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# Energy-dissipation anomaly in systems of localized waves

Basile Gallet

*Service de Physique de l'État Condensé, CEA, CNRS, Université Paris-Saclay, CEA Saclay, 91191 Gif-sur-Yvette, France*

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We study the statistics of the power $P$ dissipated by waves propagating in a one-dimensional disordered medium with damping coefficient $\nu$. An operator imposes the wave amplitude at one end, therefore injecting a power $P$ that balances dissipation. The typical realization of $P$ vanishes for $\nu \to 0$: Disorder leads to localization and total reflection of the wave energy back to the emitter, with negligible losses. More surprisingly, the mean dissipated power $\langle P \rangle$ averaged over the disorder reaches a finite limit for $\nu \to 0$. We show that this "anomalous dissipation" $\lim_{\nu\to 0} \langle P \rangle$ is directly given by the integrated density of states of the undamped system. In some cases, this allows us to compute the anomalous dissipation exactly, using properties of the undamped system only. As an example, we compute the anomalous dissipation for weak correlated disorder and for Gaussian white noise of arbitrary strength. Although the focus is on the singular limit $\nu \to 0$, we finally show that this approach is easily extended to arbitrary $\nu$.

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Anderson localization refers to the peculiar behavior of waves propagating in a disordered medium: The eigenmodes are localized in space and decay typically exponentially away from a center point [1]. This phenomenon naturally arises in quantum mechanics to describe the influence of impurities on the electrical conductivity of metals at low temperature. In macroscopic physics, wave localization is a generic phenomenon as well, which applies to light [2–4], sound [5,6], surface waves [7–9], and Rossby waves [10–12], among other examples. Waves interacting with mean flows, a common situation in oceanographic and atmospheric fluid dynamics, can sometimes be recast into a Schrödinger equation for a particle in a disordered electromagnetic field [13,14], suggesting the possible occurrence of wave localization.

Such classical waves are necessarily dissipative, and in a stationary state an energy source is needed to compensate for dissipation. The present Rapid Communication deals with the energy budget of systems of localized waves: How much power is needed to sustain the waves? What are the statistics of the dissipated power? Can we relate the dissipated power to properties of the idealized undamped system?

Although the study of localization in mesoscopic physics and in classical wave systems share many similarities, they also differ in many respects: First, classical waves are dissipated on a typical scale which can be much smaller than the system size, making the system effectively infinite from the point of view of the waves. Second, to study the effect of disorder in a mesoscopic conductor, one typically considers a wave incoming on a disordered region of finite size. The electrical conductivity is then determined from the transmission coefficients of the disordered region [15]. By contrast, in many situations classical waves are forced directly inside the disordered region. This is, for instance, the case of a storm generating waves on a moving ocean [16]. While the two situations are connected, this difference between an incoming wave of finite amplitude and an internal source inside the disordered region drastically modifies the statistics of the power dissipated in the system, as pointed out by Klyatskin and Saichev [17]: As shown in Fig. 1(b), imposing the oscillation amplitude somewhere inside the disordered region allows for rare events of extremely large dissipated power, whereas in the case of an incoming wave of fixed amplitude this dissipated power is always bounded by the finite incoming energy flux.

We address the questions raised in the outset for one-dimensional semi-infinite systems, where the wave amplitude is imposed at the origin. We emphasize the intuitive case of the viscously damped semi-infinite string with inhomogeneous density [see Fig. 1(a)], but the waves could equally be light, sound, etc. In a stationary state, the operator maintaining the sinusoidal motion of the end point provides the power dissipated by viscous friction. Intuitively, one expects this dissipated power to be reduced by disorder: Strong disorder leads to wave localization on a length scale $\ell_{\text{loc}}$ much shorter than the viscous decay length $\ell_{\text{diss}} \sim \nu^{-1}$, where $\nu$ denotes the damping coefficient; the waves are reflected back to the wavemaker by the density inhomogeneities before they undergo significant damping, and the input power is approximately zero.

Surprisingly, this intuition only holds for a typical realization of the disorder: While the most probable value of the dissipated power vanishes for vanishing damping coefficient $\nu$, the dissipated power averaged over the disorder reaches a finite limit as $\nu \to 0$ [17]. Such a singular limit of the dissipated power is reminiscent of the *dissipation anomaly* of turbulent flows, which dissipate a finite power in the limit of vanishing viscosity [18], and we therefore call it "anomalous dissipation" in the present context as well. Its precise determination is of central importance for the energetics of the systems mentioned at the outset.

In the following, we compute the anomalous dissipation in rather general situations, including correlated disorder and disorder of arbitrary strength. To wit, we show that the anomalous dissipation is given by the integrated density of states of the undamped system, establishing a connection between two *a priori* unrelated quantities. We can therefore deduce the mean dissipated power from properties of the undamped system only, by borrowing exact results on the density of states of the latter.

*The inhomogeneous string.* We consider the system sketched in Fig. 1(a): An inhomogeneous string occupies the semi-infinite region $x \in [0,\infty[$. In the rest position, it

 

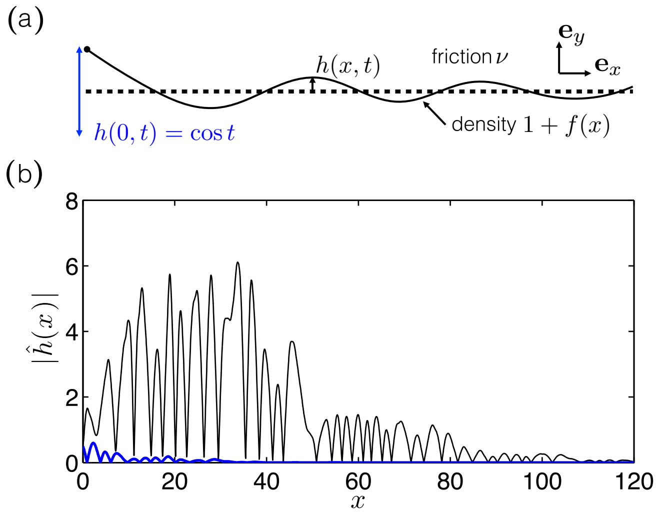


FIG. 1. (a) An operator imposes the sinusoidal motion of the end point $x = 0$ of a semi-infinite inhomogeneous string. The parameters are indicated in dimensionless form. (b) Solution of (5) for Ornstein-Uhlenbeck disorder with $\sigma = 1$, $\ell = 1$, $\nu = 0.002$, and $L = 500$. Blue thick line: A typical realization. The amplitude rapidly decreases as a result of localization. Black line: A rare event with large dissipated power.

is subject to a uniform tension $T_0\mathbf{e}_x$. It has inhomogeneous linear density $\rho(x)$ and its transverse motion $h(x,t)\mathbf{e}_y$ is subject to viscous friction with a damping coefficient $\mu$. An operator drives transverse waves of the string by imposing a periodic sinusoidal motion of the end point $x = 0$ at frequency $\omega$: $h(0,t) = a\cos(\omega t)$. Denoting the mean density as $\rho_0$ and the total density as $\rho(x) = \rho_0[1 + f(x)]$, we nondimensionalize time with $\omega^{-1}$, $x$ with $\omega^{-1}\sqrt{T_0/\rho_0}$, and $h(x,t)$ with $a$. Restricting attention to small-amplitude waves, the dimensionless linear equation of motion reads

$$[1 + f(x)]\partial_{tt}h - \partial_{xx}h + \nu\partial_t h = 0, \tag{1}$$

where the variables $x$, $t$, and $h$ are now dimensionless, and $\nu = \mu/\rho_0\omega$. The mean-zero function $f(x)$ representing the density fluctuations is a random process with homogeneous statistics in $x$. The dimensionless boundary conditions are $h(0,t) = \cos(t)$ and $\lim_{x\to\infty} h(x,t) = 0$. One can form the energy equation by multiplying (1) with $\partial_t h$ and integrating over the semi-infinite domain,

$$\frac{d\mathcal{E}}{dt} = -\partial_t h|_0\,\partial_x h|_0 - \nu\int_0^\infty \partial_t h^2 dx, \tag{2}$$

where

$$\mathcal{E} = \int_0^\infty [1 + f(x)]\frac{\partial_t h^2}{2} + \frac{\partial_x h^2}{2}dx. \tag{3}$$

The subscript $\cdot|_0$ means "evaluated at $x = 0$." The first term on the right-hand side of (2) is the work done by the force exerted by the operator to maintain the sinusoidal motion of the end point. The second term is viscous dissipation. After some transient, the system reaches a stationary state, for which the dissipated power averaged over the wave period is

$$P = -\overline{\partial_t h|_0\,\partial_x h|_0} = \nu\int_0^\infty \overline{\partial_t h^2}dx, \tag{4}$$

where the overline denotes an average over the wave period. $P$ is the focus of this Rapid Communication.

The uniform system $f(x) = 0$ deserves a few comments: The waves decay exponentially with $x$ on a length scale proportional to $\nu^{-1}$. For small $\nu$, the dissipated power is $P = \frac{1}{2}$, which corresponds to the energy that would be radiated towards $x \to \infty$ in an undamped infinite system. However, this result crucially depends on the infinite-domain limit being taken before the small-$\nu$ limit. Indeed, for a finite domain $x \in [0,L]$, when $\nu \ll 1/L$, a stationary wave forms inside the domain and the input power is approximately zero.

We study the stationary state using Fourier transform in time, $h(x,t) = \hat{h}(x)e^{it} + \hat{h}^*(x)e^{-it}$, where $^*$ denotes the complex conjugate. The equation of motion becomes

$$\hat{h}''(x) + [1 - i\nu + f(x)]\hat{h}(x) = 0, \tag{5}$$

where the prime denotes a derivative. The boundary conditions are $\hat{h}(0) = \frac{1}{2}$ and $\lim_{x\to\infty}\hat{h}(x) = 0$, and the injected power becomes $P = -\mathrm{Im}\{\hat{h}'(0)\}$.

Qualitatively, the process $P$ results from the competition between viscous damping and localization: Considering weak disorder $f(x)$ with standard deviation $\sigma \ll 1$, the waves decay exponentially on a typical scale $\min\{\ell_{\rm loc},\ell_{\rm diss}\}$, where $\ell_{\rm loc} \sim \sigma^{-2}$ is the localization length and $\ell_{\rm diss} \sim \nu^{-1}$ is the damping length. When $\ell_{\rm diss} \ll \ell_{\rm loc}$, viscous damping is the dominant process and a power close to $\frac{1}{2}$ is dissipated. By contrast, when $\ell_{\rm loc} \ll \ell_{\rm diss}$, the waves get localized and totally reflected before significant damping takes place, which leads to $P \simeq 0$. The competition between these two phenomena is governed by the ratio $S = \sigma/\sqrt{\nu}$.

To test these predictions, we first consider fluctuations $f(x)$ given by a stationary Ornstein-Uhlenbeck process with standard deviation $\sigma$ and correlation length $\ell$. We solve (5) numerically by discretizing and inverting the linear operator inside a domain $x \in [0,L]$, with $L$ large enough for the results to be independent of $L$. In Fig. 2 we show the most probable value $P_{\rm mp}$ of $P$ for various $\sigma$ and $\nu$. $P_{\rm mp}$ goes to zero in the limit of small damping $\nu$ or large disorder $\sigma \to \infty$. Plotting $P_{\rm mp}$ as a function of $S$ leads to a good collapse of the data, which indicates that $P_{\rm mp}$ is indeed governed by the competition between localization and damping. This is illustrated by the typical solution shown in Fig. 1(b), where the waves get localized on a length scale much shorter than $\nu^{-1}$.

In contrast with $P_{\rm mp}$, the mean dissipated power $\langle P\rangle$ over the realizations of the disorder has a very singular behavior (see Fig. 2): It remains nonzero and is independent of the damping coefficient $\nu$ for small $\nu$. The observation of this singular limit traces back to Klyatskin and Saichev [17], who showed that the mean energy flux is unaffected by weak uncorrelated disorder to lowest order in noise strength. Intuition on this "dissipation anomaly" can be gathered from the rare event displayed in Fig. 1: The motion of the end point excites modes which are localized around center points $x_c$ far away from the origin, resulting in a large wave amplitude over a significant distance. These rare events are reminiscent of the transmission resonances observed in finite-size systems [19,20]. They have a large wave amplitude and contribute significantly to the mean dissipated power $\langle P\rangle$. The remainder of this Rapid Communication is dedicated to the analytical computation of

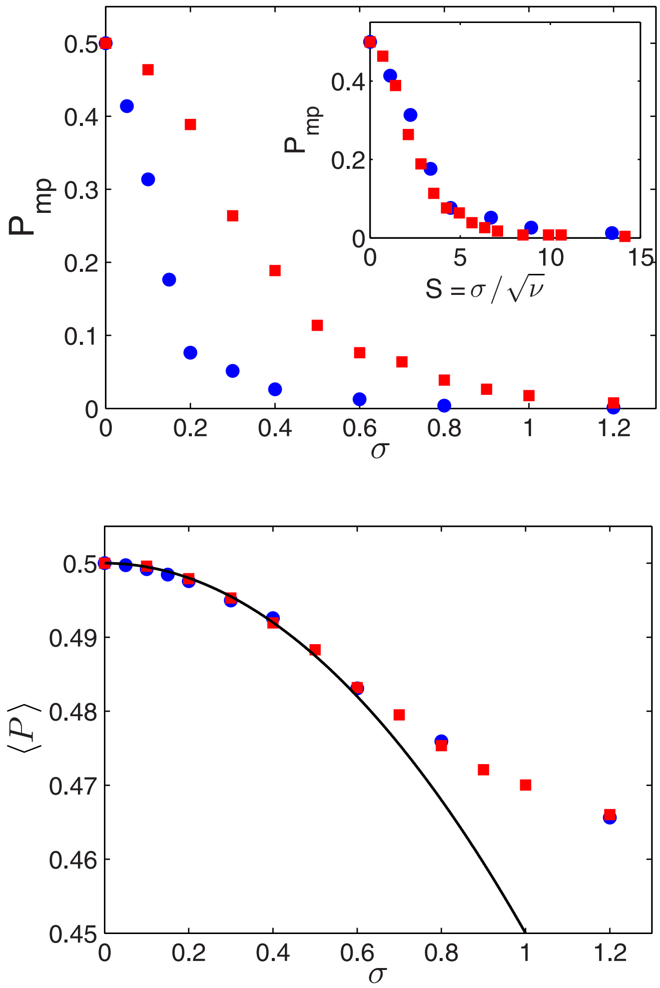


FIG. 2. Top: Most probable value of the dissipated power for Ornstein-Uhlenbeck disorder with $\ell = 1$, and damping $\nu = 0.002$ (●) and $\nu = 0.02$ (■). Rescaling $\sigma$ with $\sqrt{\nu}$ leads to a good collapse of the data, which highlights the competition between damping and localization. Bottom: For small damping, the mean dissipated power $\langle P \rangle$ over the disorder is finite and independent of $\nu$ (same symbols as above). The solid line is the weak-disorder theoretical expression (16).

the mean dissipated power $\langle P \rangle$, both for finite $\nu$ and in the singular limit $\nu \to 0$.

*A useful symmetry.* Introducing the complex Riccati variable $R(x) = \hat{h}'(x)/\hat{h}(x)$, the equation of motion (5) becomes

$$R'(x) = -1 - f(x) + i\nu - R^2(x), \tag{6}$$

and the dissipated power is

$$P = -\tfrac{1}{2}\mathrm{Im}\{R(0)\}. \tag{7}$$

The idea is then the following: Consider a solution $\hat{h}(x)$ to Eq. (5) and a given abscissa $x_0 > 0$. $\hat{h}(x)$ satisfies the boundary condition $\hat{h}(0) = \frac{1}{2}$. Define the translated and rescaled displacement $\hat{g}(y) = \hat{h}(x_0 + y)/2\hat{h}(x_0)$, which satisfies

$$\hat{g}''(y) + [1 + f(x_0 + y) - i\nu]\hat{g}(y) = 0, \tag{8}$$

together with the boundary conditions $\hat{g}(0) = \frac{1}{2}$ and $\lim_{y\to\infty} \hat{g}(y) = 0$. In other words, it satisfies the same equation and boundary conditions as $\hat{h}$, but for a translated noise $f(x_0 + x)$. The input power for this translated realization of the noise is

$$P = -\mathrm{Im}\{\hat{g}'(0)\} = -\mathrm{Im}\left\{\frac{\hat{h}'(x_0)}{2\hat{h}(x_0)}\right\} = -\frac{1}{2}\mathrm{Im}\{R(x_0)\}. \tag{9}$$

We conclude that the value of $-\frac{1}{2}\mathrm{Im}\{R(x)\}$ at each point gives a realization of $P$ for a translated version of the initial realization of the noise. This method allows one to obtain many realizations of $P$ from a single numerical integration of the Riccati equation (6), which drastically speeds up the numerical determination of the statistics of $P$. For instance, to obtain the average input power, one can integrate (6) numerically starting from some initial condition with a negative imaginary part [21], and average the resulting function to obtain

$$\langle P \rangle = -\tfrac{1}{2}\mathrm{Im}\{\langle R(x)\rangle_x\}, \tag{10}$$

where $\langle\cdot\rangle_x$ denotes an average over $x$. In the following, we assume ergodicity: After some finite-length transient, the process $R(x)$ reaches a stationary state. Whenever needed, we replace ensemble averages $\langle\cdot\rangle$ in this stationary state by spatial averages $\langle\cdot\rangle_x$, and vice versa.

*Relation to the density of states.* The relation (10) has useful consequences. Indeed, $\langle R(x)\rangle$ is a central object of the theory of localization called the characteristic exponent [22]. For nondissipative waves satisfying the equation

$$\hat{h}''(x) + [E + f(x)]\hat{h}(x) = 0, \tag{11}$$

which is identical to the undamped version of (5) up to a change of nondimensionalization, the characteristic exponent $\Omega(E)$ is an analytic function of $E$ in the upper half of the complex plane [23], which reads

$$\Omega(E) = \gamma(E) - i\pi N(E). \tag{12}$$

Here, $\gamma(E)$ is the Lyapunov exponent, i.e., the inverse localization length for waves satisfying (11), and $N(E)$ is the integrated density of states per unit length. In the limit $\nu \to 0$, Eq. (5) reduces to (11) with $E = 1$: The mean of the Riccati variable $R$ is therefore given by the characteristic exponent $\Omega(E)$ for $E = 1$. Inserting the resulting value of $\langle R\rangle$ into (10) leads to the following expression for the dissipation anomaly,

$$\lim_{\nu\to 0}\langle P \rangle = \frac{\pi}{2} N(E = 1), \tag{13}$$

where $N(E = 1)$ is the integrated density of states of Eq. (11) evaluated at $E = 1$. The relation (13) is useful in many respects. First, it establishes a connection between the mean dissipated power of the weakly damped system and the density of states of the conservative one. Second, it provides a way to readily compute the anomalous dissipation from known results about the undamped system.

*Weak-disorder expansion.* The relation (10) is well suited to compute the mean dissipated power in the weak-noise limit. Consider weak disorder $f(x) = \epsilon \tilde{f}(x)$, with $\epsilon \ll 1$, and expand the Riccati variable as $R(x) = R_0(x) + \epsilon R_1(x) + \epsilon^2 R_2(x) + O(\epsilon^3)$ before substituting into the Riccati equation (6). Solving at orders $O(1)$ to $O(\epsilon^2)$ and averaging yields, in the limit $\nu \to 0$,

$$R_0 = -i, \quad \langle \epsilon R_1 \rangle = 0, \tag{14}$$

$$\langle \epsilon^2 R_2 \rangle = -\frac{1}{4} \int_{x=0}^{\infty} e^{-2ix} \mathcal{C}(x) dx, \tag{15}$$

where $R_0$ corresponds to the uniform string, and $\mathcal{C}(y) = \langle f(x) f(x+y) \rangle$ denotes the two-point correlation function of the disorder. As an example, substituting the correlation function $\mathcal{C}(x) = \sigma^2 e^{-|x|/\ell}$ of Ornstein-Uhlenbeck fluctuations leads to the following expression for the anomalous dissipation,

$$\lim_{\nu \to 0} \langle P \rangle = \frac{1}{2} - \frac{\sigma^2 \ell^2}{16\ell^2 + 4} + O(\sigma^4). \tag{16}$$

The $O(\sigma^2)$ correction is negative, indicating that the anomalous dissipation is less than the power that would be radiated towards infinity in the absence of disorder. One can check that this correction vanishes in the white-noise limit ($\ell \to 0$, with $D = \sigma^2 \ell$ constant), in agreement with Ref. [17]. For correlated noise, (16) is in excellent agreement with the numerical values of $\langle P \rangle$ for weak noise and damping (see Fig. 2).

*White noise of arbitrary strength.* The Schrödinger equation (5) shows that our problem is analogous to particles maintained at fixed concentration and energy (i.e., frequency) at the boundary $x = 0$ of a disordered medium with random potential and weak absorption $\nu \ll 1$ occupying $x > 0$. $P$ is then the flux of particles (and energy) into the disordered medium: Surprisingly, a finite mean flux $\langle P \rangle$ is absorbed by the medium in the limit of vanishing absorption $\nu \to 0$.

Motivated by this context, we consider a second example, where the disorder is Gaussian white noise with a correlation function $C(x) = D\delta(x)$, and follow a route other than perturbative expansion: We consider disorder of arbitrary strength $D$ and borrow the exact expression of the characteristic exponent determined by Halperin [24] to evaluate (13). In our notations this leads to

$$\lim_{\nu \to 0} \langle P \rangle = -\frac{D^{1/3}}{2^{4/3}} \mathrm{Im} \left\{ \frac{\mathrm{Ai}'(\xi) - i\, \mathrm{Bi}'(\xi)}{\mathrm{Ai}(\xi) - i\, \mathrm{Bi}(\xi)} \right\}, \tag{17}$$

where Ai and Bi denote Airy functions and $\xi = -(2/D)^{2/3}$. This prediction then reduces to $\lim_{\nu \to 0} \langle P \rangle = D^{1/3} \{\pi 2^{4/3} [\mathrm{Ai}^2(\xi) + \mathrm{Bi}^2(\xi)]\}^{-1}$, which is in excellent agreement with the numerical results for arbitrary disorder strength $D$ and small $\nu$ (see the solid line in Fig. 3). Here, the anomalous dissipation departs from $\frac{1}{2}$ as $D^2$ initially: There is no term linear in $D$, in agreement with the vanishing $O(\sigma^2)$ correction in (16) in the white-noise limit. Interestingly, in contrast with the Ornstein-Uhlenbeck case, the anomalous dissipation is always greater than $\frac{1}{2}$: On average, it costs more power to sustain the waves in the disordered system than in the absence of disorder.

*Discussion.* Although the primary focus of this study is the anomalous dissipation arising in the singular limit $\nu \to 0$, the results can be easily extended to finite damping $\nu$. Indeed,

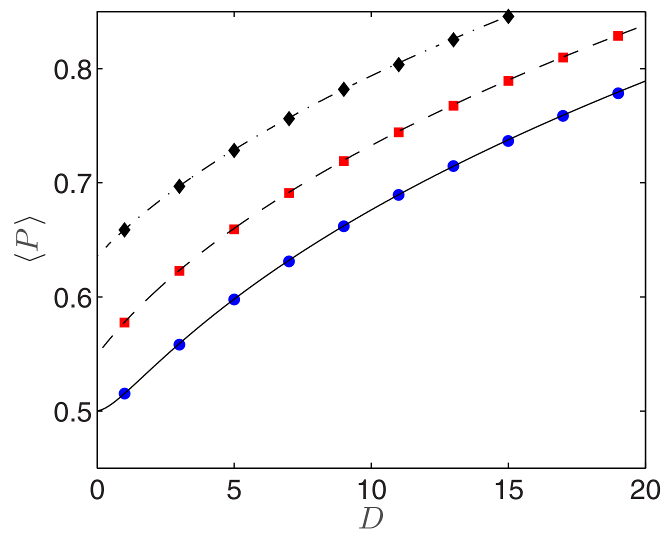


FIG. 3. Mean dissipated power as a function of the strength $D$ of the white-noise disorder. Symbols are numerical solutions of the Riccati equation for $\nu = 2$ (♦), $\nu = 1$ (■), and $\nu = 0.05$ (•). The theoretical prediction is shown for $\nu = 2$ (dashed-dotted), $\nu = 1$ (dashed) and in the limit $\nu \to 0$ (solid).

Eq. (5) corresponds to (11) for $E = 1 - i\nu$. We cannot directly inject this value into known expressions of $\Omega(E)$, because the latter are valid in the upper half of the complex plane only. We therefore introduce the function $G(x) = -R^*(-x)$, which satisfies Eq. (6) with $i\nu$ replaced by $-i\nu$ and $f(x)$ by $f(-x)$. $\langle G \rangle$ now corresponds to the characteristic exponent evaluated at $E = 1 + i\nu$, which lies in the upper half of the complex plane. Using $\mathrm{Im}\langle R \rangle = \mathrm{Im}\langle G \rangle$, we finally obtain

$$\begin{aligned} \langle P \rangle &= -\tfrac{1}{2} \mathrm{Im}\{\Omega(1 + i\nu)\} \\ &= -\tfrac{1}{2} \mathrm{Im}\{\gamma(1 + i\nu) - i\pi N(1 + i\nu)\}. \end{aligned} \tag{18}$$

For the white-noise example presented above, taking into account finite $\nu$ then simply amounts to substituting $\xi = -(2/D)^{2/3}(1 + i\nu)$ into the right-hand side of (17). We show in Fig. 3 that this prediction agrees perfectly with the numerical solutions.

We conclude by stressing the fact that (12) and (13) allow one to write Kramers-Kronig relations between the Lyapunov exponent and the mean dissipated power [23]: The latter can be written as an integral of the Lyapunov exponent for waves at various frequencies (i.e., various energies $E$). Numerically, such relations may allow one to estimate the mean dissipated power from typical realizations of the disorder only, without having to sample extensively the rare strongly dissipative events.

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