and the OPE in the second-order (two-loop) approximation. First of all, it is shown that the IR asymptotic behavior of the model deep inside the inertial interval has scaling form which is driven by the IR stable fixed point of the corresponding RG equations [see Eq. (20)] which depends explicitly on the parameter related to the description of the compressibility of the studied turbulent environment but is independent of the presence of the spatial parity violation (helicity) in the system. Note that the present model has an important feature, namely, that the IR stable fixed point is exactly defined by the one-loop calculations (by the first-order approximation in the corresponding perturbation expansion), i.e., it has no higher-loop contributions.

On the other hand, it is shown that the single-time two-point correlation functions of the magnetic field defined in Eq. (22) exhibit the anomalous scaling behavior related to the nontrivial singular asymptotic behavior of the corresponding scaling functions which is investigated by using the OPE technique. In this respect, the influence of the compressibility and the helicity of the turbulent environment on the anomalous dimensions of the leading composite operators in the OPE (28) with the smallest (the most negative) critical dimensions is analyzed up to the second-order approximation. The general two-loop analytic expression for the anomalous dimensions is found as an explicit function of the parameters which characterize the compressibility and helicity of the system [see Eqs. (39), (40), and (42)–(53)] and their anisotropic hierarchies are discussed. It is shown that for physically acceptable values of the compressibility parameter ($\alpha < 1$) as well as for all possible values of the helicity parameter ($0 \leqslant |\rho| \leqslant 1$), the hierarchy relations (59)–(61) are valid in accordance with the Kolmogorov's local isotropy restoration hypothesis.

Further, the properties of the anomalous dimensions of the composite operators (28) important for our vector model are compared to the corresponding properties of the anomalous dimensions of the leading composite operators (62) which play the central role in the Kraichnan model of a passively advected scalar field. It is shown that there is a significant difference between the scaling properties of the passively advected scalar and vector (magnetic) fields, i.e., that the internal tensor structure has nontrivial impact on the scaling properties of passively advected quantities. This difference, which takes place already in the incompressible and nonhelical case (see Ref. [48]), is even more pronounced when the compressibility and/or spatial parity violation of the turbulent environment are supposed. First of all, it is shown that while in the case of the Kraichnan model of passively advected scalar quantity the anomalous dimensions of the leading composite operators are helicity blind, the anomalous dimensions of the composite operators relevant for the Kazantsev-Kraichnan model strongly depend on the helicity parameter (see Figs. 5–16). It is also shown that there is also a nontrivial difference between them as for their dependence on the compressibility parameter (see Figs. 23–28 as well as Figs. 17–22 and 35–40). As it follows from Figs. 23–28, while the total two-loop anomalous dimensions of the composite operators (28) always decrease as functions of the compressibility parameter when its values are small enough, the leading total two-loop anomalous dimensions of the composite operators (62) of the scalar model are increasing functions of the compressibility parameter (the only exceptions are the anomalous dimensions for $N = 2$ and 3).

The calculated two-loop anomalous dimensions of the composite operators (28) are then used for analysis of the dependence of the scaling exponents of the single-time two-point correlation functions (22) of various orders on the compressibility parameter as well as on the helicity parameter. It is shown that, at least in the two-loop order of approximation, due to the validity of the anisotropy hierarchies among the anomalous dimensions of the composite operators (28), the asymptotic scaling behavior of the correlation functions (22) inside the inertial interval is driven by the anomalous dimensions of the operators from the isotropy shell (when the operator has even order) and of the operators that are the closest to the isotropic shell (in the case of the operators with odd orders) for all physically acceptable values of the compressibility and helicity parameters. It is found (see Figs. 41–52) that the presence of the spatial parity violation in the turbulent environment can significantly decrease the scaling exponents of the magnetic correlation functions (22). It means that the negative scaling exponents of these correlation functions become even more negative under the influence of helicity, i.e., the intermittent behavior of the fluctuations of the magnetic field in the conductive turbulent environment is more strongly pronounced when the mirror symmetry is broken in the system. Note that this pure theoretical result is in agreement with recent experimental observation obtained in an MHD plasma [52]. On the other hand, the influence of compressibility on the inertial-range scaling behavior of the single-time correlation functions of the magnetic field is more complicated. For small order correlation functions, namely, for $B_{1,1}$, $B_{2,1}$, $B_{3,1}$, $B_{2,2}$, $B_{4,1}$, and $B_{3,2}$, the corresponding scaling exponents decrease as functions of the compressibility parameter, at least for small enough values of the compressibility parameter, for small spatial parity violation in the system (see Figs. 41–46). However, for higher correlation functions, the corresponding scaling exponents become increasing functions of the compressibility parameter regardless of the value of the helicity parameter (see Figs. 47–52, where the critical dimensions for the correlation functions $B_{5,1}$, $B_{4,2}$, $B_{3,3}$, $B_{6,1}$, $B_{4,3}$, and $B_{5,2}$ are shown explicitly).

In addition, we have also performed a detail analysis of the influence of compressibility and spatial parity violation of the turbulent environment on the persistence of the large-scale anisotropy deep inside the inertial interval. For this purpose, the explicit asymptotic dependence of dimensionless ratios of the magnetic correlation functions defined in Eq. (93) on the Péclet number is established. Note that relevant functions for analyzing of the anisotropy persistence in the inertial range are the odd-order ratios of the correlation functions which are identically equal to zero when exact isotropy of the system is supposed. The corresponding exponents are found as explicit functions of the compressibility parameter as well as of the helicity parameter up to the two-loop approximation [see Eqs. (98)–(112)]. It is shown that, up to the one exception, namely, the odd ratio with $N = 3$ (see Fig. 53), the presence of the helicity in the system leads to most pronounced persistence of the large-scale anisotropy, i.e., the corresponding exponents are increasing functions of the helicity parameter (see Figs. 55 and 57). On the other hand, the exponent ξ_3 is an increasing function of the compressibility parameter (at least for physically relevant small values of the compressibility parameter) regardless of the value of the helicity parameter (see Fig. 53). At the same time, the exponent ξ_5 is increasing function of the compressibility parameter only near the nonhelical limit of the system and only for very small values of the compressibility parameter (see Fig. 55). Finally, the compressibility always leads to the decreasing of the exponents ξ_N for $N \geqslant 7$, i.e., to the less pronounced persistence of the anisotropy.

The analysis performed in this paper indicates that various symmetry violations in the turbulent environments can have a nontrivial impact on the asymptotic scaling properties of physically important quantities (such as the single-time correlation functions of the magnetic field studied in the paper). Although, in this paper, we have analyzed the influence of the compressibility and the spatial parity violation of the turbulent system on the statistical properties of the passive magnetic field in the framework of the Kazantsev-Kraichnan model where the statistics of the turbulent velocity field is assumed to be Gaussian, nevertheless, we suppose that they correctly describe the properties of real systems at least qualitatively.

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