Counterexamples to Moffatt's statements on vortex knots

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One of the well-known problems of hydrodynamics is studied: the problem of classification of vortex knots for ideal fluid flows. In the literature there are known Moffatt statements that all torus knots $K_{m,n}$ for all rational numbers m/n ($0 < m/n < \infty$) are realized as vortex knots for each one of the considered axisymmetric fluid flows. We prove that actually such a uniformity does not exist because it does not correspond to the facts. Namely, we derive a complete classification of all vortex knots realized for the fluid flows studied by Moffatt and demonstrate that the real structure of vortex knots is much more rich because the sets of mutually nonisotopic vortex knots realized for different axisymmetric fluid flows are all different. An exact formula for the limit of the hydrodynamic safety factor q_h at a vortex axis is derived for arbitrary axisymmetric fluid equilibria. Another exact formula is obtained for the limit of the magnetohydrodynamics safety factor q at a magnetic axis for the general axisymmetric plasma equilibria.

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I. INTRODUCTION

Euler equations of the dynamics of an ideal incompressible fluid with a constant density ρ have the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot \text{grad})\mathbf{V} = -\,\text{grad}\left(\frac{p}{\rho} + \Phi\right), \quad \text{div}\,\mathbf{V} = 0. \tag{1.1}$$

Here $\mathbf{V}(t,\mathbf{x})$ is the fluid velocity vector field, $p(t,\mathbf{x})$ the pressure, and Φ the gravitational potential. As known, the vortex field curl $\mathbf{V}$ is transformed in time by the flow diffeomorphisms (or "is frozen in the flow" [1]) and therefore any knot formed by a closed vortex line at a time t is transformed by the flow into an isotopic knot.

Since Kelvin's works there exists the problem of classification of all mutually nonisotopic vortex knots for concrete solutions to Euler equations (1.1). For the steady axisymmetric fluid flows, this problem was studied by Moffatt in [2–4], where he claimed that all torus knots $K_{m,n}$ for all rational numbers m/n ($0 < m/n < +\infty$) are realized as vortex knots for all considered flows. We show in this paper that such a uniformity actually does not exist and that the families of mutually nonisotopic vortex knots realized for different axisymmetric fluid flows are in general different.

Axisymmetric steady fluid flows have velocity vector fields

$$\mathbf{V}(r,z) = -\frac{1}{r}\frac{\partial \psi}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial \psi}{\partial r}\hat{\mathbf{e}}_z + \frac{w(r,z)}{r}\hat{\mathbf{e}}_\varphi, \tag{1.2}$$

where $\psi(r,z)$ is the Stokes stream function and $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_z$, and $\hat{\mathbf{e}}_\varphi$ are the unit orts in the cylindrical coordinates r, z, and φ. The steady axisymmetric Euler equations (1.1) are reduced to the Grad-Shafranov equation

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = r^2\frac{dH}{d\psi} - C\frac{dC}{d\psi}, \tag{1.3}$$

where $H(\psi)$ and $C(\psi)$ are arbitrary smooth functions connected with the vector field $\mathbf{V}$ [Eq. (1.2)] and pressure p by the relations [1]

$$\frac{p}{\rho} + \Phi + \frac{1}{2}|\mathbf{V}|^2 = H(\psi), \quad w(r,z) = C(\psi). \tag{1.4}$$

The Grad-Shafranov equation (1.3) was first derived for the plasma equilibria

$$\text{curl}\,\mathbf{B}\times\mathbf{B} = \text{grad}(\mu p + \rho\mu\Phi), \quad \text{div}\,\mathbf{B} = 0, \tag{1.5}$$

where $\mathbf{B}(\mathbf{x})$ is the magnetic field and μ the magnetic permeability. The hydrodynamics equilibrium equations

$$\text{curl}\,\mathbf{V}\times\mathbf{V} = -\,\text{grad}(p/\rho + \rho\mu\Phi + |\mathbf{V}|^2/2), \quad \text{div}\,\mathbf{V} = 0 \tag{1.6}$$

and Eqs. (1.5) are equivalent [5].

Kruskal and Kulsrud [6] proved for Eqs. (1.5) that surfaces $p(\mathbf{x}) = \text{const}$ "by $\mathbf{B}\cdot\nabla p = 0$ are 'magnetic surfaces', in the sense that they are made up of lines of magnetic force, and simultaneously by $\mathbf{j}\cdot\nabla p = 0$ they are 'current surfaces'. If such a surface lies in a bounded volume of space and has no edges and if either $\mathbf{B}$ or $\mathbf{j}$ nowhere vanishes on it, then by a well-known theorem [7] it must be a toroid (by which we mean a topological torus) or a Klein bottle. The latter, however is not realizable in physical space".

Newcomb [8] stated that "it is easy to verify that the lines of force on a pressure surface are closed if and only if $\iota(P)/2\pi$ is rational; if it is irrational, the lines of force cover the surface ergodically". Here $\iota(P)$ is the rotational transform connected with the safety factor $q(P)$ [9] by the relation $q(P) = 2\pi/\iota(P)$. The safety factor q and the rotational transform $\iota = 2\pi/q$ are connected with stability of the plasma equilibria [9].

The analogous results for the equivalent equations (1.6) were published by Arnold in [10,11], where he added to [6,8] a statement that if a Bernoulli surface M intersects the boundary of the invariant domain D then M has "coordinates of the ring" and "all streamlines on M are closed". The results of [6,8] imply that for the plasma equilibrium equations (1.5) [and hence for the equivalent hydrodynamics equations (1.6)] all magnetic field knots (and correspondingly all vortex knots) are torus knots $K_{m,n}$ defined by the rational values m/n of the safety factor $q(P)$. Therefore, to classify the magnetic knots in magnetohydrodynamics (MHD) and vortex knots in steady hydrodynamics it is necessary to know the ranges of the corresponding safety factors.

For the fluid velocity field $\mathbf{V}(\mathbf{x})$ [Eq. (1.2)] its vortex field is

$$\operatorname{curl}\mathbf{V} = -\frac{C(\psi)_z}{r}\hat{\mathbf{e}}_r + \frac{C(\psi)_r}{r}\hat{\mathbf{e}}_z - \frac{1}{r}\left(\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz}\right)\hat{\mathbf{e}}_\varphi. \tag{1.7}$$

The surfaces $\psi(r,z) = \text{const}$ are invariant for both flows $\mathbf{V}(\mathbf{x})$ [Eq. (1.2)] and curl $\mathbf{V}(\mathbf{x})$ [Eq. (1.7)]. The smooth axisymmetric surfaces $\psi(r,z) = \text{const}$ are either tori $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1 \subset \mathbb{R}^3$, or spheres $\mathbb{S}^2_\psi$ or cylinders $\mathbb{C}^2_\psi = R_\psi \times \mathbb{S}^1$. Here the closed curves C_ψ and infinite lines R_ψ are purely poloidal, lie in the plane (r,z), and satisfy equation $\psi(r,z) = \text{const}$. The circle $\mathbb{S}^1$ corresponds to the angular coordinate $0 \leqslant \varphi \leqslant 2\pi$ and is z axisymmetric.

If a surface $\psi(r,z) = \text{const}$ is a torus $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$ then the fluid streamlines and vortex lines on the $\mathbb{T}^2_\psi$ are either infinite helical curves or closed curves. Moffatt's definition of the pitch (see [2], pp. 128–129) is as follows: "If any one vortex line is followed in the direction of increasing φ the value of z on that line varies periodically; the *pitch* p of the vortex line may conveniently be defined as twice the increase of φ, between successive zeros of z". The Moffatt definition is not applicable to the vortex lines on tori $\mathbb{T}^2_\psi$ that are located in domains $z > 0$ or $z < 0$ because there are no successive zeros of z on those vortex lines. The definition is applicable only if the poloidal curves C_ψ (and hence the tori $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$) are invariant under the reflection $z \to -z$, because only then the words "twice" and "successive zeros of z" become meaningful. This is true for the special solutions considered in [2], but not for the general case. Therefore, we will use another definition of the pitch.

Definition 1. The pitch $p(\psi)$ of a vortex line on a torus $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$ is the change of the angle φ along the vortex line when its poloidal projection makes one complete turn around the closed curve C_ψ. The formula for the pitch is

$$p(\psi) = \int_0^{t(\psi)} \frac{d\varphi}{dt} dt, \tag{1.8}$$

where the integral is taken along the curl $\mathbf{V}$ line and $t(\psi)$ is the period of its poloidal projection C_ψ. The hydrodynamic safety factor $q_h(\psi)$ is

$$q_h(\psi) = \frac{p(\psi)}{2\pi}. \tag{1.9}$$

Note that the pitch function $p(\psi)$ [Eq. (1.8)] is positive if the clockwise rotation along the curve C_ψ is accompanied by the increase of the total angle φ and $p(\psi)$ is negative if the total change of angle φ is negative. Definition 1 conforms to Moffatt's definition [2] when the curve C_ψ is invariant under the reflection $z \to -z$. Suppose the pitch is $p(\psi) = 2\pi m/n$, where m and n are integers [i.e., the safety factor $q_h(\psi) = m/n$]. Then, after n complete turns of the vortex line around the poloidal circle C_ψ, the angle φ is changed for $np(\psi) = 2\pi m$. Hence the vortex line is a closed curve because its end and starting points coincide. In other words, the vortex line is a torus knot $K_{m,n} \subset \mathbb{R}^3$.

If a closed magnetic field $\mathbf{B}$ line on a torus $\mathbb{T}^2_\psi$ makes m turns the long way $\mathbb{S}^1$ around and n turns the short way C_ψ around then its safety factor q is, as defined in [9,12], $q = m/n$.

Definition 2. The safety factor $q(\psi)$ of a magnetic field $\mathbf{B}$ line on a torus $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$ is the divided by 2π change of the azimuthal angle φ along the magnetic field line when its poloidal projection makes one complete turn around the closed curve C_ψ. The corresponding formula is

$$q(\psi) = \frac{1}{2\pi}\int_0^{\overline{t(\psi)}} \frac{d\varphi}{dt} dt. \tag{1.10}$$

Here the integral is taken along the magnetic field $\mathbf{B}$ line on the torus $\mathbb{T}^2_\psi$ and $\overline{t(\psi)}$ is the period of its poloidal projection C_ψ.

For the closed magnetic field $\mathbf{B}$-line definition (1.10) conforms to the definition in [9,12]. For the same ψ function $\psi(r,z)$, the two safety factors $q_h(\psi)$ and $q(\psi)$ are qualitatively different because they are defined correspondingly for the curl $\mathbf{V}$ in (1.6) and the magnetic field $\mathbf{B}$ in (1.5) that is analogous to the fluid velocity $\mathbf{V}$ in the equivalent equations (1.6). For example, when $q_h(\psi)$ [Eq. (1.9)] is a rational number m/n and hence the corresponding vortex line is a torus knot $K_{m,n}$, the MHD safety factor $q(\psi)$ [Eq. (1.10)] in general is an irrational number and the corresponding magnetic field $\mathbf{B}$ line is an infinite curve that is dense on the invariant torus $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$ and vice versa. Neither the pitch p nor the safety factor q_h is defined for the vortex lines belonging to invariant spheres $\mathbb{S}^2_\psi$ or cylinders $\mathbb{C}^2_\psi = R_\psi \times \mathbb{S}^1$ where $\psi(r,z) = \text{const}$.

In Sec. II we derive for the general solutions to the Grad-Shafranov equation (1.3) the exact formula for the limit of the hydrodynamic safety factor $q_h(\psi)$ [Eq. (1.9)] at the vortex axis $\mathbb{S}^1_{\psi_m}$ $(r = r_m, z = z_m, 0 \leqslant \varphi \leqslant 2\pi)$, $\psi_m = \psi(r_m,z_m)$:

$$q_h(\psi_m) = -\frac{\Delta\psi(a_m)}{r_m C'(\psi_m)\sqrt{\mathcal{H}(a_m)}}. \tag{1.11}$$

Here Δ is the Laplace operator and $\mathcal{H}(a_m) = \psi_{rr}(a_m)\psi_{zz}(a_m) - \psi^2_{rz}(a_m) \geqslant 0$ is the Hessian of the function $\psi(r,z)$ at the point $a_m = (r_m,z_m)$.

Remark 1. Formulas (1.9) and (1.11) prove that Moffatt's statements of [2–4], that at the vortex axis $\mathbb{S}^1_{\psi_m}$ the limit of the pitch $p(\psi_m)$ is always equal to infinity (or twist 0), can be true only for the degenerate solutions to the Grad-Shafranov equation (1.3) for which either $C'(\psi_m) = 0$ or $\mathcal{H}(a_m) = 0$. For the solutions studied by Moffatt $C'(\psi_m) = \alpha \neq 0$ and $\mathcal{H}(a_m) \neq 0$.

In Sec. III we derive for the general axisymmetric plasma equilibria (1.5) the exact formula for the limit of the MHD safety factor $q(\psi)$ [Eq. (1.10)] at a magnetic axis

$$q(\psi_m) = \frac{C(\psi_m)}{r_m\sqrt{\mathcal{H}(a_m)}}. \tag{1.12}$$

The formulas (1.11) and (1.12) are evidently different. This reflects the fact that for the same stream (or flux) function $\psi(r,z)$ the topological properties of the vortex field curl $\mathbf{V}(\mathbf{x})$ and magnetic field $\mathbf{B}(\mathbf{x})$ are essentially different.

There are two cases for which Eq. (1.3) becomes linear:

$$H(\psi) = H_1 + c^2\psi^2, \quad C(\psi) = \alpha\psi, \tag{1.13}$$

$$H(\psi) = \lambda\psi, \quad C(\psi) = \alpha\psi. \tag{1.14}$$

Exact solutions for the case (1.13) that are global plasma equilibria modeling astrophysical jets are presented in[13–15], where also extensive literature for the case (1.13) is quoted.

In Secs. IV–VII we study solutions to the Grad-Shafranov equation (1.3) satisfying the conditions (1.14):

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = \alpha^2(\xi r^2 - \psi), \quad \xi = \lambda/\alpha^2. \tag{1.15}$$

Substituting formulas (1.14) and (1.15) into Eq. (1.7), we find

$$\operatorname{curl}\mathbf{V} = \alpha\mathbf{V} - \alpha^2\xi r\hat{\mathbf{e}}_\varphi. \tag{1.16}$$

Hence vector fields $\mathbf{V}(r,z)$ [Eqs. (1.2) and (1.15)] for $\xi \neq 0$ are not Beltrami flows. They satisfy the Beltrami equation

$$\operatorname{curl}\mathbf{V}_0 = \alpha\mathbf{V}_0 \tag{1.17}$$

only for $\xi = 0$. Therefore, the parameter ξ has the following physical meaning: It defines the deviation of the fluid flow $\mathbf{V}(r,z)$ [Eqs. (1.2) and (1.15)] from the Beltrami flow $\mathbf{V}_0(r,z)$ [Eq. (1.17)].

We study the fluid flows $\mathbf{V}_\xi(r,z)$ [Eqs. (1.2) and (1.15)] that have the Stokes function

$$\psi(r,z) = r^2[\xi - G_2(\alpha R)], \quad R = \sqrt{r^2+z^2}, \tag{1.18}$$

where

$$G_2(u) = u^{-2}(\cos u - u^{-1}\sin u). \tag{1.19}$$

The parameters $\alpha \neq 0$ and ξ take all real values from $(-\infty,\infty)$. The parameter $\alpha \neq 0$ in view of (1.18) specifies the scaling of the fluid equilibria $\mathbf{V}_\xi(r,z)$ and its sign indicates the direction of fluid rotation $[\mathbf{V}_\xi(r,z)]_\varphi = \alpha r^{-1}\psi$ [Eqs. (1.2) and (1.14)]. Substituting the Stokes function (1.18) and $w(r,z) = \alpha\psi$ into Eq. (1.2), we get the fluid velocity field

$$\begin{aligned}\mathbf{V}_\xi(r,z) = {} & \alpha^2 rzG_3(\alpha R)\hat{\mathbf{e}}_r + \{2[\xi - G_2(\alpha R)] \\ & - \alpha^2 r^2 G_3(\alpha R)\}\hat{\mathbf{e}}_z + \alpha r[\xi - G_2(\alpha R)]\hat{\mathbf{e}}_\varphi,\end{aligned} \tag{1.20}$$

where

$$G_3(u) = u^{-1}dG_2(u)/du = u^{-4}[(3-u^2)u^{-1}\sin u - 3\cos u]. \tag{1.21}$$

The vortex field corresponding to (1.20) has the form

$$\begin{aligned}\operatorname{curl}\mathbf{V}_\xi(r,z) = {} & \alpha^3 rzG_3(\alpha R)\hat{\mathbf{e}}_r + \{2\alpha[\xi - G_2(\alpha R)] \\ & - \alpha^3 r^2 G_3(\alpha R)\}\hat{\mathbf{e}}_z - r\alpha^2 G_2(\alpha R)\hat{\mathbf{e}}_\varphi.\end{aligned} \tag{1.22}$$

Remark 2. The vector fields (1.20) and (1.22) satisfy Eq. (1.16). Both flows have invariant spheres $\mathbb{S}^2_{a_k}$ of radii $R = a_k$ obeying the equation $G_2(\alpha a_k) = \xi$ or $\psi(r,z) = 0$. The vortex field on the invariant spheres $\mathbb{S}^2_{a_k}$ is not poloidal for any $\xi \neq 0$ because the angular velocity of $\operatorname{curl}\mathbf{V}_\xi(r,z)$ [Eq. (1.22)] on the spheres $\mathbb{S}^2_{a_k}$ is $-r\alpha^2 G_2(\alpha a_k) = -r\alpha^2\xi \neq 0$. The fluid flow $\mathbf{V}_\xi(r,z)$ is purely poloidal on the invariant spheres $\mathbb{S}^2_{a_k}$ because its angular velocity is $\alpha r[\xi - G_2(\alpha a_k)] = 0$ since $G_2(\alpha a_k) = \xi$.

To study what vortex torus knots $K_{m,n}$ are realized for the fluid flow $\mathbf{V}_\xi(r,z)$ [Eq. (1.20)] inside the first invariant spheroid $\mathbb{B}^3_{a_1}$ of radius a_1 it is necessary to study the limit behavior of the pitch $p(\psi)$ [or the safety factor $q_h(\psi)$] in two cases: (a) when the nested tori $\mathbb{T}^2_\psi \subset \mathbb{B}^3_{a_1}$ collapse onto the innermost vortex axis $\mathbb{S}^1_{\psi_m}$ having coordinates $(r = r_m, z = z_m, 0 \leqslant \varphi \leqslant 2\pi)$ and $\psi_m = \psi(r_m,z_m)$ and (b) when the invariant tori $\mathbb{T}^2_\psi \subset \mathbb{B}^3_{a_1}$ approach at $\psi \to 0$ their limit that is the union of the invariant sphere $\mathbb{S}^2_{a_1}$ and invariant diameter $r = 0$. In Moffatt's works [2–4] it is stated[1] that in case (a) the limit value of the pitch p is always equal to infinity and in case (b) the limit value of the pitch p is always equal to zero. In Secs. V and VI we prove that both statements are incorrect for the following physical reasons.

For case (a), in spite of the poloidal components of $\operatorname{curl}\mathbf{V}_\xi(r,z)$ tending to zero near a vortex axis $\mathbb{S}^1_{\psi_m}$, the angular velocity of rotation of the poloidal projection (of the helical vortex line) around the point $a_m = (r_m,z_m)$ has a nonzero limit (in the nondegenerate case). This causes a finite and nonzero limit for the pitch p (and for the safety factor q_h) at the vortex axis.

For case (b) the invariant set defined by the equation $\psi(r,z) = 0$ inside the first invariant spheroid $\mathbb{B}^3_{a_1}$ is the union of the boundary sphere $\mathbb{S}^2_{a_1}$ and its diameter I that satisfies the equation $r = 0$ and connects two poles N and S on the sphere. Therefore, poloidal curves C_ψ at $\psi \to 0$ approach the union of the semicircle $\sqrt{r^2+z^2} = a_1$, $r \geqslant 0$, and the diameter I: $r = 0$ and $-a_1 \leqslant z \leqslant a_1$. Since N and S belong to the sphere $\mathbb{S}^2_{a_1}$ where $\psi(r,z) = 0$ we get from (1.18) that at these points $G_2(\alpha a_1) = \xi$.[2] When $\psi \to 0$ the poloidal projections of the vortex filaments move along the meridians on the sphere $\mathbb{S}^2_{a_1}$ to a small neighborhood of the pole N. Then, after staying a long time T_N near the pole N the vortex filaments move along the diameter I to a small neighborhood of the pole S and the same dynamics repeats, etc. Formula (1.22) yields that the angular velocity of $\operatorname{curl}\mathbf{V}_\xi(r,z)$ is $-\alpha^2 G_2(\alpha R)$, which near the poles N and S becomes approximately equal to $-\alpha^2\xi$ because $G_2(\alpha a_1) = \xi$ at the points N and S. Therefore, during the long time T_N of a vortex line evolution near the poles N and S its pitch $p(\psi)$ [Eq. (1.8)] acquires a big change equal to

$$-2\alpha^2 G_2(\alpha R)T_N \approx -2\alpha^2\xi T_N. \tag{1.23}$$

The contribution to the pitch $p(\psi)$ [Eq. (1.8)] from the evolution of vortex filament outside the poles N and S is much smaller than (1.23). Therefore, at $\psi \to 0$ the value of the pitch $p(\psi) \to +\infty$ if $\xi < 0$ and $p(\psi) \to -\infty$ if $\xi > 0$.

The formula (1.23) becomes an uncertainty when $\xi = 0$. For this case we prove in Sec. VI that the pitch $p(\psi)$ has a finite limit at $\psi \to 0$ and derive an exact formula for it.

Remark 3. For the magnetic field $\mathbf{B}_\xi(r,z)$ that has the same form as $\mathbf{V}_\xi(r,z)$ [Eq. (1.20)] the behavior of its pitch $p(\psi)$ at $\psi \to 0$ is completely different. The distinction is connected

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[1] In [2] on p. 129 about the pitch: "This quantity clearly increases continuously from zero to infinity as ψ increases from zero (on $R = a$) to $\psi_{\max}$ (on the vortex axis)". In [3] on p. 30 about the spheromak force-free magnetic field $\mathbf{B}(\mathbf{x})$: "Each $\mathbf{B}$-line is a helix and the pitch of the helices decreases continuously from infinity on the magnetic axis to zero on the sphere $r = R$ as we move outwards across the family of toroidal surfaces".

[2] The poles N and S are the saddle stagnation points for the fluid flow $\mathbf{V}_\xi(\mathbf{x})$ (1.20) inside the spheroid $\mathbb{B}^3_{a_1}$. The diameter I is a fluid streamline that is a separatrix of the stagnation points N and S. The same points N and S are the focus-saddle stagnation points for the vortex field $\operatorname{curl}\mathbf{V}_\xi(\mathbf{x})$.

with the fact that the vortex field curl $\mathbf{V}_\xi(r,z)$ [Eq. (1.22)] is not poloidal on the invariant sphere $\mathbb{S}^2_{a_1}$, while the fluid velocity field $\mathbf{V}_\xi(r,z)$ [and hence the magnetic field $\mathbf{B}_\xi(r,z)$] is purely poloidal (see Remark 2 above). Therefore, in MHD the limit of the safety factor $q(\psi)$ [Eq. (1.10)] at $\psi \to 0$ is finite for any ξ. Its exact formula is not included here.

Since the Stokes function (1.18) is a first integral of the fluid flow $\mathbf{V}_\xi(r,z)$ [Eq. (1.20)] and its vortex field (1.22), we get from (1.18) that roots a_k of the equation

$$G_2(\alpha a_k) = \xi \tag{1.24}$$

define spheroids $\mathbb{B}^3_{a_k}$ of radii $R = a_k$ that are invariant under the flows $\mathbf{V}_\xi(r,z)$ [Eq. (1.20)] and curl $\mathbf{V}_\xi(r,z)$ [Eq. (1.22)]. The flows possess invariant spheroids $\mathbb{B}^3_{a_k}$ only if the parameter ξ belongs to the range of function $G_2(u)$ since otherwise Eq. (1.24) does not have any roots. A calculation shows that the range of the function $G_2(u)$ is the segment $I^* = [-1/3,\xi_1 \approx 0.028\,72]$. Therefore, for $\xi < -1/3$ or $\xi > \xi_1$ the fluid flow $\mathbf{V}_\xi(r,z)$ [Eq. (1.20)] does not have any invariant spheroids.

For $\xi_1 < \xi < \overline{\xi}_1 \approx 0.111\,82$ the flows $\mathbf{V}_\xi(r,z)$ [Eq. (1.20)] and curl $\mathbf{V}_\xi(r,z)$ [Eq. (1.22)] have $K(\xi)$ disjoint invariant rings $\mathcal{R}^3_i$. The number $K(\xi) = 3$ when $\xi \to \xi_1$, $\xi > \xi_1$ and $K(\xi) = 1$ for $\xi \to \overline{\xi}_1$, $\xi < \overline{\xi}_1$. We study fluid flows with $\xi_1 < \xi < \overline{\xi}_1$ in Sec. VII. For $\xi > \overline{\xi}_1$ or $\xi < -1/3$ the flows $\mathbf{V}_\xi(r,z)$ and curl $\mathbf{V}_\xi(r,z)$ do not have any invariant tori $\mathbb{T}^2$ and any closed vortex lines.

For $-1/3 < \xi < \xi_1$ and $\xi \neq 0$, the flows $\mathbf{V}_\xi(r,z)$ and curl $\mathbf{V}_\xi(r,z)$ in the whole Euclidean space $\mathbb{R}^3$ have a finite number $N(\xi)$ of nested invariant spheroids $\mathbb{B}^3_{a_k}$ and a finite number $K(\xi)$ of disjoint invariant rings $\mathcal{R}^3_i$.

For $\xi = 0$, the exact solution (1.18) describes the spheromak fluid flow that was discovered first by Woltjer [16] as a plasma equilibrium and was applied by him to model a magnetic field in the Crab Nebula. The spheromak fluid flow $\mathbf{V}_0(r,z)$ has infinitely many nested invariant spheroids $\mathbb{B}^3_{a_k}$ with radii a_k satisfying the equation $\tan(\alpha a_k) = \alpha a_k$ and hence having the asymptotics $a_k \approx |\alpha|^{-1}(k + 1/2)\pi$ at $k \to \infty$. For $\xi \neq 0$ the number $N(\xi)$ of $\mathbf{V}_\xi$-invariant spheroids $\mathbb{B}^3_{a_k}$ is finite and $N(\xi) \to \infty$ when $\xi \to 0$. For positive ξ, $0 < \xi < \xi_1$, the number $N(\xi)$ is even; for negative ξ, $-1/3 < \xi < 0$, the number $N(\xi)$ is odd. These facts follow at once from the plot of function $y_1(u) = G_2(u)$ in Fig. 1 and Eq. (1.24).

The Prendergast exact solution [17] describes plasma equilibrium with everywhere vanishing fluid velocity $\mathbf{V}(\mathbf{x}) \equiv 0$ or a hydrostatic model of a magnetic star.

Section VIII is devoted to a liquid planet model. Here the necessary boundary conditions of vanishing of fluid velocity $\mathbf{V}_\xi(\mathbf{x})$ and pressure $p(\mathbf{x})$ on the surface $\mathbb{S}^2_{a_\ell}$ ($|\mathbf{x}| = a_\ell$) of the planet are satisfied for special values of the parameter $\xi = \xi_\ell$. The numbers $\xi_\ell = G_2(u_\ell)$ are extremal values of the function $G_2(u)$ (local maxima and local minima) that are attained at the points u_ℓ satisfying the equation $G_3(u_\ell) = u^{-1}G_2'(u_\ell) = 0$.

In Secs. V–VII we give a complete classification of all vortex knots for the fluid flows $\mathbf{V}_\xi(r,z)$ [Eq. (1.20)]. Additionally, these results provide counterexamples to the Moffatt statements of [2–4] on vortex knots.

The moduli spaces of vortex knots are defined and studied in [18] for the exact Beltrami flow having stream function $\psi(r,z) = -zr^2G_3(R)$. Reference [19] is devoted to the investigation of the moduli spaces of vortex knots for the spheromak fluid flow that has the stream function $\psi(r,z) = -r^2G_2(R)$. The results of [18,19] are equally applicable to the analogous force-free plasma equilibria.

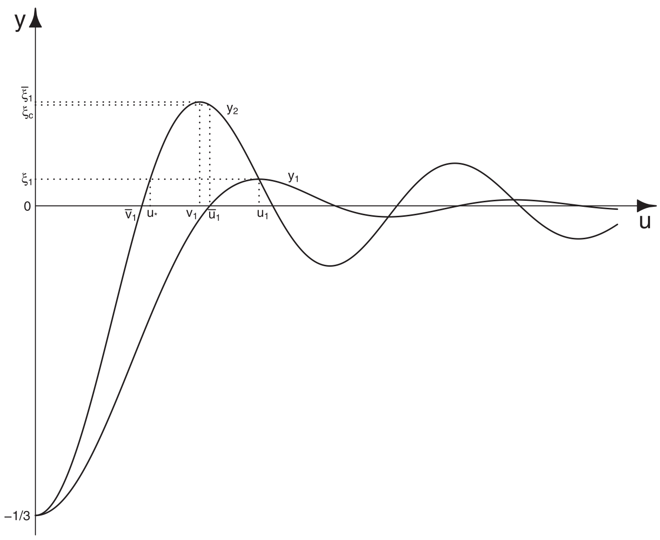

FIG. 1. Plot of the functions $y_1(u) = G_2(u)$ and $y_2(u) = -[G_1(u) + G_2(u)]/2$.

II. LIMIT OF THE PITCH FUNCTION AT A VORTEX AXIS

The dynamical system of vortex lines $d\mathbf{x}/dt = \text{curl}\,\mathbf{V}$ has the form

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z) = \dot{r}\hat{\mathbf{e}}_r + r\dot{\varphi}\hat{\mathbf{e}}_\varphi + \dot{z}\hat{\mathbf{e}}_z = \text{curl}\,\mathbf{V}. \tag{2.1}$$

This equation by virtue of Eqs. (1.3) and (1.7) yields

$$\dot{r} = -\frac{1}{r}C'(\psi)\psi_z, \quad \dot{z} = \frac{1}{r}C'(\psi)\psi_r, \tag{2.2}$$

$$\dot{\varphi} = -H'(\psi) + \frac{1}{r^2}C(\psi)C'(\psi). \tag{2.3}$$

Suppose that the stream function $\psi(r,z)$ has a local nondegenerate maximum or minimum $\psi_m = \psi(a_m)$ at a point $a_m = (r_m,z_m)$:

$$\psi_r(a_m) = 0, \quad \psi_z(a_m) = 0,$$
$$\mathcal{H}(a_m) = \psi_{rr}(a_m)\psi_{zz}(a_m) - \psi^2_{rz}(a_m) > 0. \tag{2.4}$$

Hence the system (2.2) has the center equilibrium point a_m and the system (2.2) and (2.3) has a stable trajectory-vortex axis $\mathbb{S}_{\psi_m}$: $r = r_m$, $z = z_m$, $0 \leqslant \varphi < 2\pi$, and $\psi_m = \psi(r_m,z_m)$. All trajectories of the system (2.2) near the center are closed curves C_ψ, $\psi(r,z) = \text{const}$, encircling the point a_m. The corresponding trajectories of the system (2.2) and (2.3) are either infinite helices or closed curves (knots) lying on the invariant tori $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$, where the circle $\mathbb{S}^1$ corresponds to the angular variable φ.

Substituting Eq. (2.3) into the formula (1.8), we get

$$p(\psi) = -H'(\psi)t(\psi) + C(\psi)C'(\psi)\int_0^{t(\psi)} \frac{dt}{r^2(t)}. \tag{2.5}$$

In the limit $\psi \to \psi_m$ we have $r(t) \to r_m$ for all t, hence

$$p(\psi_m) = \lim_{\psi\to\psi_m} p(\psi) = t(\psi_m)\big[-H'(\psi_m) + C(\psi_m)C'(\psi_m)/r_m^2\big], \quad (2.6)$$

where $t(\psi_m) = \lim_{\psi\to\psi_m} t(\psi)$.

The dynamical system (2.2) near the equilibrium point (r_m,z_m) is approximated by the system in variations [20]

$$\frac{d\delta r}{dt} = -a_{11}\delta z - a_{12}\delta r, \quad \frac{d\delta z}{dt} = a_{12}\delta z + a_{22}\delta r, \quad (2.7)$$

$$a_{11} = c_m\psi_{zz}(a_m), \quad a_{12} = c_m\psi_{rz}(a_m), \quad a_{22} = c_m\psi_{rr}(a_m),$$
$$c_m = \frac{C'(\psi_m)}{r_m}, \quad (2.8)$$

where $\delta r(t) = r(t) - r_m$ and $\delta z(t) = z(t) - z_m$. From Eqs. (2.4) and (2.8) we get

$$D_m = a_{11}a_{22} - a_{12}^2 = c_m^2\mathcal{H}(a_m) > 0. \quad (2.9)$$

The linear system (2.7) has the quadratic first integral $Q(\delta r,\delta z) = a_{22}(\delta r)^2 + 2a_{12}\delta r\delta z + a_{11}(\delta z)^2$ that in view of (2.9) is either positive or negative definite. Hence its level curves $Q(\delta r,\delta z) = \text{const}$ are nested ellipses and therefore all solutions to (2.7) are periodic. Due to the scaling invariance of the linear system (2.7), all its solutions have the same period $t_m = 2\pi/\sqrt{D_m}$.

From the general theory of dynamical systems [20] it follows that the limit at $\psi \to \psi_m$ of the function of periods $t(\psi)$ is the period t_m of the system in variations (2.7). Using formula (2.9) we find

$$t(\psi_m) = \lim_{\psi\to\psi_m} t(\psi) = \frac{2\pi}{\sqrt{D_m}} = \frac{2\pi r_m}{|C'(\psi_m)|\sqrt{\mathcal{H}(a_m)}}. \quad (2.10)$$

Substituting (2.10) into (2.6) we get

$$p(\psi_m) = \lim_{\psi\to\psi_m} p(\psi) = \frac{2\pi r_m}{|C'(\psi_m)|\sqrt{\mathcal{H}(a_m)}}\left[-H'(\psi_m) + \frac{1}{r_m^2}C(\psi_m)C'(\psi_m)\right]. \quad (2.11)$$

Formula (2.11) shows that the Moffatt statement of [2–4] that at the vortex axis the pitch p is always equal to infinity can be true only for degenerate solutions with either $C'(\psi_m) = 0$ or $\mathcal{H}(a_m) = 0$.

Using Eq. (1.3) we find

$$-H'(\psi_m) + \frac{1}{r_m^2}C(\psi_m)C'(\psi_m) = -\frac{1}{r_m^2}\left(\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz}\right)(a_m). \quad (2.12)$$

At the equilibrium point a_m in view of $\psi_r(a_m) = 0$ we have

$$\left(\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz}\right)(a_m) = \left(\psi_{rr} + \frac{1}{r}\psi_r + \psi_{zz}\right)(a_m) = \Delta\psi(a_m), \quad (2.13)$$

where Δ is the Laplace operator. Substituting (2.12) and (2.13) into Eq. (2.11) we get, for the safety factor $q_h(\psi_m) = p(\psi_m)/2\pi$,

$$q_h(\psi_m) = -\frac{\Delta\psi(a_m)}{r_m C'(\psi_m)\sqrt{\mathcal{H}(a_m)}}. \quad (2.14)$$

The derivatives $\psi_{rr}(a_m)$ and $\psi_{zz}(a_m)$ have the same sign since the point a_m is either a local maximum or a local minimum of the function $\psi(r,z)$. Hence, using the standard inequality we find

$$\sqrt{\mathcal{H}(a_m)} = \sqrt{\psi_{rr}(a_m)\psi_{zz}(a_m) - \psi_{rz}^2(a_m)} \leqslant \sqrt{\psi_{rr}(a_m)\psi_{zz}(a_m)} \leqslant |\psi_{rr}(a_m) + \psi_{zz}(a_m)|/2 = |\Delta\psi(a_m)|/2.$$

Applying this inequality in (2.14) we derive

$$|q_h(\psi_m)| \geqslant \frac{2}{r_m|C'(\psi_m)|}. \quad (2.15)$$

For $C(\psi) = \alpha\psi$ we get the simple formula $|q_h(\psi_m)| \geqslant 2(|\alpha|r_m)^{-1}$.

Formulas (2.14) and (2.15) prove that for the case of arbitrary functions $H(\psi)$ and $C(\psi)$ in Eq. (1.3) the safety factor $q_h(\psi)$ has a finite and nonzero limit at $\psi \to \psi_m$ provided the nondegeneracy conditions $C'(\psi_m)\neq 0, \mathcal{H}(a_m)\neq 0$ are met. The limit (2.14) is one of the two bounds of the range of safety factor $q_h(\psi)$. Hence we get one of the two bounds $q_h(\psi_m)$ for the range of the rational values m/n corresponding to the torus knots $K_{m,n}$ that can be realized as vortex knots for the considered fluid flow $\mathbf{V}(\mathbf{x})$ [Eq. (1.2)].

III. LIMIT OF THE SAFETY FACTOR AT A MAGNETIC AXIS

Equations of magnetohydrodynamics have the form

$$\frac{\partial\mathbf{V}}{\partial t} + \text{curl}\mathbf{V}\times\mathbf{V} = -\text{grad}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{V}|^2 + \Phi\right) + \frac{1}{\rho\mu}\text{curl}\mathbf{B}\times\mathbf{B} + \nu\Delta\mathbf{V},$$
$$\frac{\partial\mathbf{B}}{\partial t} = \text{curl}(\mathbf{V}\times\mathbf{B}), \quad \text{div}\mathbf{V} = 0, \quad \text{div}\mathbf{B} = 0, \quad (3.1)$$

where $\mathbf{B}$ is the magnetic field, μ the magnetic permeability, ν the kinematic viscosity, and Δ the Laplace operator. As known since the Newcomb paper [21], Eqs. (3.1) imply that magnetic field $\mathbf{B}(t,\mathbf{x})$ is transformed in time by the flow diffeomorphisms or is frozen in the flow. Therefore, in the MHD any magnetic field $\mathbf{B}$ knot is transformed in time into an isotopic $\mathbf{B}$ knot.

Plasma equilibrium equations follow from (3.1) for $\mathbf{V} = 0$:

$$\text{curl}\,\mathbf{B}\times\mathbf{B} = \text{grad}(\mu p + \rho\mu\Phi), \quad \text{div}\,\mathbf{B} = 0. \quad (3.2)$$

The hydrodynamics equilibrium equations follow from (3.1) for $\mathbf{B} = 0$:

$$\text{curl}\,\mathbf{V}\times\mathbf{V} = \text{grad}\left(-\frac{p}{\rho} - \Phi - \frac{1}{2}|\mathbf{V}|^2\right), \quad \text{div}\,\mathbf{V} = 0. \quad (3.3)$$

Since Eqs. (3.2) and (3.3) are equivalent [5], we get that for the axisymmetric solutions the same ψ functions satisfying Eq. (1.3) describe axisymmetric equilibria of ideal fluid and plasma. Topological invariants of the equilibrium of fluid are defined by the curl **V** knots, while the topological invariants of plasma equilibrium are defined by the **B** knots.

Let us show that these topological invariants for the ideal fluid and for plasma are completely different, for the same ψ function $\psi(r,z)$. Indeed, the axisymmetric magnetic field $\mathbf{B}(\mathbf{x})$ has a form analogous to (1.2),

$$\mathbf{B}(r,z) = -\frac{1}{r}\frac{\partial\psi}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\psi}{\partial r}\hat{\mathbf{e}}_z + \frac{C(\psi)}{r}\hat{\mathbf{e}}_\varphi, \tag{3.4}$$

where we used the second of Eqs. (1.4), which follows from (3.2). The corresponding dynamical system of magnetic field lines $d\mathbf{x}/dt = \mathbf{B}(\mathbf{x})$ is

$$\dot{r} = -\psi_z/r, \quad \dot{z} = \psi_r/r, \tag{3.5}$$

$$\dot{\varphi} = C(\psi)/r^2. \tag{3.6}$$

For closed magnetic field lines on an invariant torus $\mathbb{T}^2_\psi$ that go around the torus m times the long way around and n times the short way around, the *safety factor* q is defined as $q = m/n$ [9,12,22]. Mercier had demonstrated in [9] that the value of the safety factor q is connected with stability of the MHD equilibria: For example, if $q = 1/n$, where n is an integer, then the MHD equilibrium is unstable.

For the axisymmetric magnetic fields the definition of [9,12,22] conforms to the definition (1.10), which after substituting Eq. (3.6) gives

$$q(\psi) = \frac{1}{2\pi}C(\psi)\int_0^{\overline{t(\psi)}}\frac{dt}{r^2(t)}. \tag{3.7}$$

The analogous safety factor in hydrodynamics $q_h(\psi) = p(\psi)/(2\pi)$ due to Eq. (2.5) is

$$q_h(\psi) = \frac{1}{2\pi}\left[-H'(\psi)t(\psi) + C(\psi)C'(\psi)\int_0^{t(\psi)}\frac{dt}{r^2(t)}\right]. \tag{3.8}$$

For the same closed trajectory C_ψ, $\psi(r,z) = \text{const}$, the two safety factors $q(\psi)$ and $q_h(\psi)$ are qualitatively different. For example, when $q(\psi)$ [Eq. (3.7)] is a rational number m/n and hence the corresponding magnetic field **B** line is a torus knot $K_{m,n}$ [9,12], the safety factor $q_h(\psi)$ [Eq. (3.8)] in general is an irrational number and the corresponding curl **V** line is an infinite curve dense on the invariant torus $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$ and vice versa. Here the circle $\mathbb{S}^1$ corresponds to the angular variable φ. Another possibility appears when both safety factors are rational numbers, $q(\psi) = m/n$ and $q_h(\psi) = p/q$, and therefore define topologically nonequivalent torus knots: the magnetic field **B** knot $K_{m,n}$ and the vortex knot $K_{p,q}$.

Let the magnetic field $\mathbf{B}(\mathbf{x})$ have a family of nested invariant tori $\mathbb{T}^2_\psi$ defined by the equation $\psi = \text{const}$. Then the innermost torus $\mathbb{T}^2_{\psi_m}$ is a circle $\mathbb{S}^2_{\psi_m}$, which is called a magnetic axis $\psi = \psi_m$. The magnetic axis corresponds to a center equilibrium point $a_m = (r_m,z_m)$ of the system (3.5), which is defined by the same equations (2.4) as for the stable vortex axis in hydrodynamics. The safety factor (3.7) at $\psi \to \psi_m$ due to $r(t) \to r_m$ has the limit

$$q(\psi_m) = \lim_{\psi\to\psi_m} q(\psi) = \frac{1}{2\pi r_m^2}\overline{t(\psi_m)}C(\psi_m), \tag{3.9}$$

where $\overline{t(\psi_m)} = \lim_{\psi\to\psi_m}\overline{t(\psi)}$. To derive the limit value $\overline{t(\psi_m)}$ we consider the system in variations for the system (3.5) near its stable equilibrium a_m:

$$\frac{d\delta r}{dt} = -b_{11}\delta z - b_{12}\delta r, \quad \frac{d\delta z}{dt} = b_{12}\delta z + b_{22}\delta r, \tag{3.10}$$

$$b_{11} = \psi_{zz}(a_m)/r_m, \quad b_{12} = \psi_{rz}(a_m)/r_m, \quad b_{22} = \psi_{rr}(a_m)/r_m. \tag{3.11}$$

Here $\delta r(t) = r(t) - r_m$ and $\delta z(t) = z(t) - z_m$. From (2.4) and (3.11) we find

$$\overline{D_m} = b_{11}b_{22} - b_{12}^2 = \mathcal{H}(a_m)/r_m^2 > 0. \tag{3.12}$$

Due to the inequality (3.12) and the scaling invariance of the linear system (3.10), all its trajectories have the same period $\overline{t_m} = 2\pi/\sqrt{\overline{D_m}} = 2\pi r_m/\sqrt{\mathcal{H}(a_m)}$.

From the general theory of dynamical systems [20] we get that the limit at $\psi \to \psi_m$ of the function of periods $\overline{t(\psi)}$ is the period $\overline{t_m}$ of the system in variations (3.10). Hence $\overline{t(\psi_m)} = \lim_{\psi\to\psi_m}\overline{t(\psi)} = \overline{t_m} = 2\pi r_m/\sqrt{\mathcal{H}(a_m)}$. Substituting into (3.9) we get the limit value of the safety factor

$$q(\psi_m) = \frac{C(\psi_m)}{r_m\sqrt{\mathcal{H}(a_m)}} \tag{3.13}$$

at the magnetic axis $\psi = \psi_m$.

At the vortex axis $\psi = \psi_m$, the hydrodynamic safety factor $q_h(\psi) = p(\psi)/2\pi$ in view of (2.11) and (3.13) has the limit

$$q_h(\psi_m) = q(\psi_m) - \frac{r_m H'(\psi_m)}{C'(\psi_m)\sqrt{\mathcal{H}(a_m)}}. \tag{3.14}$$

The limit safety factors (3.13) and (3.14) are evidently different.

For the solutions satisfying Eqs. (1.14) and (1.15) we find

$$q(\psi_m) = \frac{\alpha\psi_m}{r_m\sqrt{\mathcal{H}(a_m)}}, \quad q_h(\psi_m) = \frac{\alpha\left(\psi_m - \xi r_m^2\right)}{r_m\sqrt{\mathcal{H}(a_m)}}. \tag{3.15}$$

Formulas (3.15) show that the two safety factors $q(\psi_m)$ and $q_h(\psi_m)$ are different if $\xi \neq 0$ and coincide if $\xi = 0$.

IV. MAIN DYNAMICAL SYSTEM

Moffatt studied in [2] vortex lines for the exact solutions to Eq. (1.15):

$$\psi(r,z) = r^2\left[\frac{\lambda}{\alpha^2} + A\left(\frac{a}{R}\right)^{3/2}J_{3/2}(\alpha R)\right], \quad R = \sqrt{r^2 + z^2}, \tag{4.1}$$

where $J_\nu(u)$ is the Bessel function of order ν. The solutions [Eq. (54) of [2]] are considered in [2] for

$$\frac{\lambda}{\alpha^2} = -AJ_{3/2}(\alpha a) \tag{4.2}$$

[Eq. (57) of [2]] in the first invariant spheroid $\mathbb{B}_a$ ($R \leqslant a$), where the vortex lines on the invariant tori $\mathbb{T}^2_\psi \subset \mathbb{B}_a$ are either infinite helices or torus knots $K_{m,n}$. The solutions (4.1) and (4.2) are matched continuously with an irrotational flow for $R > a$ having the stream function

$$\psi(r,z) = -\frac{1}{3}AJ_{5/2}(\alpha a)r^2\left(1 - \frac{a^3}{R^3}\right).$$

The solutions (4.1) and (4.2) with $J_{5/2}(\alpha a) = 0$ were derived by Prendergast [17] as a model of equilibrium of a magnetic star; the continuously matched exterior magnetic field **B** vanishes in view of the condition $J_{5/2}(\alpha a) = 0$. For $\lambda = 0$, the flux function (4.1) describes the spheromak magnetic field **B** [by the same formula as (1.2)] that was discovered first by Woltjer [16] and applied by him to model the Crab Nebula (see also [23]).

In Ref. [2] (pp. 128–129) Moffatt writes about the *pitch* $p(\psi)$ of the helical vortex lines: "This quantity clearly increases continuously from zero to infinity as ψ increases from zero (on $R = a$) to $\psi_{\max}$ (on the vortex axis)". Thus Moffatt states here that for all values of the parameters A, α, and $\lambda/\alpha^2 = -AJ_{3/2}(\alpha a)$ [formula (57) of [2]] the limits of the pitch function $p(\psi)$ at $\psi \to 0$ and at $\psi \to \psi_{\max}$ are

$$p(0) = \lim_{\psi\to 0} p(\psi) = 0, \quad p(\psi_{\max}) = \lim_{\psi\to\psi_{\max}} p(\psi) = \infty.$$

The same results for the Beltrami vector fields [having stream functions (4.1) with $\lambda = 0$] are stated in [3] (pp. 30–31).

Our formula (2.11) $\lim_{\psi\to\psi_m} p(\psi) = p(\psi_m) < \infty$ is rigorously proven in Sec. II for any axisymmetric flows (1.2). Formula (2.11) demonstrates that presented in [2,3] Moffatt's statement that $\lim_{\psi\to\psi_{\max}} p(\psi) = \infty$ does not correspond to the facts.

The known formula for the Bessel functions $J_{n+1/2}(u)$ yields (see [24], p. 56)

$$J_{3/2}(u) = -\frac{1}{\sqrt{\pi/2}\sqrt{u}}\left(\cos u - \frac{\sin u}{u}\right).$$

Therefore, the solution (54) of [2] [formula (4.1) above] takes the form

$$\psi(r,z) = r^2\left[\frac{\lambda}{\alpha^2} - \frac{A(|\alpha|a)^{3/2}}{\sqrt{\pi/2}(\alpha R)^2}\left(\cos(\alpha R) - \frac{\sin(\alpha R)}{(\alpha R)}\right)\right]. \tag{4.3}$$

Instead of Bessel's functions $J_{n+1/2}(u)$ we will use the elementary functions

$$G_0(u) = -\cos u, \quad G_1(u) = \frac{d}{udu}G_0(u) = \frac{\sin u}{u} = \frac{\sqrt{\pi/2}}{u^{1/2}}J_{1/2}(u),$$

$$G_2(u) = \frac{d}{udu}G_1(u) = \frac{1}{u^2}\left[\cos u - \frac{\sin u}{u}\right] = -\frac{\sqrt{\pi/2}}{u^{3/2}}J_{3/2}(u),$$

$$G_3(u) = \frac{d}{udu}G_2(u) = \frac{1}{u^4}\left[(3-u^2)\frac{\sin u}{u} - 3\cos u\right] = \frac{\sqrt{\pi/2}}{u^{5/2}}J_{5/2}(u),$$

$$G_4(u) = \frac{1}{u}\frac{dG_3(u)}{du} = \frac{1}{u^6}\left[(6u^2-15)\frac{\sin u}{u} - (u^2-15)\cos u\right]. \tag{4.4}$$

All functions $G_n(u)$ are analytic everywhere and have the values

$$G_1(0) = 1, \quad G_2(0) = -1/3, \quad G_3(0) = 1/15, \quad G_4(0) = -1/105. \tag{4.5}$$

The functions $G_n(u)$ [Eq. (4.4)] satisfy the easily verifiable identities

$$G_0 + G_1 + u^2G_2 = 0, \quad G_1 + 3G_2 + u^2G_3 = 0. \tag{4.6}$$

The general identity is $G_n + (2n+1)G_{n+1} + u^2G_{n+2} = 0$, with $G_{n+1} = u^{-1}dG_n/du$.

The stream function (4.3) in view of (4.4) takes the form

$$\psi(r,z) = Br^2[\xi - G_2(\alpha R)], \quad B = \frac{A(|\alpha|a)^{3/2}}{\sqrt{\pi/2}},$$

$$\xi = \frac{\lambda}{\alpha^2 B} = \frac{\lambda\sqrt{\pi/2}}{A|\alpha|^{7/2}a^{3/2}}. \tag{4.7}$$

Hence we get, for $H(\psi) = \lambda\psi$ and $C(\psi) = \alpha\psi$,

$$-H'(\psi) + \frac{1}{r^2}C(\psi)C'(\psi) = -\lambda + \frac{\alpha^2}{r^2}\psi = -\alpha^2BG_2(\alpha R). \tag{4.8}$$

Substituting $C(\psi) = \alpha\psi$ and (4.8) into the system (2.2) and (2.3) we get, for the main dynamical system (2.1),

$$\dot r = -\frac{\alpha}{r}\psi_z, \quad \dot z = \frac{\alpha}{r}\psi_r, \quad \dot\varphi = -\alpha^2BG_2(\alpha R).$$

Substituting here formulas (4.7) and (4.4) we get

$$\dot r = \alpha^3BrzG_3(\alpha R),$$
$$\dot z = 2\alpha B[\xi - G_2(\alpha R)] - \alpha^3Br^2G_3(\alpha R), \tag{4.9}$$

$$\dot\varphi = -\alpha^2BG_2(\alpha R). \tag{4.10}$$

The parameters α and B are nonessential here because they can be removed by the scaling transformation $r_1 = |\alpha|r$, $z_1 = |\alpha|z$, and the time change $d\tau/dt = \alpha^2B$. Therefore, we assume $B > 0$, which means $A > 0$ [Eq. (4.7)]. The only essential parameter of the problem is the parameter $\xi = \lambda/\alpha^2B$ [Eq. (4.7)].

Let us consider the system (4.9) and (4.10) in the first spheroid $\mathbb{B}_a$ ($R \leqslant a$) that is invariant under the system. That means $\psi(r,z) = 0$ on the sphere $\mathbb{S}_a^2$ of radius $R = a$. Hence we get the condition $|\alpha|a = \bar u_1(\xi)$ where $\bar u_1(\xi)$ is the first root of the equation

$$G_2(u) = \xi. \tag{4.11}$$

Figure 1 shows that the range of the function $G_2(u)$ for all positive u is the segment I^*, $[-1/3,\xi_1]$, where ξ_1 is the maximal value of the function $G_2(u)$. The maximum is attained at a point u_1 where the derivative $G_2'(u)$ vanishes, $\xi_1 = G_2(u_1)$. Hence, from (4.4) we get $G_3(u_1) = u_1^{-1}G_2'(u_1) = 0$. The first root of the equation $G_3(u) = 0$ is $u_1 \approx 5.7635$. Hence we find the value of $\xi_1 = G_2(u_1) \approx 0.028\,72$. As a consequence we get that Eq. (4.11) has no solutions if $\xi < -1/3$ or $\xi > \xi_1$. Hence the fluid flow (1.2) and (4.1) does not have any invariant spheroid $\mathbb{B}_c$ ($R \leqslant c$) if $\xi < -1/3$ or $\xi > \xi_1$.

From Fig. 1 we see that $\bar u_1(\xi) \in [0,u_1 \approx 5.7635]$. Figure 1 yields that Eq. (4.11) for $\xi \neq 0$ has a finite number $N(\xi)$ of roots and that $N(\xi) \to \infty$ when $\xi \to 0$. This means that the system (4.9) and (4.10) in the whole space $\mathbb{R}^3$ can have, for $\xi \neq 0$, a finite number $N(\xi)$ of invariant spheroids $\mathbb{B}_{R_i}$ and has infinitely many invariant spheroids when $\xi = 0$. We consider the systems (4.9) and (4.10) in the spheroid $\mathbb{B}_{R_1}$,

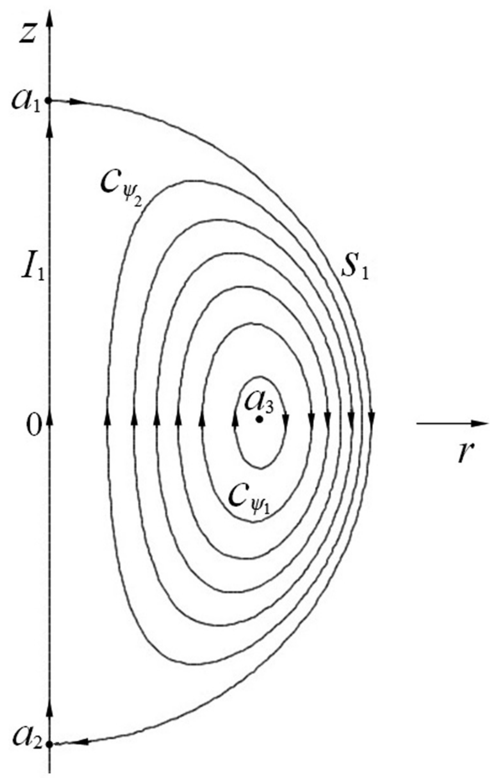

FIG. 2. Poloidal contours of stream surfaces inside the first invariant spheroid $\mathbb{B}^3_{a_1}$ for $-1/3 < \xi \leqslant \xi_1$. Rotation of contours around the z axis defines $\mathbf{V}$-invariant tori $\mathbb{T}^2$. Rotation of the separatrix S_1 gives the $\mathbf{V}$-invariant sphere $\mathbb{S}^2_{a_1}$. Rotation of point a_3 produces the stable vortex axis $a_3 \times \mathbb{S}^1$.

$R_1 = a$, corresponding to the first root $\overline{u}_1(\xi)$ of Eq. (4.11); then $|\alpha| = \overline{u}_1(\xi)/a$. The first zero of the function $G_2(u)$ is $\overline{u}_1(0) = \overline{u}_1 \approx 4.4931$.

For $\xi \in I^*$ the system (4.9) and (4.10) has an invariant spheroid $\mathbb{B}_a$, $|\alpha|a = \overline{u}_1(\xi)$. The system (4.9) in the invariant semidisk D_1 $[\psi(r,z) \geqslant 0, r \geqslant 0, R \leqslant a]$ has three equilibrium points: $a_1(r = 0, z = a)$, $a_2(r = 0, z = -a)$, $a_3(r = u_m/|\alpha|, z = 0)$. The equilibria a_1 and a_2 are nondegenerate saddles if $G_3[\overline{u}_1(\xi)] \neq 0$. Their separatrices are the interval I_1, $(r = 0, -a < z < a)$, and the arc S_1, $(r^2 + z^2 = a^2, r > 0)$ (see Fig. 2). The equilibrium a_3 is a center, its coordinate $r = R_m = u_m(\xi)/|\alpha|$, where $u_m(\xi)$ is the first positive root of the equation $2G_2(u) + u^2 G_3(u) = 2\xi$ (the condition that $\dot{z} = 0$ in (4.9) at the point $[a_3(r = R_m, z = 0), u = \alpha R]$). The latter in view of the second identity (4.6) takes the form

$$-[G_1(u) + G_2(u)]/2 = \xi. \tag{4.12}$$

All equilibrium points of the system (4.9) with $r \neq 0$ satisfy the equation $z = 0$ and Eq. (4.12) for $u = |\alpha| r$.

From Fig. 1 it follows that the range of the function $-[G_1(u) + G_2(u)]/2$ for all positive u is the segment $[-1/3, \overline{\xi}_1]$ where $\overline{\xi}_1$ is its maximum that is attained at a point v_1; hence we have $G_1'(v_1) + G_2'(v_1) = 0$. Equations (4.4) yield $G_1'(u) + G_2'(u) = u[G_2(u) + G_3(u)]$; therefore $G_2(v_1) + G_3(v_1) = 0$. The first root of this equation is $v_1 \approx 4.2329$. Hence we get $\overline{\xi}_1 = -2^{-1}[G_1(v_1) + G_2(v_1)] \approx 0.111\,82$.

V. KEY FUNCTION $f(\xi)$

Let us derive exact formulas for $p(\psi_m)$ [Eq. (2.11)] for solutions (4.7). Substituting $r^2 = R^2 - z^2$ into (4.7) and using the first identity (4.6) we find

$$\psi(r,z) = B\left[\xi r^2 + \frac{1}{\alpha^2}[G_0(\alpha R) + G_1(\alpha R)] + z^2 G_2(\alpha R)\right]. \tag{5.1}$$

Differentiating the function $\psi(r,z)$ [Eq. (5.1)] and using formulas (4.4) we get

$$\begin{aligned} \frac{\partial \psi}{\partial r} &= rB[2\xi + G_1(\alpha R) + G_2(\alpha R) + \alpha^2 z^2 G_3(\alpha R)], \\ \frac{\partial \psi}{\partial z} &= zB[G_1(\alpha R) + 3G_2(\alpha R) + \alpha^2 z^2 G_3(\alpha R)]. \end{aligned} \tag{5.2}$$

The function $\psi(r,z)$ [Eq. (4.7)] achieves its maximal in the semidisk D_1 value ψ_m at the point $a_3 = (r_m, z_m)$. Since $\psi_r(a_3) = 0$ and $\psi_z(a_3) = 0$, we get from (5.2) $z_m = 0$ and $u_m(\xi) = |\alpha| r_m$ satisfies Eq. (4.12). For the second derivatives of the function $\psi(r,z)$ at the point a_3 we find from (5.2)

$$\begin{aligned} \psi_{rr}(a_3) &= u_m^2 B[G_2(u_m) + G_3(u_m)], \\ \psi_{zz}(a_3) &= B[G_1(u_m) + 3G_2(u_m)], \end{aligned} \tag{5.3}$$

with $\psi_{rz}(a_3) = 0$. Hence we get the Hessian (2.4),

$$\mathcal{H}(a_m) = u_m^2 B^2 [G_2(u_m) + G_3(u_m)][G_1(u_m) + 3G_2(u_m)]. \tag{5.4}$$

Using formula (4.8) we find, on the vortex axis $\mathbb{S}_m$ $[\alpha R = u_m(\xi), z = 0]$,

$$-H'(\psi_m) + \frac{1}{r_m^2} C(\psi_m) C'(\psi_m) = -\alpha^2 B G_2(u_m(\xi)). \tag{5.5}$$

Substituting (5.4) and (5.5) and $r_m = u_m(\xi)/|\alpha|$ into (2.11), we find for the exact solutions (4.7)

$$\begin{aligned} p(\psi_m) &= \lim_{\psi \to \psi_m} p(\psi) \\ &= -\frac{2\pi G_2(u_m)}{\sqrt{[G_2(u_m) + G_3(u_m)][G_1(u_m) + 3G_2(u_m)]}}. \end{aligned} \tag{5.6}$$

It is evident that formula (5.6) does not contain the two nonessential parameters α and B. Therefore, expression (5.6) is a function of the parameter ξ only, since $u_m = u_m(\xi)$ is a function of ξ defined as the first root of Eq. (4.12). Thus we arrive at the key function $f(\xi) = p(\psi_m(\xi))/2\pi = q_h(\psi_m(\xi))$. Applying the second identity (4.6), we get from (5.6)

$$f(\xi) = -\frac{G_2(u_m(\xi))}{u_m(\xi)\sqrt{-G_3(u_m(\xi))[G_2(u_m(\xi)) + G_3(u_m(\xi))]}}. \tag{5.7}$$

The function $f(\xi)$ describes one of the two boundaries of the range of fractions m/n that correspond to the vortex knots $K_{m,n}$ realized for the fluid flow (1.2) defined by the stream function $\psi(r,z)$ [Eq. (4.7)] for the given ξ.

Equation (4.12) for $\xi = \overline{\xi}_1 \approx 0.111\,82$ has the root $u_m(\overline{\xi}_1) = v_1 \approx 4.2329 < \overline{u}_1(0) = \overline{u}_1 \approx 4.4931$. Hence we

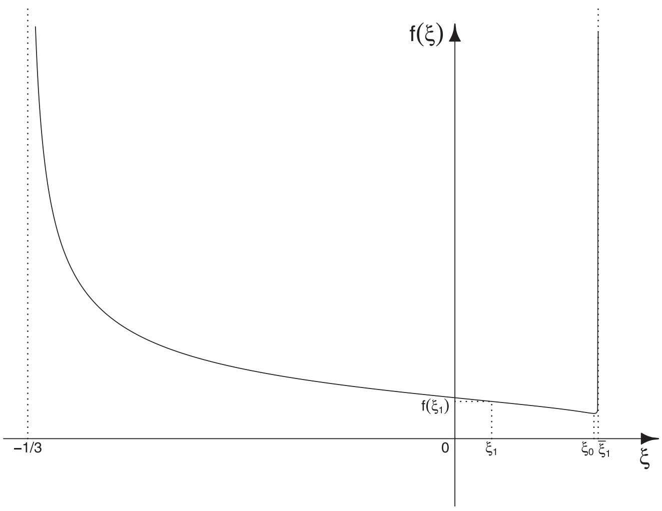

FIG. 3. Plot of the function $f(\xi)$.

get $G_2(u_m(\bar{\xi}_1)) = G_2(v_1) < 0$. Numerical calculation gives $G_2(v_1) \approx -0.0140$. Since $u_m(\xi)$ is a monotonically increasing function of $\xi \in [-1/3,\bar{\xi}_1 \approx 0.111\,82]$ [see the plot of the function $y_2(u)$ in Fig. 1] and $G_2(u)$ is a monotonically increasing function of $u \in [0,u_1 \approx 5.7635]$, we find that $G_2(u_m(\xi))$ is a monotonically increasing function of $\xi \in [-1/3,\bar{\xi}_1]$. Hence, from $G_2(u_m(\bar{\xi}_1)) = G_2(v_1) \approx -0.0140 < 0$ it follows that for all $\xi \in [-1/3,\bar{\xi}_1]$ we have $G_2(u_m(\xi)) < 0$. Hence formulas (5.6) and (5.7) give $p(\psi_m) > 0$ and $f(\xi) > 0$ for all $\xi \in [-1/3,\bar{\xi}_1 \approx 0.111\,82]$.

In view of the formulas (4.4) we get that Eq. (4.12) near $\xi = -1/3$ is $u_m^2(\xi) \approx 15(\xi + 1/3)$ and $u_m(-1/3) = 0$. From (4.5) we find

$$G_2(0) = -1/3, \quad G_2(0) + G_3(0) = -4/15, \quad G_3(0) = 1/15.$$

Hence we get from (5.7) the asymptotic formula

$$f(\xi) \approx \frac{\sqrt{5/3}}{2\sqrt{\xi + 1/3}} \approx \frac{0.6455}{\sqrt{\xi + 1/3}}, \quad \xi \to -\frac{1}{3}. \tag{5.8}$$

Hence $f(\xi) \to \infty$ at $\xi \to -1/3$.

The point $u_m(\bar{\xi}_1) = v_1$ is the point of maximum of the function $-[G_1(u) + G_2(u)]/2$ (see Fig. 1). Hence, using Taylor's formula we get that Eq. (4.12) near $\bar{\xi}_1$ takes the form $\bar{\xi}_1 - \xi \approx \frac{1}{4}v_1^2[G_3(v_1) + G_4(v_1)][u_m(\xi) - v_1]^2$. We have from (4.4) $G_2(v_1) + G_3(v_1) = v_1^{-1}[G_1'(v_1) + G_2'(v_1)] = 0$. Therefore, for $u_m(\xi) \approx v_1$ we have $G_2(u_m(\xi)) + G_3(u_m(\xi)) \approx v_1[G_3(v_1) + G_4(v_1)][u_m(\xi) - v_1]$. Substituting this into (5.7), we find the asymptotes

$$f(\xi) \approx -\frac{G_2(v_1)}{v_1\sqrt{2G_3(v_1)}[G_3(v_1) + G_4(v_1)]^{1/4}(\bar{\xi}_1 - \xi)^{1/4}}, \quad \xi \to \bar{\xi}_1. \tag{5.9}$$

Substituting $v_1 \approx 4.2329$, $G_2(v_1) = -G_3(v_1) \approx -0.0140$, and $G_4(v_1) \approx -0.0031$ into (5.9), we find $f(\xi) \approx 0.0612(\bar{\xi}_1 - \xi)^{-1/4} \to \infty$ at $\xi \to \bar{\xi}_1$. The plot of the function $f(\xi)$ is shown in Fig. 3.

To find the minimum of the function $f(\xi)$ we consider the equation $df(\xi)/d\xi = 0$. The equation in view of (5.7) and (4.4) has the form

$$[G_1(u) + 3G_2(u)]\big[G_3^2(u) - G_2(u)G_4(u)\big] = [G_2(u) + G_3(u)]\big[G_2^2(u) - G_1(u)G_3(u)\big],$$

where $u = u_m(\xi)$. The latter equation has the root $\xi_0 \approx 0.1085$; the corresponding value $f(\xi_0) \approx 0.5079$ is the minimal value of the function $f(\xi)$ on the segment $-1/3 \leqslant \xi \leqslant \bar{\xi}_1 \approx 0.111\,82$.

Substituting numerical values $\xi_1 \approx 0.028\,72$ and $u_m(\xi_1) = u_* \approx 2.9570$ into (5.7), we find $f(\xi_1) \approx 0.7502$, which is the minimal value of the function $f(\xi)$ on the segment I^*: $-1/3 \leqslant \xi \leqslant \xi_1 \approx 0.028\,72$ (see Fig. 3). Substituting the value $u_m(0) = \bar{v}_1 \approx 2.7437$ into formula (5.7), we get $f(0) \approx 0.825\,24$. The corresponding value of $p(\psi_m)$ is $p(\psi_m) = 2\pi f(0) \approx 5.1849$.

VI. LIMIT OF THE PITCH FUNCTION $p(\psi)$ AT $\psi \to 0$

All trajectories of the system (4.9) inside the domain D_1 for $-1/3 < \xi \leqslant \xi_1$ are closed curves C_ψ, $\psi(r,z) = \psi = \text{const}$, $0 < \psi < \psi_m$, encircling the center equilibrium point a_3 and having periods $t(\psi)$. The corresponding trajectories of the system (4.9) and (4.10) are helices moving on invariant tori $\mathbb{T}_\psi^2 = C_\psi \times \mathbb{S}^1 \subset \mathbb{R}^3$ (the circle $\mathbb{S}^1$ corresponds to the angle φ). In view of (4.10), the pitch $p(\psi)$ of the helices is

$$p(\psi) = \int_0^{t(\psi)} \frac{d\varphi}{dt}dt = -\alpha^2 B \int_0^{t(\psi)} G_2(\alpha R(t))dt. \tag{6.1}$$

For $\xi \in I^*$, the closed trajectories C_ψ at $\psi \to 0$ approach the cycle of two separatrices I_1 and S_1 that satisfy the equation $\psi(r,z) = 0$ (see Fig. 2). Since the dynamics along each separatrix takes an infinite time [20] we get

$$\lim_{\psi \to 0} t(\psi) = \infty. \tag{6.2}$$

At the saddle equilibrium points a_1 and a_2 we have $G_2(a_i) = \xi$. Let $\mathcal{O}_\varepsilon(a_1)$ and $\mathcal{O}_\varepsilon(a_2)$ be small neighborhoods of the points a_1 and a_2 such that inside them $|G_2(\alpha R) - \xi| < \varepsilon$. The velocity $\mathbf{v}$ of the dynamics of any trajectory C_ψ outside $\mathcal{O}_\varepsilon = \mathcal{O}_\varepsilon(a_1) \cup \mathcal{O}_\varepsilon(a_2)$ satisfies $|\mathbf{v}| > \text{const} > 0$. Hence any trajectory C_ψ spends, during one period $t(\psi)$, only a limited time $t_{\text{out}}(\psi) < C \ll t(\psi)$ outside $\mathcal{O}_\varepsilon$ and time $t_{\text{in}}(\psi)$ inside $\mathcal{O}_\varepsilon$. In view of (6.2) we have $t_{\text{in}}(\psi) = [t(\psi) - t_{\text{out}}(\psi)] \to \infty$ at $\psi \to 0$ and $t_{\text{out}}(\psi) < \text{const}$. Therefore, using (6.1) with $B > 0$ and $G_2(\alpha R) \approx \xi$ in $\mathcal{O}_\varepsilon$ we get

$$\lim_{\psi \to 0} p(\psi) = \begin{cases} +\infty & \text{for } -1/3 < \xi < 0 \\ -\infty & \text{for } 0 < \xi \leqslant \xi_1. \end{cases} \tag{6.3}$$

For $-1/3 < \xi < 0$ the function $p(\psi)$ monotonically decreases from its limit $p(0) = +\infty$ at $\psi = 0$ to its positive limit value $p(\psi_m) = 2\pi f(\xi)$ [Eq. (5.6)] at $\psi = \psi_m$ (see Fig. 4). Hence all vortex knots for the fluid flow (1.2) with a given ξ $(-1/3 < \xi < 0)$ are torus knots $K_{m,n}$ for which the rational numbers m/n are not arbitrary and satisfy the inequalities

$$f(\xi) < \frac{m}{n} < +\infty. \tag{6.4}$$

For $0 < \xi \leqslant \xi_1$, the function $p(\psi)$ monotonically increases in view of (6.3) from its limit $p(0) = -\infty$ at $\psi = 0$ to its positive limit value (5.6) at $\psi = \psi_m$ (see Fig. 5). Hence, for the fluid flow (1.2) with $0 < \xi \leqslant \xi_1$ all vortex knots are torus

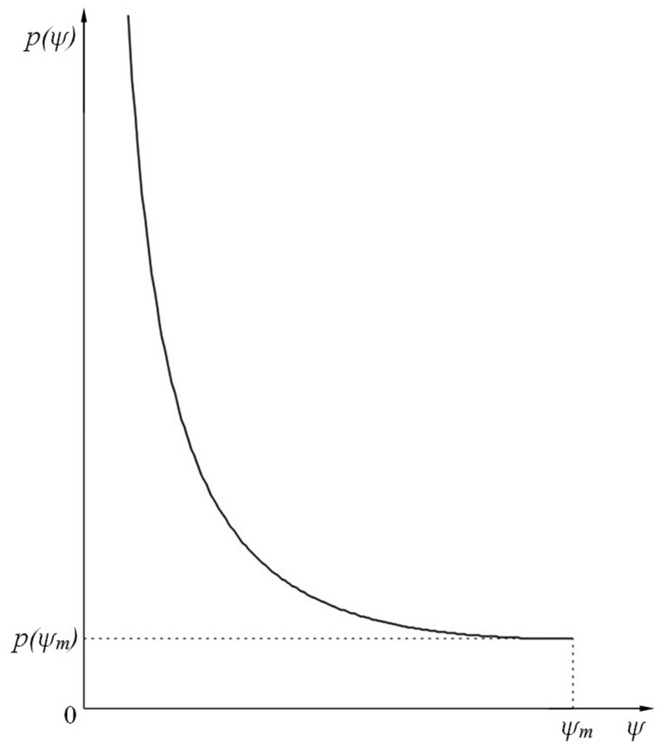

FIG. 4. Plot of the pitch function $p(\psi)$ for $-1/3 < \xi < 0$.

knots $K_{|m|,n}$ (where the integer m can be negative) with the ratio m/n satisfying the inequalities

$$-\infty < \frac{m}{n} < f(\xi). \tag{6.5}$$

In Sec. V we proved that $p(\psi_m) = 2\pi f(\xi) > 0$ for all ξ (see Fig. 3). Hence the function $p(\psi)$ vanishes at some point ψ_0: $p(\psi_0) = 0$. At some point $\psi_1 < \psi_0$ we have $p(\psi_1) = -p(\psi_m)$

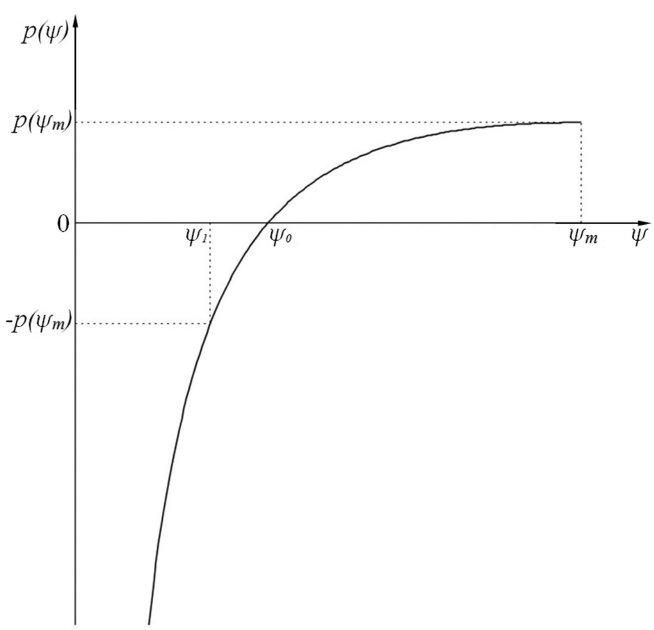

FIG. 5. Plot of the pitch function $p(\psi)$ for $0 < \xi \leqslant \xi_1$.

(see Fig. 5). For $\psi = \psi_0$ all trajectories of the dynamical system (4.9) and (4.10) on the torus $\mathbb{T}^2_{\psi_0}$ are closed curves that are embedded into $\mathbb{R}^3$ as unknots because $p(\psi_0) = 0$. All torus knots $K_{m,n}$ for $0 < m/n < f(\xi)$ are realized on two different tori $\mathbb{T}^2_{\psi}$: one for ψ in the interval (ψ_0,ψ_m), where $p(\psi) = 2\pi m/n > 0$, and another one on torus $\mathbb{T}^2_{\tilde\psi}$ for $\tilde\psi$ in the interval (ψ_1,ψ_0), where $p(\tilde\psi) = -2\pi m/n < 0$. The torus knots $K_{m,n}$ corresponding to $p(\psi) = 2\pi m/n > 0$ and $p(\tilde\psi) = -2\pi m/n < 0$ are nonisotopic mirror images of each other.

For the special case $\xi = \lambda = 0$ the vector field $\mathbf{V}$ [Eq. (1.2)] satisfies the Beltrami equation $\operatorname{curl}\mathbf{V} = \alpha\mathbf{V}$ and the analogous magnetic field $\mathbf{B}$ is force-free. This is the well-known spheromak equilibrium solution derived by Woltjer [16] and Chandrasekhar [23] and later studied in [25–30]. The term "spheromak" was first introduced in [25]. For $\xi = 0$, the parameter $\overline{u}_1(0) = \overline{u}_1$, where $\overline{u}_1 \approx 4.4931$ is the first positive root of the equation $G_2(u) = 0$.

The evaluation of the limit of $p(\psi)$ at $\psi \to 0$ by formula (6.1) for $\xi = 0$ leads to an uncertainty because $G_2(\alpha R) = 0$ at the equilibrium points a_1 and a_2 and on the separatrix S_1. To resolve this uncertainty we use another method based on the invariance of the pitch function $p(\psi)$ under the rescaling of r, z, and a reparametrization of time t. In the new coordinates $r_1 = |\alpha|r$, $z_1 = |\alpha|z$, and $u = \sqrt{r_1^2 + z_1^2} = |\alpha|R$ the system (4.9) and (4.10) for $\xi = 0$ is transformed into [$\sigma = \pm 1 = \operatorname{sgn}(\alpha)$]

$$\begin{aligned} \sigma\dot r_1 &= \alpha^2 B r_1 z_1 G_3(u), \\ \sigma\dot z_1 &= -2\alpha^2 B G_2(u) - \alpha^2 B r_1^2 G_3(u), \end{aligned} \tag{6.6}$$

$$\dot\varphi = -\alpha^2 B G_2(u). \tag{6.7}$$

After the change of time $d\tau/dt = -\alpha^2 B G_2(u)$ the system (6.6) and (6.7) turns into a system that does not depend on the parameters α and B:

$$\sigma\frac{dr_1}{d\tau} = -r_1 z_1 \frac{G_3(u)}{G_2(u)}, \quad \sigma\frac{dz_1}{d\tau} = 2 + r_1^2\frac{G_3(u)}{G_2(u)}, \quad \frac{d\varphi}{d\tau} = 1. \tag{6.8}$$

As above, the closed trajectories C_ψ at $\psi \to 0$ are approximated by the interval I_1 ($r_1 = 0$ and $-\overline{u}_1 < z_1 < \overline{u}_1$) and the arc S_1 ($r_1^2 + z_1^2 = \overline{u}_1^2$ and $r_1 > 0$). The system (6.8) yields, on the interval I_1, $r_1 = 0$ and $|dz_1/d\tau| = 2$. Hence the dynamics of the trajectory C_ψ along the interval I_1 takes the time $\tau_1(\psi) \approx 2\overline{u}_1/2 = \overline{u}_1$.

The dynamics of C_ψ along the arc S_1 takes an infinitesimally small time $\tau_2(\psi)$. Indeed, since $G_3(\overline{u}_1) = (\overline{u}_1^2\sqrt{1+\overline{u}_1^2})^{-1} \neq 0$ and $G_2(u) \to 0$ as $u \to \overline{u}_1$, we get, for the velocity $\mathbf{v}$ of the dynamics near the arc S_1 ($u = \overline{u}_1$), $|\mathbf{v}| \to \infty$ when $\psi \to 0$. Hence $\tau_2(\psi) \to 0$. Therefore, we get, for the pitch function $p(\psi)$ and for the function of periods $\tau(\psi) = \tau_1(\psi) + \tau_2(\psi)$,

$$\lim_{\psi\to 0} p(\psi) = \int_0^{\tau(\psi)} \frac{d\varphi}{d\tau}d\tau = \lim_{\psi\to 0}\tau(\psi) = \overline{u}_1 \approx 4.4931. \tag{6.9}$$

The stream function $\psi(r,z)$ [Eq. (4.7)] for $\xi = 0$, $B = 1$, and $\alpha = 1$ attains its maximal value $\psi_m \approx 1.0631$ at the point $r_m \approx 2.7437$, $z_m = 0$. Numerical integration of the system (6.8) shows that on the interval $(0,\psi_m)$ the pitch function $p(\psi)$

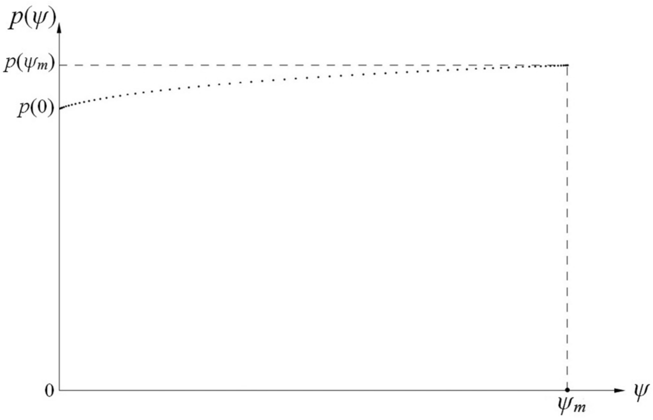

FIG. 6. Numerical calculation of the pitch function $p(\psi)$ for $\xi = 0$.

monotonically increases from its value $p(0) = \overline{u}_1 \approx 4.4931$ [Eq. (6.9)] at $\psi = 0$ to its value $p(\psi_m) = 2\pi f(0) \approx 5.1849$ [Eq. (5.9)] at $\psi = \psi_m$ (see Fig. 6). These results prove that for the spheromak Beltrami flows corresponding to $\xi = 0$ [or $\lambda = 0$ in (4.1)] only those torus knots $K_{m,n}$ are realized for which the fractions m/n satisfy the inequalities

$$\frac{1}{2\pi}4.4931 \approx 0.7151 < \frac{m}{n} < f(0) \approx 0.8252. \qquad (6.10)$$

VII. FLUID FLOWS (1.2) AND (4.7) WITH INVARIANT RINGS

For $\xi_1 < \xi < \overline{\xi}_1$, Eq. (4.11) does not have any roots. Hence the flow (1.2) and (4.7) does not have invariant spheroids $\mathbb{B}_c$. Equation (4.12) for the same ξ has two roots $u_m = \alpha r_m$ and $u_s = \alpha r_s$, $u_m < u_s$ (see Fig. 1). Therefore, the corresponding dynamical system (4.9) has two equilibrium points: a stable center a_3 [$(r = r_m, z = 0)$] and an unstable saddle a_4 [$(r = r_s, z = 0)$]. The saddle equilibrium a_4 has a loop separatrix S_2 that begins and ends at the point a_4 and satisfies the equation

$$\psi(r,z) = Br^2[\xi - G_2(\alpha R)] = Br_s^2[\xi - G_2(u_s)] = \psi_s. \qquad (7.1)$$

The phase portrait of the system (4.9) for $\xi = (\xi_1 + \overline{\xi}_1)/2 \approx 0.070\,27$ is shown in Fig. 7, where $r_m = 3.374\,02$ and $r_s = 5.189\,61$.

The loop separatrix S_2 bounds a domain D_2 that is filled with closed trajectories C_ψ, which define invariant tori of the dynamical system (4.9) and (4.10), $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1 \subset D_2 \times \mathbb{S}^1$. Here the circle $\mathbb{S}^1$ corresponds to the angular variable φ: $0 \leqslant \varphi \leqslant 2\pi$. The closure of the product $D_2 \times \mathbb{S}^1$ is an invariant ring $\mathcal{R}^3 \subset \mathbb{R}^3$.

Let us study the structure of the vortex knots inside the ring $\mathcal{R}^3$. The first root of the function $G_2(u)$ [Eq. (4.4)] is $\overline{u}_1 \approx 4.4931$. Let $\xi_c = -[G_1(\overline{u}_1) + G_2(\overline{u}_1)]/2 \approx 0.1088$. From Fig. 1 it follows that for $\xi_c < \xi < \overline{\xi}_1$ the roots of Eq. (4.12) satisfy the inequalities $u_m < u_s < \overline{u}_1$ and hence $G_2(u_s) < 0$. For $\xi_1 < \xi < \xi_c$ the inequalities are $u_m < \overline{u}_1 < u_s$ and hence $G_2(u_s) > 0$. Using formula (6.1) for the pitch $p(\psi)$ and the fact that the dynamics along the loop separatrix S_2 takes an infinite time, we get (the proof is the same as in

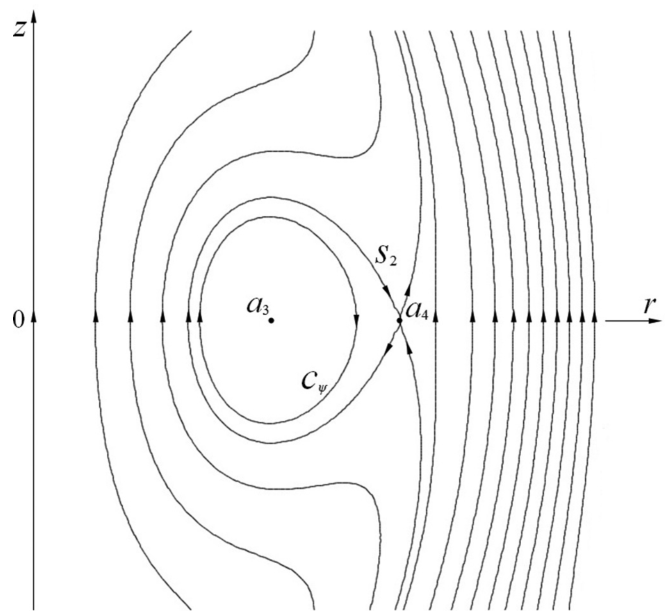

FIG. 7. Poloidal contours of stream surfaces for $\xi = 0.070\,27$. Rotation of closed contours around the z axis defines the **V**-invariant ring $\mathcal{R}$ bounded by the **V**-invariant torus $\mathbb{T}^2 = S_2 \times \mathbb{S}^1$. Rotation of points a_3 and a_4 gives the stable vortex axis $a_3 \times \mathbb{S}^1$ and the unstable one $a_4 \times \mathbb{S}^1$. All vortex knots are located inside the ring $\mathcal{R}$.

Sec. VI) the following limits of the pitch function $p(\psi)$ for trajectories in the domain D_2 at $\psi \to \psi_s$ [Eq. (7.1)]:

$$\lim_{\psi \to \psi_s} p(\psi) = \begin{cases} +\infty, & \xi_c < \xi < \overline{\xi}_1 \\ -\infty, & \xi_1 < \xi < \xi_c. \end{cases}$$

We proved in Sec. V that near a center equilibrium point of the system (4.9) at which the function $\psi(r,z)$ has its maximum or minimum ψ_m the limit of the pitch function $p(\psi)$ at $\psi \to \psi_m$ is a finite positive number $p(\psi_m) = 2\pi f(\xi)$. Hence we get that for the vortex knots $K_{m,n}$ realized in the ring $\mathcal{R}^3$ the ratios m/n satisfy the inequalities

$$f(\xi) < \frac{m}{n} < +\infty, \quad \xi_c < \xi < \overline{\xi}_1 \qquad (7.2)$$

$$-\infty < \frac{m}{n} < f(\xi), \quad \xi_1 < \xi < \xi_c, \qquad (7.3)$$

where $f(\xi) \geqslant f(\xi_0) \approx 0.5079$.

The equilibria a_3 and a_4 define, respectively, the stable and unstable vortex axes of the flow (1.2) and (4.7) in the invariant ring $\mathcal{R}^3$; both axes are also the exact streamlines because the tori $\mathbb{T}^2_\psi = C_\psi \times \mathbb{S}^1$ are invariant under both flows curl $\mathbf{V}(\mathbf{x})$ and $\mathbf{V}(\mathbf{x})$. All trajectories of the system (4.9) outside the domain D_2 are infinite curves satisfying the equation $Br^2[\xi - G_2(\alpha R)] = \psi = \text{const}$. Hence, for these trajectories at $t \to \pm\infty$ we have $z \to \pm\infty$ and $r \to r_\xi = \sqrt{\psi/B\xi}$ (see Fig. 7).

For $\xi > \overline{\xi}_1$ or $\xi < -1/3$ both Eqs. (4.11) and (4.12) have no solutions (see Fig. 1). Therefore, the dynamical system (4.9) has no equilibrium points and no closed trajectories. For all its trajectories at $t \to \pm\infty$ we have $z \to \pm\infty$ and $r \to r_\xi = \sqrt{\psi/B\xi}$. Hence the fluid flows (1.2) and (4.7) for $\xi > \overline{\xi}_1$ or $\xi < -1/3$ have no vortex knots.

VIII. LIQUID PLANET MODEL

The fluid velocity field $\mathbf{V}_\xi(r,z)$ [Eq. (1.20)] for $-1/3 < \xi \leqslant \xi_1 \approx 0.028\,72$, $\xi \neq 0$, has a finite number $N(\xi)$ of invariant spheres $\mathbb{S}^2_{a_k}$ of radii $R = a_k$ satisfying the equation $G_2(\alpha a_k) = \xi$. On each sphere $\mathbb{S}^2_{a_k}$ the flow takes the form

$$\overline{\mathbf{V}}_\xi(r,z) = \alpha^2 r G_3(\alpha a_k)[z\hat{\mathbf{e}}_r - r\hat{\mathbf{e}}_z] \tag{8.1}$$

and is evidently tangent to the sphere $\mathbb{S}^2_{a_k}$. Hence each spheroid $\mathbb{B}^3_{a_k}$ defined by the inequality $R \leqslant a_k$ is invariant under the flow (1.20).

The velocity field (8.1) vanishes for the special values of ξ when $G_3(\alpha a_\ell) = 0$. Since $G_3(u) = u^{-1} dG_2(u)/du$ [Eq. (4.4)], the equation $G_3(u_\ell) = 0$ means that u_ℓ is a point of extremum of the function $G_2(u)$. Using formula (4.4) we find that the equation $G_3(u) = 0$ is equivalent to the equation

$$\tan u = \frac{3u}{3 - u^2}. \tag{8.2}$$

Hence the roots u_ℓ of the equation $G_3(u) = 0$ or Eq. (8.2) have the asymptotes $u_\ell \approx (\ell + 1)\pi$ and $u_\ell < (\ell + 1)\pi$ at $\ell \to \infty$. The first eight roots of Eq. (8.2) are

$$u_1 \approx 5.7635, \quad u_2 \approx 9.0950, \quad u_3 \approx 12.3229,$$
$$u_4 \approx 15.5146, \quad u_5 \approx 18.6890, \quad u_6 \approx 21.8539,$$
$$u_7 \approx 25.0128, \quad u_8 \approx 28.1678.$$

Let us define $\xi_\ell = G_2(u_\ell)$. Substituting $G_3(u_\ell) = 0$ into the second identity (4.6), we find $\xi_\ell = -G_1(u_\ell)/3 = -(3u_\ell)^{-1} \sin u_\ell$. Formula (8.2) yields $-(3u_\ell)^{-1} \sin u_\ell = \cos u_\ell/(u_\ell^2 - 3)$. Hence we get the exact formula

$$\xi_\ell = \frac{\cos u_\ell}{u_\ell^2 - 3}. \tag{8.3}$$

Using $u_\ell \approx (\ell + 1)\pi$, we find from (8.3) the asymptotes $\xi_\ell \approx (-1)^{\ell+1}(u_\ell)^{-2}$ at $\ell \to \infty$.

The first eight values of $\xi_\ell = G_2(u_\ell)$ [Eq. (8.3)] are

$$\xi_1 = G_2(u_1) \approx 0.02872, \quad \xi_2 \approx -0.0119, \quad \xi_3 \approx 0.0065,$$
$$\xi_4 \approx -0.0041, \quad \xi_5 \approx 0.0029, \quad \xi_6 \approx -0.0021,$$
$$\xi_7 \approx 0.0016, \quad \xi_8 \approx -0.0013.$$

The positive values $\xi_\ell > 0$ are local maxima of the function $G_2(u)$ and the negative values $\xi_\ell < 0$ are local minima (see Fig. 1).

For the special values of $\xi = \xi_\ell$ flow (8.1) vanishes on the sphere $\mathbb{S}^2_{a_\ell}$ of radius $a_\ell = \alpha^{-1}u_\ell$, because $G_3(u_\ell) = 0$. Therefore, the flow inside the invariant spheroid $\mathbb{B}^3_{a_\ell}$, $R \leqslant a_\ell$, can be continuously matched with empty space. Indeed, from the equilibrium equations (1.6) we find, on the sphere $\mathbb{S}^2_{a_\ell}$,

$$\rho^{-1} p(\mathbf{x}) + \rho\mu\Phi(\mathbf{x}) + |\mathbf{V}_{\xi_\ell}(\mathbf{x})|^2/2 = C, \tag{8.4}$$

where C is an arbitrary constant. Since the fluid density ρ is constant by assumption, we get that the Newtonian potential $\Phi(\mathbf{x})$ is spherically symmetric and therefore $\Phi(\mathbf{x}) = \Phi(a_\ell)$ on the sphere $|\mathbf{x}| = a_\ell$. Therefore, using the vanishing of velocity $\mathbf{V}_{\xi_\ell}(r,z)$ [Eq. (8.1)] on the sphere $\mathbb{S}^2_{a_\ell}$ and choosing the arbitrary constant $C = \rho\mu\Phi(a_\ell)$, we get from (8.4) that pressure $p(\mathbf{x}) = 0$ on the sphere $\mathbb{S}^2_{a_\ell}$. Therefore, all conditions at the boundary with empty space are satisfied and thus the flow (1.2) for $\xi = \xi_\ell$ is correctly matched with empty space outside the invariant spheroid $\mathbb{B}^3_{a_\ell}$. We propose these exact solutions as a liquid planet model.

The fluid flow $\mathbf{V}_{\xi_1}(r,z)$ [Eq. (1.20)], $\xi_1 = G_2(u_1)$, has no other invariant spheroids inside the $\mathbb{B}^3_{a_1}$, where $a_1 = \alpha^{-1}u_1$. For all other flows $\mathbf{V}_{\xi_\ell}(r,z)$ [Eq. (1.20)] there exists at least one interior invariant spheroid $\mathbb{B}^3_{a_k} \subset \mathbb{B}^3_{a_\ell}$. The plot of the function $y_1(u) = G_2(u)$ in Fig. 1 shows that for $\xi_\ell > 0$ the number of interior invariant spheroids is even; for $\xi_\ell < 0$ the number is odd.

Remark 4. The first invariant sphere $\mathbb{S}^2_{a_k} \subset \mathbb{B}^3_{a_\ell}$ can be considered as the interior boundary of the flow, for $\ell > 1$. In this case the fluid moves with velocity $\mathbf{V}_{\xi_\ell}(r,z)$ [Eq. (1.20)] in the spherical shell $a_k < R \leqslant a_\ell$ and the spheroid $\mathbb{B}^3_{a_k}$ (where $R \leqslant a_k$) is the rigid core of the planet with a spherically symmetric distribution of mass. For the ideal fluid model the no-slip condition at the interior boundary $\mathbb{S}^2_{a_k}$ can be relaxed because it is necessary only for viscous fluids.

The problem of equilibrium of magnetic stars with a dipole field outside was studied by Chandrasekhar and Fermi in [31]. The first example of the magnetic field that is continuously matched with an empty space was derived by Prendergast under the condition $J_{5/2}(\alpha a) = 0$ [17]. Our condition $G_3(\alpha a) = 0$ coincides with the Prendergast one in view of the formula for the Bessel function

$$J_{5/2}(u) = \frac{1}{\sqrt{\pi/2}u^{3/2}}\left[(3 - u^2)\frac{\sin u}{u} - 3\cos u\right] = \frac{u^{5/2}}{\sqrt{\pi/2}} G_3(u),$$

which follows from [24] (p. 56).

IX. CONCLUSION

As known [32,33], the torus knots $K_{m,n}$ are nontrivial unless $m/n = N$ or $m/n = 1/N$, where N is an integer. All other torus knots are mutually nonisotopic. A torus knot $K_{m,n}$ with $m/n \neq N, 1/N$ and its mirror image [which corresponds to the pair $(-m,n)$] are nonisotopic to each other. Therefore, we obtain from (6.4) that the set of mutually nonisotopic vortex torus knots $K_{m,n}$ for fluid flows (1.2) and (4.7) inside the first invariant spheroid $\mathbb{B}_{a_1}$ with the parameter ξ in the range $-1/3 < \xi < 0$ is equivalent to the set of all rational numbers m/n satisfying relations

$$f(\xi) < \frac{m}{n} < +\infty, \quad \frac{m}{n} \neq N, \tag{9.1}$$

where N is any integer. Here the lower bound $f(\xi) > f(0) \approx 0.8252$ and $f(\xi) \approx 0.6455(\xi + 1/3)^{-1/2} \to \infty$ when $\xi \to -1/3$ [see (5.8)]. Formula (9.1) implies that all flows (1.2) and (4.7) for $-1/3 < \xi < 0$ in view of $f(\xi) > 0.8252$ are counterexamples to the Moffatt statement (see [2], p. 129): "It is interesting that *every* torus knot is represented once and only once among all the vortex lines of each member of the family of flows represented by the stream function (54), together with circulation (52)".

From Fig. 3 it follows that for any real number $x > 0.8252$ there exist $\xi(x) \in [-1/3,0]$ such that $f(\xi(x)) = x$ (see Fig. 3). For the flow (1.2) and (4.7) with the parameter $\xi = \xi(x)$ only torus knots $K_{m,n}$ with $m/n > x$ are realized as vortex knots and all torus knots $K_{\overline{m},\overline{n}}$ with $\overline{m}/\overline{n} \leqslant x$ are not realized.

If the parameter ξ is in the range $0 < \xi < \xi_1 \approx 0.028\,72$ then Eq. (6.5) yields that the set of mutually nonisotopic vortex torus knots $K_{m,n}$ for the flows (1.2) and (4.7) inside the first invariant spheroid $\mathbb{B}_{a_1}$ is equivalent to the set of rational numbers m/n defined by the conditions

$$-\infty < \frac{m}{n} < f(\xi), \quad \frac{m}{n} \neq \pm N, \quad \frac{m}{n} \neq \pm\frac{1}{N}, \tag{9.2}$$

where N is any integer and $f(0) \approx 0.8252 > f(\xi) > f(\xi_1) \approx 0.7502$. Note that torus knots $K_{|m|,|n|}$ with $|m|/|n| < f(\xi)$ appear as vortex knots twice: For the positive rational numbers m/n satisfying $0 < m/n < f(\xi)$ and for the negative ones satisfying $-f(\xi) < -m/n < 0$, the corresponding knots $K_{m,n}$ and $K_{-m,n}$ are nonisotopic mirror images of each other. These results also are counterexamples to the above Moffatt statement. Indeed, here we have torus knots $K_{|m|,|n|}$ realized on infinitely many pairs of different tori $\mathbb{T}^2_{\psi}$ and $\mathbb{T}^2_{\tilde{\psi}}$ on that the pitch function $p(\psi) = -p(\tilde{\psi})$, so realized *twice* and not "once and only once" as it is claimed in [2]. Note that the existence of the pitch functions $p(\psi)$ changing their sign on the segment $0 \leqslant \psi \leqslant \psi_m$ (as in Fig. 5) was never discussed in [2–4].

For $\xi = 0$ the fluid flow (1.2) and (4.7) coincides with the spheromak Beltrami field [16,23,25]. From Eq. (6.10) we get that the set of mutually nonisotopic vortex torus knots $K_{m,n}$ inside the first invariant spheroid $\mathbb{B}_{a_1}$ is equivalent to the set of all rational numbers m/n satisfying the inequalities

$$0.7151 < \frac{m}{n} < f(0) \approx 0.8252. \tag{9.3}$$

This result is a counterexample to the Moffatt statement in [3] (pp. 30–31): "It is an intriguing property of this **B**-field that if we take a subset of the **B**-lines consisting of one **B**-line on each toroidal surface, then every torus knot is represented once and only once in this subset, since $p/2\pi$ passes through every rational number m/n once as it decreases continuously from infinity to zero".

Note that here the magnetic field **B** knots are discussed, but since the considered field satisfies the Beltrami equation $\operatorname{curl}\mathbf{B} = \alpha\mathbf{B}$, the **B** lines coincide with the $\operatorname{curl}\mathbf{B}$ lines. Therefore, the above statement of [3] is applicable also to the vortex knots of the steady fluid flow with $\mathbf{V}(\mathbf{x}) \equiv \mathbf{B}(\mathbf{x})$ [Eqs. (1.2) and (4.7)] with $\xi = 0$ (and also is incorrect).

Since the function $f(\xi)$ [Eq. (5.7)] for $-1/3 < \xi < \xi_1$ is monotonically decreasing (see Fig. 3), we get that all families of nonisotopic vortex torus knots $K_{m,n}$ [Eqs. (9.1)–(9.3)] are different for flows (1.2) and (4.7) with different values of the parameter ξ. Among these families of vortex knots in view of $f(\xi) \geqslant 0.5079$ for all ξ no one family coincides with Moffatt's description in [2,3]. Therefore, Moffatt's statements of [2,3] on vortex knots do not correspond to the facts.

For any axisymmetric steady fluid flows [defined by arbitrary smooth functions $H(\psi)$ and $C(\psi)$ in Eq. (1.3)] for which the function $\psi(r,z)$ has a nondegenerate maximum or minimum ψ_m at a point $a_m(r_m,z_m)$, we have proved in Sec. II that the pitch function $p(\psi)$ has a finite and nonzero limit at $\psi \to \psi_m$:

$$p(\psi_m) = \lim_{\psi\to\psi_m} p(\psi) = \frac{2\pi r_m}{|C'(\psi_m)|\sqrt{\mathcal{H}(a_m)}}\left[-H'(\psi_m) + \frac{1}{r_m^2}C(\psi_m)C'(\psi_m)\right]. \tag{9.4}$$

Formula (9.4) demonstrates that presented in [2,3] Moffatt's statements that for the special flows (1.2) and (4.1) $\lim_{\psi\to\psi_{\max}} p(\psi) = \infty$ are incorrect because for these flows $C'(\psi_m) = \alpha \neq 0$ and $\mathcal{H}(a_m) \neq 0$.

Formula (9.4) yields a plethora of counterexamples to the concluding part of Moffatt's statement of [4] (p. 29): "The streamlines within these vortices are topologically similar to those of the special case when $F(\psi)$ and $G(\psi)$ are linear in ψ, [i.e.,] they are helices wrapped on the family of nested tori [$\psi = \text{const}(0 < \psi < \psi_{\max})$], the pitch of the helix varying continuously from zero . . . to infinity . . .". Indeed, for the generic functions $F(\psi)$ and $G(\psi)$ [or $H(\psi)$ and $C(\psi)$ in (1.3)] the limit value $p(\psi_m)$ [Eq. (9.4)] evidently is nonzero and is one of the two exact bounds for the range of the function $p(\psi)$. Since the bound $p(\psi_m) \neq 0$ we get that the pitch function $p(\psi)$ does not change "continuously from zero to infinity".

For some functions $H(\psi)$ and $C(\psi)$ the stream function $\psi(r,z)$ satisfying Eq. (1.3) can have no maxima or minima at all. Then the fluid flow (1.2) does not have invariant tori $\mathbb{T}^2_{\psi}$, vortex knots, or pitch function $p(\psi)$. This is realized for the exact fluid flows (1.2) and (4.7), with the parameter $\xi < -1/3$ or $\xi > \bar{\xi}_1 \approx 0.111\,82$, and for some exact solutions to the Grad-Shafranov equation (1.3) derived in [13,14].

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