Evidence for equivalence of diffusion processes of passive scalar and magnetic fields in anisotropic Navier-Stokes turbulence

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The influence of the uniaxial small-scale anisotropy on the kinematic magnetohydrodynamic turbulence is investigated by using the field theoretic renormalization group technique in the one-loop approximation of a perturbation theory. The infrared stable fixed point of the renormalization group equations, which drives the scaling properties of the model in the inertial range, is investigated as the function of the anisotropy parameters and it is shown that, at least at the one-loop level of approximation, the diffusion processes of the weak passive magnetic field in the anisotropically driven kinematic magnetohydrodynamic turbulence are completely equivalent to the corresponding diffusion processes of passively advected scalar fields in the anisotropic Navier-Stokes turbulent environments.

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It was shown recently in the framework of the model of a passive scalar field advected by the Navier-Stokes turbulent velocity field with the presence of strong uniaxial small-scale anisotropy that the anisotropically driven turbulent environments can have serious impacts on the diffusion processes of the scalar field [1], where the dependence of the isotropic part of the turbulent Prandtl number, i.e., the ratio of the turbulent kinematic viscosity of the fluid to the corresponding coefficient of turbulent diffusivity [2], on the parameters which describe the form of the uniaxial anisotropy was investigated using the field theoretic renormalization group (RG) technique in the first-order perturbation approximation (the one-loop approximation in the field theoretic language). In addition, a few years ago it was also shown in the two-loop approximation in the framework of the field theoretic approach [3] that the turbulent magnetic Prandtl number of the passive weak magnetic field in the kinematic magnetohydrodynamic (MHD) turbulence without any symmetry breaking is equal to the turbulent Prandtl number of passively advected scalar field by the fully symmetric isotropic turbulent velocity field driven by the stochastic Navier-Stokes equation [4,5]. In this respect, let us also note that at the leading one-loop level of approximation this equivalence between diffusion processes in these two models for arbitrary space dimensions $d > 2$ was known for a long time (see, e.g., Refs. [6,7] as well as recent Refs. [8]). In addition, it is also well known that in two dimensions these two problems are equivalent beyond any approximation (see, e.g., Refs. [9] and references cited therein).

It means that the diffusion processes in these two physically different turbulent environments are in fact completely equivalent. However, this conclusion is valid only for fully symmetric isotropic turbulent systems because, as was shown in Ref. [10] in the framework of the two-loop field theoretic RG approximation, when the spatial parity violation (helicity) is present in these turbulent environments then the corresponding turbulent Prandtl numbers, and therefore also the corresponding diffusion coefficients, become different. Thus, it seems that symmetries (more precisely their violation) play significant roles in diffusion processes in various turbulent systems.

The following question immediately and naturally arises: Is there any difference between diffusion processes of scalar and magnetic fields passively advected by the corresponding anisotropic turbulent environments, e.g., with the presence of the simplest uniaxial small-scale anisotropy, in the same sense as in the case with the presence of the spatial parity violation? The answer on this, in fact, nontrivial question is the aim of the present paper. As we shall see, the answer is quite surprising, namely, that there is no difference between these two diffusion processes, i.e., that all anisotropic turbulent Prandtl numbers, and therefore also all anisotropic diffusion coefficients, are the same in both models. This result means that the internal tensor structure of admixtures plays no role as for the properties of diffusion processes of passive scalar and magnetic fields advected by the corresponding anisotropically driven stochastic Navier-Stokes equations.

Thus, let us consider a passive solenoidal magnetic field $\mathbf{b} \equiv \mathbf{b}(x)$ $[x \equiv (t,\mathbf{x})]$ in the framework of the kinematic MHD turbulence described by the following system of stochastic equations:

$$\partial_t \mathbf{b} = \nu_0 u_0 \triangle \mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \tag{1}$$

$$\partial_t \mathbf{v} = \nu_0 \triangle \mathbf{v} - (\mathbf{v}\cdot\partial)\mathbf{v} - \partial\mathcal{P} + \mathbf{f}^{\mathbf{v}}, \tag{2}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, ν_0 is kinematic viscosity coefficient (subscript 0 always denotes bare parameter of the unrenormalized theory), $\nu_0 u_0 = c^2/(4\pi\sigma_0)$ represents the magnetic diffusivity, u_0 is dimensionless reciprocal magnetic Prandtl number, c is the speed of light, σ_0 is the conductivity, $\mathcal{P} \equiv \mathcal{P}(x)$ is the pressure, and $\mathbf{v} \equiv \mathbf{v}(x)$ is a solenoidal (owing to the incompressibility) velocity field. Thus, both $\mathbf{v}$ and $\mathbf{b}$ are divergence-free vector fields: $\partial\cdot\mathbf{v} = \partial\cdot\mathbf{b} = 0$.

Random noises $\mathbf{f}^{\mathbf{v}}$ and $\mathbf{f}^{\mathbf{b}}$ in Eqs. (1) and (2) simulate the kinematic and magnetic energy pumping into the dissipative turbulent system to maintain its steady state. We shall suppose that the magnetic energy pumping is realized by a transverse Gaussian random noise $\mathbf{f}^{\mathbf{b}} = \mathbf{f}^{\mathbf{b}}(x)$ with zero mean and the correlation function in the following form:

$$D^b_{ij}(x;0) \equiv \langle f^b_i(x) f^b_j(0)\rangle = \delta(t) C_{ij}(|\mathbf{x}|/L), \tag{3}$$

which represents the source of the fluctuations of the magnetic field. In Eq. (3), L is an integral scale related to the corresponding stirring and C_{ij} is a function finite in the limit $L \to \infty$. Its detailed form is not important in what follows. The only condition which must be satisfied is that C_{ij} decreases rapidly for $|\mathbf{x}| \gg L$.

The statistics of the random force $\mathbf{f}^{\mathbf{v}} = \mathbf{f}^{\mathbf{v}}(x)$ in Eq. (2) is also taken in a Gaussian form with zero mean and with the pair correlation function (see, e.g., Refs. [11–13] for all details)

$$D^v_{ij}(x;x') \equiv \langle f^v_i(x) f^v_j(x')\rangle = \delta(t-t') \int \frac{d^d\mathbf{k}}{(2\pi)^d} g_0 \nu_0^3 k^{4-d-2\varepsilon} R_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{4}$$

where d denotes the spatial dimension of the studied system, g_0 plays the role of the coupling constant of the present model, i.e., it is a formal small parameter of the ordinary perturbation theory and is related to the characteristic ultraviolet (UV) momentum scale Λ (or inner length $l \sim \Lambda^{-1}$) by relation $g_0 \simeq \Lambda^{2\varepsilon}$, and the physical value of formally small parameter $0 < \varepsilon \leqslant 2$ is $\varepsilon = 2$. The geometric properties of the energy pumping (4) are completely determined by the form of the transverse projector $R_{ij}(\mathbf{k})$ and it is taken in the following form:

$$R_{ij}(\mathbf{k}) = \left(1 + \alpha_1 \frac{(\mathbf{n}\cdot\mathbf{k})^2}{k^2}\right) P_{ij}(\mathbf{k}) + \alpha_2 P_{is}(\mathbf{k}) n_s n_t P_{tj}(\mathbf{k}), \tag{5}$$

which represents the simplest special case of a general anisotropic transverse tensor structure for considering the uniaxial anisotropy presented at all scales (see, e.g., Refs. [14–17]). Here, $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the ordinary transverse projector, n_i is the ith component of the unit vector $\mathbf{n}$, which defines the direction of the axis of the uniaxial anisotropy, and the anisotropy parameters α_1 and α_2 must satisfy inequalities $\alpha_1 > -1$ and $\alpha_2 > -1$ to have positively defined correlation function (4). The summations over dummy indices is understood.

For completeness, let us note that in Eq. (4) the necessary IR regularization is performed by the restriction of the integration from below, i.e., $k \geqslant m$, where m represents another integral scale. In what follows, it is always supposed that $L \gg 1/m$.

Note also that due to the vector character of both random forces $\mathbf{f}^{\mathbf{v}}$ and $\mathbf{f}^{\mathbf{b}}$ it is possible to introduce the mixed correlator between them even in the pure isotropic case, which is not possible in the case of the model with the passively advected scalar quantity where the mixed correlator can be constructed only when anisotropy is present [18,19]. However, for simplicity and for the correspondence with the analysis performed in Ref. [1], we shall not consider the presence of this mixed term here.

Using the well-known theorem [20], the stochastic problem given by Eqs. (1) and (2) corresponds to the field theoretic model with double set of fields $\Phi = \{\mathbf{v},\mathbf{b},\mathbf{v}',\mathbf{b}'\}$ and with the action functional

$$\begin{aligned} S(\Phi) = {} & \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \\ & \times \left[v'_i(t_1,\mathbf{x_1}) D^v_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2}) v'_j(t_2,\mathbf{x_2}) \right. \\ & \left. + b'_i(t_1,\mathbf{x_1}) D^b_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2}) b'_j(t_2,\mathbf{x_2}) \right] \\ & + \int dt\, d^d\mathbf{x} \{ \mathbf{b}'[-\partial_t - \mathbf{v}\cdot\partial + \nu_0 u_0 (\Delta + \tau_{10}(\mathbf{n}\cdot\partial)^2)]\mathbf{b} \\ & + \nu_0 u_0 \tau_{20} \mathbf{n}\cdot\mathbf{b}' \Delta\, \mathbf{n}\cdot\mathbf{b} + \mathbf{b}'(\mathbf{b}\cdot\partial)\mathbf{v} \\ & + \mathbf{v}'[-\partial_t - \mathbf{v}\cdot\partial + \nu_0(\Delta + \chi_{10}(\mathbf{n}\cdot\partial)^2)]\mathbf{v} \\ & + \nu_0\, \mathbf{n}\cdot\mathbf{v}'[\chi_{20}\Delta + \chi_{30}(\mathbf{n}\cdot\partial)^2]\mathbf{n}\cdot\mathbf{v}\}, \end{aligned} \tag{6}$$

where $D^\theta_{ij}(x_1;x_2)$ and $D^v_{ij}(x_1;x_2)$ are the correlation functions given in Eqs. (3) and (4) for the random forces $\mathbf{f}^{\mathbf{b}}$ and $\mathbf{f}^{\mathbf{v}}$, respectively. Besides, to make the field theoretic model with the presence of the small-scale anisotropy multiplicatively renormalizable, it is necessary to enlarge the model by additional terms with new unrenormalized (bare) parameters τ_{10}, τ_{20}, χ_{10}, χ_{20}, and χ_{30}, which are not present in the original stochastic problem defined by Eqs. (1) and (2) (see, e.g., Refs. [15,16] for details, where the anisotropically driven MHD turbulence in the weak anisotropy limit and the Navier-Stokes turbulence with the presence of strong uniaxial anisotropy were investigated, respectively).

The necessity of introduction of new terms to the action functional (6) with all aforementioned bare parameters means that, in the case when the energy pumping into the system is driven by the correlator (4) with the presence of the small-scale uniaxial anisotropy in the form (5), the original stochastic equations (1) and (2) must be taken in the following enlarged form:

$$\begin{aligned} \partial_t \mathbf{b} = {} & \nu_0 u_0 [\triangle \mathbf{b} + \tau_{10}(\mathbf{n}\cdot\partial)^2 \mathbf{b} + \tau_{20}\mathbf{n}\,\Delta(\mathbf{n}\cdot\mathbf{b})] \\ & - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \end{aligned} \tag{7}$$

$$\begin{aligned} \partial_t \mathbf{v} = {} & \nu_0 [\triangle \mathbf{v} + \chi_{10}(\mathbf{n}\cdot\partial)^2 \mathbf{v} + \chi_{20}\mathbf{n}\,\Delta(\mathbf{n}\cdot\mathbf{v}) \\ & + \chi_{30}\mathbf{n}\,(\mathbf{n}\cdot\partial)^2(\mathbf{n}\cdot\mathbf{v})] - (\mathbf{v}\cdot\partial)\mathbf{v} - \partial\mathcal{P} + \mathbf{f}^{\mathbf{v}}, \end{aligned} \tag{8}$$

from which the action functional (6) is directly obtained. Physically, the introduced dimensionless parameters χ_{10}, χ_{20}, χ_{30}, τ_{10}, and τ_{20} in Eqs. (7) and (8) describe the relative impact of the different anisotropic tensor structures on the viscous dissipation processes and on the turbulent diffusion of the passive magnetic field, respectively.

Note also that the pressure term $-\partial\mathcal{P}$ in Eqs. (2) and (8) is omitted in the action functional (6) due to the transverse character of the auxiliary field $\mathbf{v}'$.

The field theoretic model defined by the action functional (6) is self-consistent for which the standard Feynman diagrammatic technique can be introduced with the following new bare propagator (in the frequency-momentum representation):

$$\langle b'_i(\omega,\mathbf{k}) b_j(\omega,\mathbf{k})\rangle_0 = \frac{1}{L_1}\left[P_{ij} - \frac{L_2 P_{is} n_s n_t P_{tj}}{L_1 + L_2(1-\xi_k^2)} \right], \tag{9}$$

where $\xi_k^2 = (\mathbf{n}\cdot\mathbf{k})^2/k^2$ and

$$L_1 = \mathrm{i}\omega + \nu_0 u_0 k^2 + \nu_0 u_0 \tau_{10}(\mathbf{n}\cdot\mathbf{k})^2, \tag{10}$$

$$L_2 = \nu_0 u_0 \tau_{20} k^2. \tag{11}$$

The explicit form of the other two important propagators, which are related to the velocity field, can be found, e.g., in Ref. [1] (see Eqs. (8) and (9) in Ref. [1]). At the same time, the model contains two interaction vertices of the forms $b_i'(-v_j\partial_j b_i + b_j\partial_j v_i) = b_i' v_j V_{ijl} b_l$ and $-v_i' v_j \partial_j v_i = v_i' v_j W_{ijl} v_l/2$, where $V_{ijl} = i(k_j\delta_{il} - k_l\delta_{ij})$ and $W_{ijl} = i(k_l\delta_{ij} + k_j\delta_{il})$ (again in the momentum-frequency representation).

The general RG analysis [13] shows that the model contains two superficially divergent one-irreducible Green's functions, namely, $\langle v'v\rangle_{1-ir}$ and $\langle b'b\rangle_{1-ir}$, and all divergences can be removed by multiplicative renormalization of the bare parameters $g_0,\ u_0,\ \nu_0,\ \tau_{i0},\ i = 1,2,$ and $\chi_{j0}, j = 1,\ 2,\ 3$ in the following form:

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon} Z_g, \quad u_0 = u Z_u, \tag{12}$$

$$\tau_{i0} = \tau_i Z_{\tau_i}, \quad \chi_{j0} = \chi_j Z_{\chi_j}, \tag{13}$$

where parameters $g,\ u,\ \nu,\ \tau_i$, and χ_j are dimensionless renormalized counterparts of the bare parameters, μ is the so-called renormalization mass, and $Z_y = Z_y(g,u,\tau_i,\chi_j;d;\varepsilon)$ for $y = \nu,\ g,u,\ \tau_i,\ \chi_j$ are the corresponding renormalization constants which absorb all divergences and which can be expressed through a set of seven independent renormalization constants $Z_i, i = 1,\ldots,7$ of the renormalized action functional

$$S_R(\Phi) = \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2}\big[v_i'(t_1,\mathbf{x_1})D_{ij}^v(t_1,\mathbf{x_1};t_2,\mathbf{x_2})v_j'(t_2,\mathbf{x_2}) + b_i'(t_1,\mathbf{x_1})D_{ij}^b(t_1,\mathbf{x_1};t_2,\mathbf{x_2})b_j'(t_2,\mathbf{x_2})\big]$$
$$+ \int dt\, d^d\mathbf{x}\{\mathbf{b}'[-\partial_t - \mathbf{v}\cdot\partial + \nu u(Z_5\Delta + \tau_1 Z_6(\mathbf{n}\cdot\partial)^2)]\mathbf{b} + \nu u\tau_2 Z_7 \mathbf{n}\cdot\mathbf{b}'\Delta\,\mathbf{n}\cdot\mathbf{b} + \mathbf{b}'(\mathbf{b}\cdot\partial)\mathbf{v}$$
$$+ \mathbf{v}'[-\partial_t - \mathbf{v}\cdot\partial + \nu(Z_1\Delta + \chi_1 Z_2(\mathbf{n}\cdot\partial)^2)]\mathbf{v} + \nu\,\mathbf{n}\cdot\mathbf{v}'[\chi_2 Z_3\Delta + \chi_3 Z_4(\mathbf{n}\cdot\partial)^2]\mathbf{n}\cdot\mathbf{v}\}, \tag{14}$$

in the following way:

$$Z_\nu = Z_1, \quad Z_g = Z_1^{-3}, \quad Z_u = Z_5 Z_1^{-1}, \tag{15}$$

$$Z_{\tau_i} = Z_{i+5}Z_5^{-1}, \quad Z_{\chi_j} = Z_{j+1}Z_1^{-1}, \tag{16}$$

where again $i = 1,2$ and $j = 1,2,3$. Finally, in the framework of the one-loop approximation, which we are interested in, and in the minimal substraction (MS) scheme [13,21], the renormalizetion constants can be expressed as follows:

$$Z_i = 1 + g\frac{z_i}{\varepsilon} + O(g^2), \quad i = 1,\ldots,7, \tag{17}$$

where the coefficients $z_i \equiv z_{11}^{(i)}$ are determined by the calculation of the corresponding one-loop Feynman diagrams (see, e.g., Ref. [22]).

The known coefficients z_i, for $i = 1,\ldots,4$, which are related to the 1-irreducible Green's function $\langle v_i' v_j\rangle_{1-ir}$, can be written in the following integral form:

$$z_1 = -\frac{1}{8}\frac{S_{d-1}}{(2\pi)^d(d^2-1)}\int_{-1}^{1}dx\frac{(1-x^2)^{\frac{d-3}{2}}}{(M_1M_2M_3)^3}b_1, \tag{18}$$

$$z_{j+1} = -\frac{1}{8}\frac{S_{d-1}}{(2\pi)^d(d^2-1)}\int_{-1}^{1}dx\frac{(1-x^2)^{\frac{d-3}{2}}}{(M_1M_2M_3)^3}\frac{b_{j+1}}{\chi_j}, \tag{19}$$

for $j = 1,2,3$, where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface area of the d-dimensional unit sphere, $\Gamma(x)$ represents the Euler's Γ function, $M_i, i = 1,2,3$ are defined as

$$M_1 = 2(1+\chi_1 x^2) + (\chi_2 + \chi_3 x^2)(1-x^2), \tag{20}$$

$$M_2 = 1 + \chi_1 x^2 + (\chi_2 + \chi_3 x^2)(1-x^2), \tag{21}$$

$$M_3 = 1 + \chi_1 x^2, \tag{22}$$

and the explicit form of huge coefficients $b_i, i = 1,\ldots,4$ can be found in Appendix I in Ref. [17].

On the other hand, the one-loop coefficients $z_i,\ i = 5,6,7,$ for the remaining renormalization constants $Z_i,\ i = 5,6,7$ in Eq. (17), related to the 1-irreducible Green's function $\langle b_i' b_j\rangle_{1-ir}$ of the magnetic field, are unknown and therefore must be determined. The calculation of the corresponding one-loop diagram finally gives

$$z_5 = -\frac{1}{4u}\frac{S_{d-1}}{(2\pi)^d(d-1)}\int_{-1}^{1}dx\frac{(1-x^2)^{\frac{d-3}{2}}}{M}b_5, \tag{23}$$

$$z_{5+i} = -\frac{1}{4u}\frac{S_{d-1}}{(2\pi)^d(d-1)}\int_{-1}^{1}dx\frac{(1-x^2)^{\frac{d-3}{2}}}{M}\frac{b_{5+i}}{\tau_i}, \tag{24}$$

where $i = 1,2$, $M = M_1M_2M_3N_1N_2N_4N_5$, and

$$b_5 = -2\alpha_2 M_2M_3N_1N_2N_5x^2(x^2-1) + N_4N_5\big((d-1)M_1M_2N_1R_1 + (x^2-1)\big\{M_2^3R_1 - M_3(M_3+N_3)R_2R_3x^2$$
$$+ M_2^2R_1(M_3+R_2x^2) + M_2R_1[M_1N_3 + (2M_3+N_3)R_2x^2]\big\}\big) - \tau_2 ux^2(1-x^2)\big(M_1M_2^2(M_2+N_7)R_1$$
$$+ M_2^3R_1R_2(x^2-1) + 2M_2^2M_3R_1R_2(x^2-1) - M_3\big(M_3^2 + N_3N_6 + M_3N_7\big)R_2R_3(x^2-1) + M_2R_1\big\{2M_3^2R_2(x^2-1)$$
$$- N_7R_2^2(x^2-1)^2 + M_3[2N_3N_6 + 3N_7R_2(x^2-1)]\big\}\big), \tag{25}$$

$$b_6 = \tau_2 u(1-3x^2+2x^4)\big(2\alpha_2 M_2 M_3 N_2 N_5(x^2-1) - M_1 M_2^3 R_1 - M_2\big(M_2^2+2M_2M_3+2M_3^2\big)R_1R_2(x^2-1) + M_3^3 R_2 R_3(x^2-1) - M_3N_3N_6[2M_2R_1 - R_2R_3(x^2-1)] - N_7\big\{M_1M_2^2R_1 - R_2(x^2-1)\big[M_3^2R_3 - M_2R_1(3M_3+R_2-R_2x^2)\big]\big\}\big) + N_4N_5\big(-R_2\big[M_2^2R_1 + M_2(2M_3+N_3)R_1 - M_3(M_3+N_3)R_3\big](d-3+2x^2) + 2\alpha_2M_2M_3N_2(x^2-1)[1+d(x^2-1)] + (dx^2-1)\{-M_1M_2R_1(1+N_3+R_2+\chi_1x^2-R_2x^2) + R_2(x^2-2)[M_3(M_3+N_3)R_3 - M_2R_1(3M_3+N_3+R_2-R_2x^2)]\}\big), \quad (26)$$

$$b_7 = -(d-2)\tau_2 u(1-x^2)^2\big[M_2^4R_1 - M_3\big(M_3^2+N_3N_5+M_3N_6\big)R_2R_3x^2 + M_2^3R_1(M_3+N_6+R_2x^2) + M_2^2R_1[N_6R_2x^2 + M_3(N_6+2R_2x^2)] + M_2\big(M_1N_3N_4R_1 + x^2\big\{-2\alpha_2M_3N_2N_5 + R_1R_2\big[2M_3^2 + M_3(3N_3+2N_6) + N_3(N_6+R_2-R_2x^2)\big]\big\}\big)\big], \quad (27)$$

with

$$N_1 = M_2 + N_3, \quad N_2 = M_3 + N_3, \quad (28)$$

$$N_3 = u(1+\tau_1 x^2), \quad N_4 = M_2 + N_3 + N_8, \quad (29)$$

$$N_5 = M_3 + N_3 + N_8, \quad N_6 = N_3 + N_8, \quad (30)$$

$$N_7 = 2N_3 + N_8, \quad N_8 = u\tau_2(1-x^2), \quad (31)$$

$$R_1 = 1 + \alpha_1 x^2, \quad R_2 = \chi_2 + \chi_3 x^2, \quad (32)$$

$$R_3 = -R_1 - \alpha_2(1-x^2). \quad (33)$$

In addition, $M_i, i = 1,2,3$ are defined in Eqs. (20)–(22).

The infrared (IR) scaling properties of the present model are driven by the corresponding IR-stable fixed point of the RG equations, the coordinates of which are given by the requirement of simultaneous vanishing of all RG β functions of the model, i.e.,

$$\beta_i(g_*,\chi_{j*},u_*,\tau_{l*};\alpha_1,\alpha_2,d,\varepsilon) = 0, \quad i = g,\chi_j,u,\tau_l, \quad (34)$$

where $j = 1,2,3$ and $l = 1,2$. Here, variables with asterisks (*) represent coordinates of the fixed point and the β functions are defined as follows:

$$\beta_g \equiv \mu\partial_\mu g = -2g(\varepsilon + gz_1), \quad (35)$$

$$\beta_{\chi_i} \equiv \mu\partial_\mu \chi_i = 2g\chi_i(z_{i+1} - z_1), \quad i = 1,2,3, \quad (36)$$

$$\beta_u \equiv \mu\partial_\mu u = 2gu(z_5 - z_1), \quad (37)$$

$$\beta_{\tau_i} \equiv \mu\partial_\mu \tau_i = 2g\tau(z_{i+5} - z_5), \quad i = 1,2, \quad (38)$$

where functions $z_i, i = 1,\dots,7$ are given in Eqs. (18), (19), (23), and (24). Note also that the fixed point is IR stable if and only if all eigenvalues of the matrix of the first derivatives of the β functions have positive real parts (see, e.g., Ref. [13] for all details of the RG technique).

The technique for finding the IR-stable fixed points of the model under consideration is based on the analysis of the so-called flow RG equations related to the β functions (35)–(38) and is described, e.g., in Refs. [1,16,17]. In our case, the system of flow equations for running variables $\bar{g}$, $\bar{\chi}_i$, $\bar{u}$, $\bar{\tau}_j$, $i = 1,2,3$, $j = 1,2$ as functions of the scale parameter $t = k/\Lambda$ reads

$$t\frac{d\bar{g}}{dt} = \beta_g(\bar{g},\bar{\chi}_j;\alpha_1,\alpha_2,d,\varepsilon), \quad (39)$$

$$t\frac{d\bar{\chi}_i}{dt} = \beta_{\chi_i}(\bar{g},\bar{\chi}_j;\alpha_1,\alpha_2,d,\varepsilon), \quad i = 1,2,3, \quad (40)$$

$$t\frac{d\bar{u}}{dt} = \beta_u(\bar{g},\bar{\chi}_j,\bar{u},\bar{\tau}_l;\alpha_1,\alpha_2,d,\varepsilon), \quad (41)$$

$$t\frac{d\bar{\tau}_i}{dt} = \beta_{\tau_i}(\bar{g},\bar{\chi}_j,\bar{u},\bar{\tau}_l;\alpha_1,\alpha_2,d,\varepsilon), \quad i = 1,2, \quad (42)$$

where $j = 1,2,3$, $l = 1,2$, the initial conditions are taken in $t = 1$, and the IR-stable fixed point (if exists) is obtained in the limit $t \to 0$, i.e., $\{\bar{g},\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3,\bar{u},\bar{\tau}_1,\bar{\tau}_2\}|_{t\to 0} = \{g_*,\chi_{1*},\chi_{2*},\chi_{3*},u_*,\tau_{1*},\tau_{2*}\}$.

The numerical analysis of the system of differential equations (39)–(42) gives three important facts: (i) the region of the anisotropy parameters α_1 and α_2 where the stable Kolmogorov scaling regime exists is the same as in the case of the anisotropic pure Navier-Stokes turbulence [1,17], (ii) the coordinate τ_{2*} of the IR-stable fixed point is always equal to zero, i.e., $\tau_{2*} = 0$ for all possible values of α_1 and α_2 for which the IR-stable fixed points exist, and (iii) all the other coordinates of the IR-stable fixed point of the system of Eqs. (39)–(42) are completely the same as in the case of the passively advected scalar quantity studied in detail in Ref. [1] (of course, when parameter τ_1 is changed to τ because τ_1 in the present problem corresponds to τ in the scalar problem studied in Ref. [1]). The last result also follows from the fact that the quantities b_5 and b_6 in Eqs. (25) and (26), respectively, are reduced to the corresponding quantities for the scalar problem for $\tau_2 = 0$ (see Eqs. (31) and (32) in Ref. [1]).

It means, however, that the diffusion processes of the passive magnetic field in the Navier-Stokes conductive turbulent environment with the presence of the uniaxial small-scale anisotropy are completely equivalent to the diffusion processes of passive scalar quantities advected by the corresponding Navier-Stokes turbulence [1], i.e., all diffusion coefficients (isotropic and anisotropic) as well as the corresponding dimensionless Prandtl numbers are the same in both models. It also means that the universality of diffusion processes in these two models which holds in fully symmetric isotropic cases even in the two-loop approximation [3–5] remains valid in the situation when the corresponding turbulent systems are anisotropic. For example, in Fig. 1 the well-defined isotropic

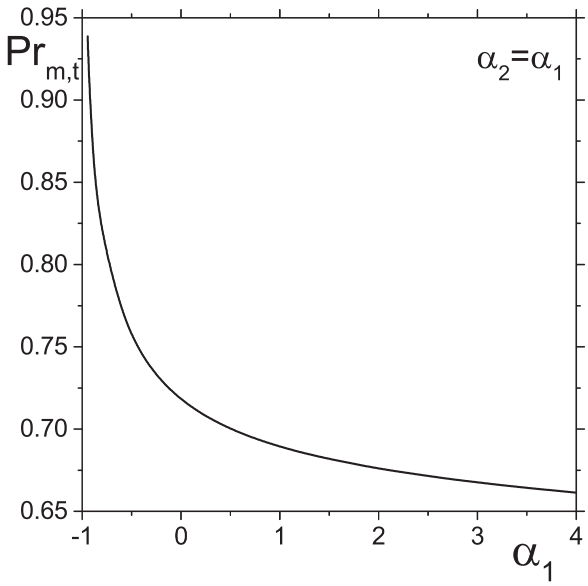

FIG. 1. The turbulent magnetic Prandtl number $\mathrm{Pr}_{\mathrm{m,t}} = 1/u_*$ as the function of the anisotropy parameters $\alpha_1 = \alpha_2$ for three-dimensional case $d = 3$ and $\varepsilon = 2$.

part of the turbulent magnetic Prandtl number $\mathrm{Pr}_{\mathrm{m,t}} \equiv 1/u_*$ is shown as the function of the anisotropy parameters $\alpha_1 = \alpha_2$, which is completely the same as the corresponding isotropic part of the turbulent Prandtl number in the scalar problem (see Fig. 6 in Ref. [1]).

It is important to stress that this is a rather surprising result because, as was already mentioned, it is known that when, for example, the spatial parity violation (helicity) of turbulent environments is supposed, the diffusion processes of passive scalar and magnetic fields become essentially different [10].

Let us also note that, due to the strong nonlinearity of the studied problem with respect to the anisotropy parameters, the crucial mathematical result of the present analysis, namely, that $\tau_{2*} = 0$ always holds for the IR-stable fixed points, from which the universality of diffusion processes follows, cannot be obtained without performing calculations; i.e., it is not obvious at the first sight.

Finally, it is still necessary to bear in mind that all conclusions are made on basis of the one-loop approximation only; therefore there still exists a possibility that higher-loop corrections will destroy this perfect one-loop equivalence between anisotropic diffusion processes of the passive scalar and magnetic fields. Nevertheless, we do not believe this scenario because when the strong small-scale anisotropy is considered, as in the present paper, the nontrivial nonlinear anisotropy corrections are already included at the one-loop level of approximation; i.e., in this case, the one-loop approximation does not mean that only linear anisotropy corrections are taken into account. As a result, it is hardly possible that this kind of equivalence between these two models is accidental and holds only at the one-loop level of approximation. Nevertheless, for ultimate verdict in this question, of course, at least two-loop calculations are needed. But this kind of calculations is enormously complicated and was not performed yet even in the much simpler weak small-scale uniaxial anisotropy limit where only linear anisotropy corrections are taken into account.

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