

Viscosity of a classical gas: The rare-collision versus the frequent-collision regime

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The shear viscosity η for a dilute classical gas of hard-sphere particles is calculated by solving the Boltzmann kinetic equation in terms of the weakly absorbed plane waves. For the rare-collision regime, the viscosity η as a function of the equilibrium gas parameters—temperature T , particle number density n , particle mass m , and hard-core particle diameter d —is quite different from that of the frequent-collision regime, e.g., from the well-known result of Chapman and Enskog. An important property of the rare-collision regime is the dependence of η on the external (“nonequilibrium”) parameter ω , frequency of the sound plane wave, that is absent in the frequent-collision regime at leading order of the corresponding perturbation expansion. A transition from the frequent to the rare-collision regime takes place when the dimensionless parameter $nd^2(T/m)^{1/2}\omega^{-1}$ goes to zero. The scaled absorption coefficient for sound waves calculated in the rare and frequent-collision regimes is found to be in qualitative agreement with the experimental data.

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As well known, the transport coefficients, such as the thermal conductivity, diffusion, and viscosity, can be defined as the linear responses on small perturbations of equilibrium systems [1–4]. Let us consider a classical gas system of hard-sphere particles. An important quantity, crucial for the description of all transport properties of this system, is the particle mean-free path estimated as $l \sim 1/(n\pi d^2)$, where n is the particle number density and d is the hard-core particle diameter. This is an internal property of the equilibrium system. For external (dynamical) perturbations, one has to introduce another scale L which is the size of the space region where the gas properties (e.g., temperature, mean particle density, and collective velocity) undergo essential changes. Usually, the inequality, $l \ll L$, is satisfied. This corresponds to the so-called frequent-collision regime (FCR), and the transport coefficients can be calculated as a perturbation expansion over a small parameter l/L (see, e.g., Refs. [5–9]). The leading terms of these expansions are found to be independent of the scale L . For example, the shear viscosity η , calculated by Maxwell within concepts of the molecular kinetic theory, reads $\eta \sim n m v_T l$, where m is the particle mass and $v_T = \sqrt{2T/m}$ is the thermal particle velocity.¹ Thus, in the FCR, η is a function of only internal parameters of the equilibrium gas. Since $l \propto 1/(nd^2)$, one finds $\eta \sim \sqrt{mT}/d^2$ in the FCR, i.e., the shear viscosity is independent of particle number density at the leading order in l/L . Similar results are valid for other transport coefficients in the FCR.

Accurate expressions for the transport coefficients in a gas of classical hard-sphere particles were obtained by Chapman

and Enskog (CE) by using the Boltzmann kinetic equation (BKE) and ideal hydrodynamic equations for calculations of a time evolution in terms of almost the local-equilibrium distribution function within the FCR [5]. The following expression for η was obtained [5–9]:

$$\eta_{\text{CE}} = \frac{5}{16\sqrt{\pi}} \frac{\sqrt{mT}}{d^2}. \quad (1)$$

This result was extended to a multicomponent hadron gas in Ref. [10]. Several investigations were devoted to go beyond the standard approach; see, e.g., Refs. [11–37]. In the recent paper [37], the shear viscosity η was calculated for a gas of particles with both the short-range repulsive and long-range attractive interactions described by the van der Waals equation of state [38]. This was realized within the FCR in terms of a strong suppression of the damping plane waves.

Many theoretical [11,19–21,23,29] and experimental [14,39,40] investigations were devoted to a weak absorption of the sound wave in dilute gases. For the sound absorption coefficient γ , one can use the famous Stokes expression in terms of the viscosity and thermal conductivity coefficients (see, e.g., Refs. [29,41]). Most results for the absorption coefficient γ were also obtained in the FCR.

Much less attention was paid to the rare-collision regime (RCR) which takes place at $l \gg L$; i.e., the so-called Knudsen parameter l/L becomes large (see, e.g., Refs. [6–8]). Different analytical methods [20,21,29] and numerical Monte Carlo simulations [29] (see also the textbook [6]) were used to calculate the absorption coefficient γ by solving approximately the BKE. They improved significantly an agreement with experimental data [39,40] for small sound frequencies. For high frequencies a progress was mainly achieved within numerical simulations [29].

The conditions of the RCR can be fulfilled in different ways. For a small particle number density, a typical RCR

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¹We use the units where the Boltzmann constant is $\kappa_B = 1$.

situation arises because of the finite system size. Note that for $nd^3 \rightarrow 0$ the mean-free path behaves as $l \rightarrow \infty$. Therefore, Eq. (1) fails for any finite physical system in the limits, $n \rightarrow 0$ and/or $d \rightarrow 0$, where a density and/or a diameter of particles is vanishing. In both these limits, one has $nd^3 \ll 1$ and, thus, one results in the relationship $l \gg L$. Thus, the FCR is transformed to the RCR; and Eq. (1), obtained within the FCR, becomes invalid. Such a situation always appears for a gas expanding to a free space: the linear size of the system L increases due to a gas expansion, but the mean-free path $l \propto n^{-1} \propto L^3$ increases much faster. Thus, the RCR always takes place at the latest stages of the gas expansion to a free space.

The purpose of the present paper is to develop a general perturbation method which can be applied in both the FCR and the RCR for the weakly absorbed plane-wave (WAPW) solutions of the BKE. The WAPWs play an important role in both the theoretical studies and physical applications. They correspond to the plane waves with a given frequency ω for which the amplitude only slightly decreases within the wave length λ . The WAPW can take place within both the FCR and RCR. Thus, the perturbation expansion can be developed for small and large parameter l/λ for the FCR and RCR, respectively.

The FCR and RCR can be defined in alternative terms. Let us consider the sound plane-wave propagation in the infinite gas system. In this case ($L \sim \lambda$), different regimes take place because of different relationships between the mean-free path l and the wavelength λ of the propagating plane wave: $l \ll \lambda$ and $l \gg \lambda$ correspond to the FCR and RCR, respectively [7,11,14]. Introducing the wave frequency ω and frequency of two-particle collisions $\tau^{-1} \sim v_T/l$, one finds another equivalent classification for different collision regimes: $\omega\tau \ll 1$ corresponds to the FCR, and $\omega\tau \gg 1$ to the RCR. A dimensionless quantity $\omega\tau$ plays the same role as the Knudsen parameter mentioned above. Small (large) values of $\omega\tau$ correspond to the collision-term (inertial-terms) dominance in the BKE. Important fields of the RCR applications are the ultrasonical absorption (see, e.g., Refs. [14,42,43] and special phenomena in the electronic plasma [7]. In the FCR we may use the perturbation expansion in power series over $\omega\tau$ whereas in the RCR the parameter $\omega\tau$ becomes large, and the perturbation expansion over $1/(\omega\tau)$ should be successful. Therefore, in these two collision regimes, one can expect a different dependence of η on the equilibrium gas parameters, n and T , and proper particle parameters, d and m . In addition, a dependence of η on external (“nonequilibrium”) parameter ω , absent in Eq. (1) for the FCR at the leading order of the perturbation approach, is expected to appear in the RCR. To our knowledge, no explicit analytical expressions for η were so far presented for the RCR. For completeness, we present also the viscosity for a small absorption within the FCR using the same perturbation method but in small parameter $\omega\tau \ll 1$.

The paper is organized as follows. In Sec. II, the BKE approach with the relaxation time approximation for the collision term through the WAPW is discussed. In Secs. III and IV, derivations of the expressions for η in the FCR and RCR are presented, respectively. The obtained results are used to calculate the scaled absorption coefficient for a sound wave propagation in the FCR and RCR. Section V is devoted to the discussions of the results. Section VI summarizes the

paper, and Appendixes A and B show some details of our calculations.

II. THE KINETIC APPROACH IN A RELAXATION TIME APPROXIMATION

For a classical system of hard spheres, the single-particle distribution function $f(\mathbf{r}, \mathbf{p}, t)$, where \mathbf{r} , \mathbf{p} , and t are the particle phase-space coordinates, and the time variable, respectively, is assumed to satisfy the BKE. The global equilibrium of this system can be described by the Maxwell distribution as a function of the modulus of the particle momentum \mathbf{p} ($p \equiv |\mathbf{p}|$):

$$f_{\text{eq}}(p) = \frac{n}{(2\pi mT)^{3/2}} \exp\left(-\frac{p^2}{2mT}\right). \quad (2)$$

The particle number density n and temperature T are constants independent of the spacial coordinates \mathbf{r} and time t . For dynamical variations of the equilibrium distribution (2), $\delta f(\mathbf{r}, \mathbf{p}, t) = f - f_{\text{eq}}$, one obtains at $|\delta f|/f_{\text{eq}} \ll 1$ the BKE linearized over δf ,

$$\frac{\partial \delta f}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial \delta f}{\partial \mathbf{r}} = \delta St. \quad (3)$$

The standard form of the Boltzmann collision integral δSt is used for hard spherical particles [5,7,9,37]. For simplicity, the attractive long-range interactions discussed in Ref. [37] are discarded.

In line with Refs. [17,26,36,37], the solutions of Eq. (3) for $\delta f(\mathbf{r}, \mathbf{p}, t)$ can be sought in terms of the WAPW,

$$\delta f(\mathbf{r}, \mathbf{p}, t) = f_{\text{eq}}(p) A(\hat{p}) \exp(-i\omega t + ikz), \quad (4)$$

where $A(\hat{p})$ is a yet unknown function of the angles, $\hat{p} \equiv \mathbf{p} \cdot \mathbf{k}/(pk)$. Here, ω and \mathbf{k} are, respectively, the frequency and a wave vector of the WAPW directed along the z axis ($k = |\mathbf{k}|$). For convenience, the spherical phase-space coordinates with the polar axis directed to the unit wave vector \mathbf{k}/k can be used. The quantities ω and k are connected by the equation,

$$\omega = kc v_T = kc \sqrt{2T/m}. \quad (5)$$

In Eq. (5), ω is a given real frequency, whereas a dimensionless sound velocity c and wave number k are presented, in general, as complex numbers,

$$c = c_r + i c_i, \quad k = k_r + i \gamma. \quad (6)$$

A parameter γ denotes the absorption coefficient. The imaginary quantities in Eq. (6) are responsible for a description of the dissipative process.

We use the standard definition of the shear viscosity [37,41],

$$\eta = \frac{3}{4} \text{Re} \frac{\delta \sigma_{zz}}{\partial u_z / \partial z}, \quad (7)$$

through the dynamical components of the stress tensor,

$$\delta \sigma_{zz} = \int d\mathbf{p} \left(\frac{p^2 - 3p_z^2}{3m} \right) \delta f(\mathbf{r}, \mathbf{p}, t), \quad (8)$$

and the z component of the collective velocity,

$$u_z = \frac{1}{n} \int d\mathbf{p} \frac{p_z}{m} \delta f(\mathbf{r}, \mathbf{p}, t). \quad (9)$$

In what follows, it will be convenient to expand the plane-wave amplitude $A(\hat{\mathbf{p}})$ in Eq. (4) over the spherical harmonics $Y_{\ell 0}(\hat{p})$ [44],

$$A(\hat{p}) = \sum_{\ell=0}^{\infty} A_{\ell} Y_{\ell 0}(\hat{p}). \quad (10)$$

To solve uniformly the BKE (3) in both the FCR and RCR, the integral collision term $\delta S t$ will be expressed in the form of the relaxation time approximation [12,17,22,26–28,33,35–37,43]:

$$\delta S t \approx -\frac{1}{\tau} \sum_{\ell \geq 2}^{\infty} \delta f_{\ell}, \quad (11)$$

with the relaxation time [37],

$$\tau \approx \frac{2}{3} \frac{1}{n v_T \sigma}, \quad (12)$$

where σ is the cross section for the two-particle collisions, which is given by $\sigma = \pi d^2$ for the case of the hard-sphere particles of the diameter d . Note that $\tau \sim l/v_T$ determines (up to a constant factor) the average time of the motion between successive collisions of the particles, and thus, $1/\tau$ is approximately the collision frequency in the molecular kinetic theory [9]. The summation over ℓ in Eq. (11) starts from $\ell = 2$ to obey the particle number and momentum conservation in two-body collisions [17,26,28,36,37]. The shear viscosity η [Eq. (7)] can be calculated analytically in the two limiting cases: $\omega\tau \ll 1$ (FCR) and $\omega\tau \gg 1$ (RCR).

III. FREQUENT COLLISIONS

In the FCR, the dispersion equation for a dimensionless velocity c reads (see Appendix A and Ref. [37])

$$c \left[c^2 \left(1 + \frac{i}{\omega\tau} \right) - \frac{3}{5} - \frac{i}{3\omega\tau} \right] = 0. \quad (13)$$

For nonzero solutions, $c \neq 0$, at first order in Knudsen parameter, $\omega\tau \ll 1$, one approximately finds from Eq. (13),

$$c_r = \frac{1}{\sqrt{3}}, \quad c_i = -\frac{2\omega\tau}{5\sqrt{3}}. \quad (14)$$

According to Eq. (6) at first order of the perturbation expansion, one then has

$$k_r = \frac{\omega}{c_r v_T}, \quad \frac{\gamma}{k_r} = -\frac{c_i}{c_r}. \quad (15)$$

Therefore, for the scaled absorption coefficient γ/k_r in the FCR, one obtains

$$\frac{\gamma_{\text{FCR}}}{k_r} = \frac{2}{5} \omega\tau. \quad (16)$$

Thus, for the scaled absorption coefficient γ/k_r in the FCR, one obtains (Appendix A)

$$\eta_{\text{FCR}} = \frac{3\sqrt{\pi} n T}{10 \omega} \omega\tau = \frac{\sqrt{2\pi m T}}{10\sigma}, \quad (17)$$

where $\sigma = \pi d^2$ is the same cross section for collisions of two hard-core spherical particles as in Eq. (12). In Eq. (17), the frequency ω is canceled at leading first-order perturbation expansion over $\omega\tau$. The shear viscosity η_{FCR} in the FCR

behaves always as $\eta_{\text{FCR}} \propto \sqrt{mT}/\sigma$. However, the numerical factor in this formula appears to be different for different physical processes. For the strongly suppressed plane waves (see Ref. [37]), one has $|c_i| \gg |c_r|$, and the shear viscosity approaches that in Eq. (1). For the WAPW in the FCR, one finds another relation $|c_i| \ll |c_r|$ which leads to a different numerical factor in η_{FCR} [Eq. (17)] as compared to Eq. (1).

IV. RARE COLLISIONS

Within the RCR, one can use the perturbation expansion over a small parameter $1/(\omega\tau) \ll 1$. After the substitution of Eqs. (4) and (11) into Eq. (3) we derive the RCR dispersion equation for the WAPW sound velocity c (see Appendix B),

$$1 - \frac{ic}{\xi\omega\tau} - \left[\frac{ic(3\xi^2 + 1)}{\xi\omega\tau} - \epsilon_0 \right] Q_1(\xi) = 0, \quad (18)$$

where $\xi = c[1 + i/(\omega\tau)]$, $\epsilon_0 = +0$, and $Q_1(\xi)$ is the Legendre function of a second kind given by Eq. (B6). From Eqs. (18), (6), and (15), one approximately finds at $1/(\omega\tau) \ll 1$ quite a different solution as compared to Eq. (14),

$$c_r = 1, \quad c_i = -\frac{1}{\omega\tau}. \quad (19)$$

Using Eqs. (5), (6), (15), and (19), one arrives at

$$k_r = \frac{\omega}{v_T}, \quad \frac{\gamma_{\text{RCR}}}{k_r} = \frac{1}{\omega\tau}. \quad (20)$$

For the shear viscosity (7), one straightforwardly obtains (Appendix B)

$$\eta_{\text{RCR}} = \frac{9\sqrt{\pi} n T}{4\omega^2 \tau} = \frac{27\sqrt{2\pi}}{8} \frac{n^2 T^{3/2} \sigma}{\sqrt{m} \omega^2}, \quad (21)$$

where $\sigma = \pi d^2$ as above. Notice that this viscosity is the leading first-order term of the perturbation expansion over a small parameter $1/(\omega\tau)$, and therefore, $\eta \propto 1/\omega^2\tau$. Comparing the expressions (21) and (17) for the shear viscosity, one observes several distinct features of the RCR: (i) the dependence on parameters m , T , and σ in the RCR [Eq. (21)] is completely different from that in the FCR [Eq. (17)]; (ii) a dependence of η on particle number density n appears in the RCR and it is absent in the FCR; (iii) a dependence of η_{RCR} on the external parameter ω exists in the RCR [Eq. (21)] whereas η_{FCR} [Eq. (17)] depends only on the internal gas properties and is independent of ω .

V. DISCUSSION OF THE RESULTS

Let us compare the expressions for the shear viscosity in the RCR [Eq. (21)] and FCR [Eq. (17)]. From this comparison, one observes quite a different dependence of η_{RCR} and η_{FCR} on the quantities n , T , d , and m which describe the equilibrium classical gas (i) and (ii). Most striking is the difference between two regimes in the limit of pointlike particles $d \rightarrow 0$. In this limit, $\eta_{\text{FCR}} \rightarrow \infty$, whereas $\eta_{\text{RCR}} \rightarrow 0$. As even a more remarkable property, the difference between the RCR and the FCR is that the leading [first-order in $1/(\omega\tau)$] term of η_{RCR} [Eq. (21)] depends on a frequency ω , $\eta_{\text{RCR}} \propto 1/\omega^2$, while η_{FCR} (at the same first order but in $\omega\tau$) is independent of ω (iii).

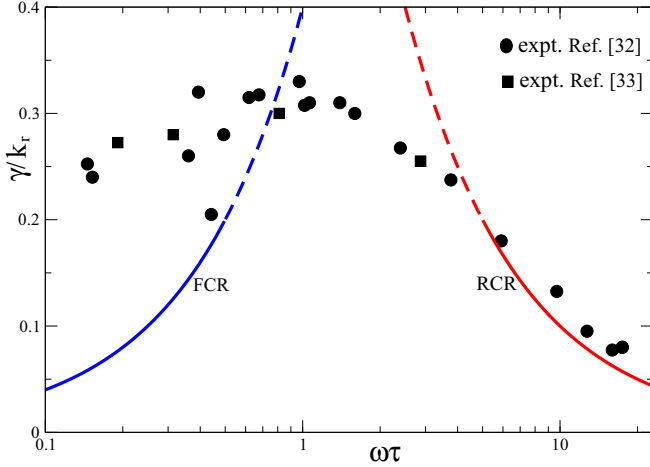


FIG. 1. The scaled absorption coefficient γ/k_r [Eqs. (20) and (16)] (in units of the wave number k_r) as a function of the Knudsen parameter $\omega\tau$ for the FCR ($\omega\tau \ll 1$, blue line) and RCR ($\omega\tau \gg 1$, red line). Full dots are the experimental data from Ref. [39] and full squares are taken from Ref. [40] for ^{40}Ar at normal conditions (see also Refs. [21,29]).

The definition of δf given by Eq. (4) is the same for both FCR and RCR. However, as explained in Appendixes A and B, one has essentially different dispersion equations—(13) in the FCR and (18) in the RCR—and their solutions c for the wave velocity [cf. Eqs. (14) and (19)]. As the result, from the same Eqs. (7)–(9), one obtains different final expressions: Eqs. (17) for η_{FC} and (21) for η_{RC} . As we consider only the leading terms in the corresponding perturbation expansions for both the RCR and FCR, these expressions are valid for dilute gases. This is a general feature of the kinetic approach. Adding higher order terms, one can hope to reach a wider range of applicability, including particularly gases at higher density.

According to Eq. (20), the dimensionless (scaled) absorption coefficient γ/k_r has a simple universal behavior in the RCR [$1/(\omega\tau) \ll 1$]. This dependence is very different from that of Eq. (16) in the FCR ($\omega\tau \ll 1$).

Figure 1 shows by solid curves the scaled absorption coefficients γ/k_r [Eqs. (20) and (16)] as functions of the Knudsen parameter $\omega\tau$. Dashed lines in Fig. 1 correspond to the values of parameter $\omega\tau \sim 1$, where neither the RCR nor the FCR condition of the leading first-order approximation is valid, i.e., the expansions neither over $\omega\tau$ nor over $1/(\omega\tau)$ can be applied. The specific fixed values of $\omega\tau = [0.5, 5]$ which denote this region, where the leading terms of these expansions cannot be sufficient, are not really strictly fixed, and they are used in Fig. 1 for illustrative purposes, only. The points and squares are the experimental data from Refs. [39] and [40] (see also Refs. [14,21,29]). The measurements were done with ^{40}Ar mono-atomic gas at the atmospheric pressure and room temperature conditions for a wide range of the sound frequencies. It is shown also from Fig. 1, a dramatic change of the absorption coefficient γ/k_r as a function of the Knudsen parameter: γ/k_r increases in the FCR like $\omega\tau$ at small $\omega\tau \ll 1$, according to Eq. (16), and decreases as $1/(\omega\tau)$ at large $\omega\tau \gg 1$ in the RCR. As seen from Fig. 1 our result in the RCR

[Eq. (20)] reproduces fairly well the experimental data² for a small sound-wave absorption. A main difference between the FCR curve [Eq. (16)] and the experimental data, can be perhaps explained by an additional thermal-conductivity contribution to γ_{FC}/k_r .

Equations (21) for η_{RC} and (17) for η_{FC} can be also used within the Stokes formula (see, e.g., Ref. [41]) for a weak absorption of sound waves. Substituting these equations into the Stokes equation and neglecting the thermal conductivity, one finds an agreement with the results for γ/k_r , presented in Fig. 1. Thus, the whole picture looks self-consistent.

VI. SUMMARY

The shear viscosity η is derived for the damping sound in terms of the plane waves, spreading in a dilute equilibrium gas of classical particles described by hard spheres. In the rare-collision regime the leading order of the perturbation expansion over parameter $1/(\omega\tau) \ll 1$ for the shear viscosity η_{RC} is quite different from the first-order result in the frequent-collision regime. First, very different dependencies of η_{FC} and η_{RC} on the internal (equilibrium) gas quantities n , T , d , and m are found [(i) and (ii)]. Second, a basic difference is that η_{FC} is independent of the nonequilibrium (external) frequency ω , whereas $\eta_{\text{RC}} \propto 1/\omega^2$ (iii).

For small and large values of the Knudsen parameter $\omega\tau$, one finds the scaled absorption coefficient γ/k_r growing proportionally to $\omega\tau$ at $\omega\tau \ll 1$ and decreasing as $1/(\omega\tau)$ at $\omega\tau \gg 1$, as well in the Stokes approach mentioned above. Therefore, one can predict a maximum of γ/k_r for the transition between these two collision regimes at $\omega\tau \approx 1$. This seems to be in a reasonable agreement with the experimental data [39,40]. Thermal conductivity calculations for both the FCR and RCR can be done within a more general approach, e.g., the linear response theory for solving the BKE. Within this formalism, one can formulate the extended presentation for all kinetic coefficients suitable for their calculations in the non-perturbative region, too. We plan to consider these problems in the forthcoming publications. The results for the kinetic coefficients can be improved by accounting for higher order terms in the perturbation expansions and numerical calculations, towards the range of $\omega\tau$ close to one. Our analytical results in the rare-collision regime are universal and do not include any fitting parameters. Their accuracy increases with increasing of $\omega\tau$. They can be extended to more general interactions between particles as well as, with the help of the linear response theory, to other transport coefficients such as the thermal and electric conductivity, and the diffusion coefficients.

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²Note that neither the conditions of the RCR, $\omega\tau \gg 1$, nor the FCR, $\omega\tau \ll 1$, are fully satisfied for the data presented in Fig. 1, except for a few very right points.

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APPENDIX A: THE FREQUENT-COLLISION REGIME

For the shear viscosity η [Eq. (7)], one should calculate the mean-velocity u_z [Eq. (9)] and the stress-tensor $\delta\sigma_{zz}$ [Eq. (8)] component. Using Eqs. (4), (10) and (2), one finds (see Ref. [37])

$$\eta = \text{Re} \left(\frac{9i\sqrt{\pi}}{4\sqrt{15}} \frac{nTc}{\omega} \frac{A_2}{A_1} \right). \quad (\text{A1})$$

For calculations of the ratio A_2/A_1 and derivations of the dispersion equation to obtain the velocity c in the FCR, we substitute the plane-wave solution (4) for the distribution function δf with the multipole expansion (10) for $A(\hat{p})$ into the BKE (3). After simple algebraic transformations, one finally arrives at the following linear equations for A_ℓ [37]:

$$\sum_{\ell=0}^{\infty} \mathcal{B}_{\mathcal{L}\ell}(c) A_\ell = 0, \quad (\text{A2})$$

where

$$\mathcal{B}_{\mathcal{L}\ell}(c) \equiv c\delta_{\mathcal{L}\ell} - C_{\ell 1;\mathcal{L}} + i\Upsilon \delta_{\ell\mathcal{L}}(1 - \delta_{\ell 0})(1 - \delta_{\ell 1}). \quad (\text{A3})$$

Here, $\delta_{\ell\mathcal{L}}$ is the Kronecker symbol,

$$\Upsilon = \frac{c}{\omega\tau}, \quad (\text{A4})$$

$$\begin{aligned} C_{\ell 1;\mathcal{L}} &= \sqrt{\frac{4\pi}{3}} \int d\Omega_p Y_{\mathcal{L}0}(\hat{p}) Y_{10}(\hat{p}) Y_{\ell 0}(\hat{p}) \\ &= \sqrt{\frac{2\ell+1}{2\mathcal{L}+1}} (C_{\ell 0,10}^{\mathcal{L}0})^2, \end{aligned} \quad (\text{A5})$$

$C_{\ell 0,10}^{\mathcal{L}0}$ are the Clebsh-Gordan coefficients [44].

To derive the dispersion equation (13) for the ratio $c = \omega/(kv_T)$ in the FCR and, then, calculate the amplitude ratio in Eq. (A1) for the viscosity η , one has to specify a small perturbation parameter $\omega\tau$ in the perturbation expansion of $\delta f(\mathbf{r}, \mathbf{p}, t)$. Then, in the FCR (small $\omega\tau$), one can truncate the expansion of $A(\hat{p})$ (10) over spherical functions $Y_{\ell 0}(\hat{p})$, and relatively, the linear system of equations (A2) at the quadrupole value of ℓ , $\ell \leq 2$, because of a fast convergence of the sum (10) over ℓ at $\omega\tau \ll 1$ [17]. Then, for the amplitude ratios $A_{\ell+1}/A_\ell$, one finds from Eq. (A2) (see Ref. [37])

$$\frac{A_0}{A_1} = \frac{1}{\sqrt{3}c}, \quad (\text{A6})$$

$$\frac{A_2}{A_1} = \frac{2}{\sqrt{15}(c+i\Upsilon)}, \quad (\text{A7})$$

where Υ is given by Eq. (A4). Within the FCR, because of large Υ [Eq. (A4)], one notes the convergence of the coefficients A_ℓ of the expansion in multiplicities (10) [see Eq. (A7) and Refs. [17,26,37]]. Therefore, from the zero determinant of the 3×3 matrix at the quadrupole value $\ell \leq 2$ and $\mathcal{L} \leq 2$ for nontrivial solutions of the truncated system of Eq. (A2), we derive the cubic dispersion equation (13) (see Ref. [37]). Substituting the underdamped (WAPW) solution (14) for the sound velocity c , from Eq. (A1) one obtains Eq. (17) for the shear viscosity η_{FC} .

The volume viscosity ζ can be calculated in a similar way:

$$\zeta = \text{Re} \frac{\delta\mathcal{P}}{\partial u_z / \partial z}, \quad (\text{A8})$$

where u_z and $\delta\mathcal{P}$ are, respectively, the mean velocity u_z [Eq. (9)] and the dynamical variation of the isotropic kinetic pressure,

$$\delta\mathcal{P} = \int \frac{p^2}{3m} d\mathbf{p} \delta f(\mathbf{r}, \mathbf{p}, t). \quad (\text{A9})$$

Using the WAPW variations δf given by Eq. (4) and multipolarity expansion (10) ($\omega = k_r c_r v_T$ is real), one finds

$$\zeta = \frac{\sqrt{3\pi} nT}{2\omega} \text{Re} \left(\frac{cA_0}{iA_1} \right). \quad (\text{A10})$$

According to Eq. (A6) for A_0/A_1 , the complex sound velocity c is canceled, and therefore, for a weak plane-wave absorption, one obtains $\zeta = 0$ (see also Refs. [9,29]).

APPENDIX B: THE RARE-COLLISION REGIME

For the integral collision term δSt of the BKE (3) in the τ approximation, one writes [28]

$$\begin{aligned} \delta St &= -\frac{\delta f}{\tau} + \frac{1}{\tau} [A_0 Y_{00}(\hat{p}) + A_1 Y_{10}(\hat{p})] \\ &\quad \times f_{\text{eq}}(p) \exp(-i\omega t + ikz). \end{aligned} \quad (\text{B1})$$

We introduce now new notations, $\hat{p} = \cos\theta = x$ and $\xi = c + i\Upsilon$ [see Eq. (A4)]. Using the expansion (10) of the amplitude $A(\hat{p})$ over spherical functions $Y_{\ell 0}(\hat{p})$ with their orthogonal properties, and the explicit expressions for Y_{00} and Y_{10} , one obtains

$$A(x) = -\frac{i}{\sqrt{4\pi}(x-\xi)} [(\Upsilon - ix\epsilon_0)A_0 + \sqrt{3}x\Upsilon A_1]. \quad (\text{B2})$$

For convenience of calculations we introduced also $\epsilon_0 = +0$ as an infinitesimally small parameter. Integrating over the spherical angles of $d\Omega_p = \sin\theta d\theta d\varphi$ with the spherical functions $Y_{\ell 0}(\hat{p})$, one has

$$\begin{aligned} A_\ell &= \int A(\hat{p}) Y_{\ell 0}(\hat{p}) d\Omega_p \\ &= \sqrt{\pi(2\ell+1)} \int_{-1}^1 A(x) P_\ell(x) dx, \end{aligned} \quad (\text{B3})$$

where $P_\ell(x) = (4\pi/(2\ell+1))^{1/2} Y_{\ell 0}(\hat{p})$ is the standard Legendre polynomials [44]. Substituting Eq. (B2) into Eq. (B3), one finds

$$A_\ell = i\sqrt{2\ell+1} \{ \Upsilon A_0 Q_\ell(\xi) - (\sqrt{3}\Upsilon A_1 - i\epsilon_0 A_0) [\delta_{\ell 0} - \xi Q_\ell(\xi)] \}, \quad (\text{B4})$$

where $Q_\ell(\xi)$ are the Legendre functions of a second kind,

$$Q_\ell(\xi) = \frac{1}{2} \int \frac{P_\ell(x) dx}{\xi - x}, \quad (\text{B5})$$

in particular,

$$Q_1(\xi) = \frac{\xi}{2} \ln \left(\frac{\xi + 1}{\xi - 1} \right) - 1. \quad (\text{B6})$$

These functions obey the recurrence equations,

$$\begin{aligned} Q_1 &= \xi Q_0(\xi) - 1, \\ 2Q_2(\xi) &= 3\xi Q_1(\xi) - Q_0(\xi), \dots \end{aligned} \quad (\text{B7})$$

For $\ell = 0$ and 1 one gets from Eq. (B4) the following system of linear equations with respect to A_0 and A_1 :

$$\begin{aligned} [1 - i\Upsilon Q_0(\xi) - \epsilon_0 Q_1(\xi)] A_0 - i\sqrt{3}\Upsilon Q_1(\xi) A_1 &= 0, \\ i\sqrt{3}(\Upsilon - i\epsilon_0\xi) Q_1(\xi) A_0 - [1 - 3i\xi\Upsilon Q_1(\xi)] A_1 &= 0. \end{aligned} \quad (\text{B8})$$

Nonzero solutions of this system of linear equations exist under the condition of zero for its determinant. This leads to the dispersion equation (18) for the sound velocity c through ξ .

Using Eqs. (B4) and (B7), one has

$$\frac{A_\ell}{A_{\ell-1}} = \sqrt{\frac{2\ell+1}{2\ell-1}} \frac{Q_\ell}{Q_{\ell-1}} \quad \text{for } \ell \geq 1. \quad (\text{B9})$$

With the help of Eq. (A1), we arrive at [37]

$$\eta = \text{Re} \left[\frac{3i\sqrt{\pi}}{4} \frac{nTc}{\omega} \frac{Q_2(\xi)}{Q_1(\xi)} \right]. \quad (\text{B10})$$

Taking into account the recurrence equations [Eq. (B7)], one can rewrite Q_2/Q_1 in terms of $Q_1^{-1}(\xi)$ and ξ , which are given by the RCR dispersion equation (18). In the limit $\epsilon_0 \rightarrow +0$ at first order in $\Upsilon \sim 1/(\omega\tau)$, one then finds from Eq. (18)

$$\frac{1}{Q_1(\xi)} \approx i\Upsilon \left(3\xi + \frac{1}{\xi} \right) \approx \frac{4i}{\omega\tau}. \quad (\text{B11})$$

We used solution (19) of the sound velocity c to the dispersion equation (18) in the second expression, valid at first-order perturbation expansion over $1/(\omega\tau)$. Using this expression and solution (19) for c , from Eq. (B10) for the RCR shear viscosity η , one obtains Eq. (21).

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