

Rogue waves in nonlocal mediaTheodoros P. Horikis¹ and Mark J. Ablowitz²¹*Department of Mathematics, University of Ioannina, Ioannina 45110, Greece*²*Department of Applied Mathematics, University of Colorado, 526 UCB, Boulder, Colorado 80309-0526, USA*

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The generation of rogue waves is investigated in a class of nonlocal nonlinear Schrödinger (NLS) equations. In this system, modulation instability is suppressed as the effect of nonlocality increases. Despite this fact, there is a parameter regime where the number and amplitude of the rogue events increase as compared to the standard NLS equation, which is a limit of the system when nonlocality vanishes. Furthermore, the nature of these waves is investigated; while no analytical solutions are known to model these events, it is shown, numerically, that these rogue events differ significantly from the rational soliton (Peregrine) solution of the limiting NLS equation. The universal structure of the associated rogue waves is discussed and a local description is presented. These results can help in the experimental realization of rogue waves in these media.

DOI: [10.1103/PhysRevE.95.042211](https://doi.org/10.1103/PhysRevE.95.042211)**I. INTRODUCTION**

Abnormally large waves have long been observed in the ocean; such “rogue” waves can be extremely dangerous even to large ships. This has motivated wide ranging research on rogue and extreme phenomena that spans across sciences [1–9]. Understanding of these phenomena has often been mainly driven by studies of the well-known integrable nonlinear Schrödinger equation (NLS) with cubic or Kerr nonlinearity and small perturbations of this equation [10].

The NLS system provides a unique balance between the critical effects that govern propagation in dispersive media, namely dispersion or diffraction on the one hand and nonlinearity on the other hand. This balance leads to the formation of solitons, which are characterized by their stability and robustness as they maintain their shape and velocity even when they interact. Rogue waves, on the other hand, and for the purpose of this article, can appear anywhere and then disappear. They are often described by the so-called Peregrine soliton [11, 12], which is a special type of solitary wave formed on top of a continuous wave (cw) background; in contrast to other well-known soliton solutions of the NLS equation it is written in terms of rational functions with the property of having relatively large amplitude and being localized in both time and space. These properties make these solutions useful to describe such events [13]. Notably, in the context of oceanic waves criteria for a wave to be called “rogue” are summarized in Ref. [14]. Here, our criteria for rogue waves, much like nonlinear optics and related nonlocal media, is that these rogue waves grow to a factor of 3 or more compared to the maximum of the initial conditions. As is common in nonlinear optics, cf. [1], we take initial conditions to be a wide unit gaussian.

The specific conditions that cause their formation is still a subject of enormous interest; it is generally recognized that modulation instability (MI) is among the important mechanisms which lead to rogue wave excitation [15–19]. MI is the nonlinear mechanism of the self-wave interactions, called the Benjamin-Feir instability [20] in water wave physics. In nonlinear optics, it is considered a basic process that classifies the qualitative behavior of modulated waves [21]. Rogue waves, as a result of an MI process, can be identified as high-contrast peaks of random intensity and are the result

of the unstable growth of weak wave modulations. Mathematically, MI is a fundamental property of many nonlinear dispersive systems and is a well documented and understood phenomenon [22].

Studies of the integrable NLS and weakly perturbed NLS equations have provided important information about rogue phenomena. However, this NLS equation does not model a range of phenomena; e.g., interacting water waves [23], or gain and loss which are inevitable in any physical system [24]. Hence, in order to model different classes of physical systems often it is necessary to go beyond the standard NLS description. There are, for example, important systems that display nonlocal nonlinear mechanisms. Such media include nematic liquid crystals [25,26], thermal nonlinear optical media [27,28], and plasmas [29,30]. Here we will study a specific class of nonlocal equations which has been shown to describe liquid crystals [25,26,31]. In this system the nonlocality yields important differences from the integrable focusing NLS equation.

The effect of the nonlocality of the type we are considering is significant; here the nonlocal term replaces the previously local cubic nonlinearity. The integrable nature of the equation is likely lost and while soliton solutions may also be found they can lack the freedom of various parameters describing the soliton’s properties (amplitude, velocity, etc.). In terms of rational (rogue type) solutions, none are known, to our knowledge. In terms of the MI properties in the model we investigate, the cw solutions are always unstable with the nonlocality suppressing the instability (although it does not eliminate the effect) [32] as compared to the standard NLS equation. It has also been shown that some types of nonlocality eliminates collapse in all physical dimensions [33]. These observations suggest that certain types of nonlocality may have a stabilizing effect. We find, here, that the particular type of nonlocality we examine in this article does not always suppress the number and size of the rogue events.

It is important to understand the nature of the rogue wave and its origins; some work in this direction has been done [22,34,35]. Here, we find that rogue waves of the nonlocal system we investigate have some properties that are similar to those in the integrable NLS equation. Some details are different: e.g., the Peregrine rational solution of the integrable

NLS system has no direct analog in our (likely nonintegrable) system. Nevertheless there are certain local universal features and solution structure associated with the rogue events of the nonlocal system we investigate that are similar.

II. BASIC EQUATIONS AND MI ANALYSIS

The normalized system that governs propagation in nonlocal media reads [26,36]

$$i \frac{\partial u}{\partial z} + d \frac{\partial^2 u}{\partial x^2} + 2g\theta u = 0, \quad (1a)$$

$$\nu \frac{\partial^2 \theta}{\partial x^2} - 2q\theta = -2|u|^2. \quad (1b)$$

Depending on the physical situation the system and its coefficients correspond to different physical quantities. For example, in the context of nematic liquid crystals, u is the complex valued, slowly varying envelope of the optical electric field and θ is the optically induced deviation of the director angle. Diffraction is represented by d and nonlinear coupling by g . The effect of nonlocality ν measures the strength of the response of the nematic in space, with a highly nonlocal response when $\nu (> 0)$ is large. The parameter $q > 0$ is related to the square of the applied static field which pretilts the nematic dielectric [31]. In this context, d, g, q are $O(1)$ while ν is large ($\nu \sim 10^2$) [26,31].

In order to investigate the stability properties of system (1) consider its cw wave solution

$$u(z) = u_0 e^{2ig\theta_0 z}, \quad \theta_0 = \frac{1}{q} u_0^2,$$

where u_0 is a real constant. Adding a small perturbation to this cw solution,

$$u(x, z) = [u_0 + u_1(x, z)] e^{2ig\theta_0 z},$$

which is assumed to behave as $\exp[i(kx - \omega z)]$, leads to the dispersion relation:

$$\omega^2 = \frac{dk^2(d\nu k^4 + 2dqk^2 - 8gu_0^2)}{\nu k^2 + 2q}. \quad (2)$$

It is clear that when $dg > 0$ the system is unstable (and is termed focusing) whereas when $dg < 0$ the system is spectrally stable (and is termed defocusing). Also, when $\nu = 0$ the equation reduces to the dispersion relation of the relative NLS equation, which has the same stability criteria. From this dispersion relation we can identify three critical values that characterize the instability, namely the maximum growth rate $\text{Im}\{\omega_{\max}\}$ its location k_{\max} , and the width of the instability region k_c . The value $\text{Im}\{\omega_{\max}\}$ is a measure of the propagation distance needed for the instability to occur (the larger its value the faster the instability occurs) and k_c defines the range of possible wave numbers that can yield instability; the larger the value of k_c , the more unstable the system is, as more wave numbers can lead to unstable propagation. By differentiating Eq. (2), with respect to k , we find that k_{\max} is the solution of the algebraic equation

$$d(\nu k^3 + 2qk)^2 - 8gqu_0^2 = 0,$$

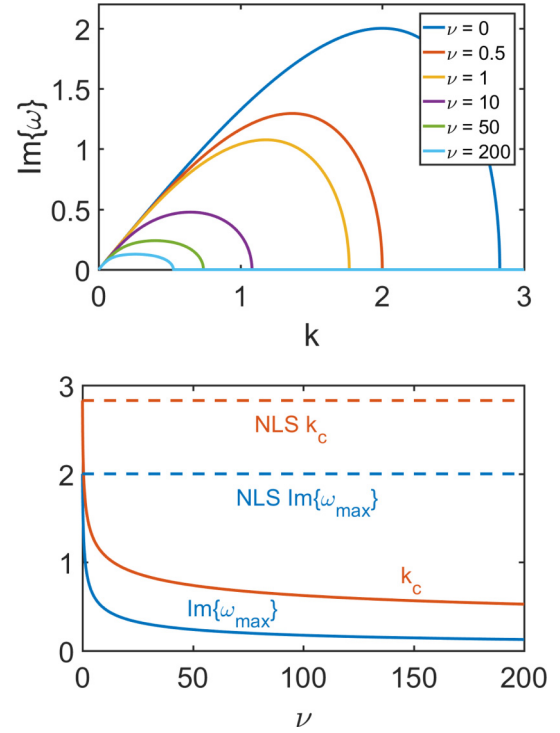


FIG. 1. Top: Growth rates for different values of the nonlocal parameter ν . The far right curve corresponds to $\nu = 0$ and the curves become smaller in amplitude and width until the value $\nu = 200$, the far left curve. Bottom: The change of critical values $\text{Im}\{\omega_{\max}\}$ and k_c with the nonlocality ν .

while k_c satisfies

$$d\nu k^4 + 2dqk^2 - 8gu_0^2 = 0.$$

Both equations can be solved in closed form (they are biquadratics) to give the relative dependance of $\text{Im}\{\omega_{\max}\}$ and k_c with the nonlocality ν . Hereafter we fix $d = 1/2$ and $g = q = u_0 = 1$. We illustrate the situation in Fig. 1.

This figure agrees with the findings of Ref. [32] that nonlocality has an increasingly stabilizing effect on the system. Indeed, both critical values that characterize the instability, $\text{Im}\{\omega_{\max}\}$ and k_c , decrease as ν increases. This means that the effect of MI will need more distance to be exhibited; and if ν is large enough this distance can be larger than the experimental scales. Further, a smaller range of wave numbers will cause an instability. While both values decrease, the effect, in the focusing case, is always present, just more suppressed as ν increases. The limiting NLS system is, by these values, significantly more unstable.

III. ROGUE WAVE FORMATION

A. Rogue wave numbers vs growth rates

To see how these observations affect the generation of rogue waves, we integrate Eqs. (1) numerically using a pseudospectral method in space and exponential Runge-Kutta for the evolution [37] we use the computational domain $x \in [-100, 100]$, $z \in [0, 20]$. An appropriate initial condition, often used in nonlinear optics, is a wide unit gaussian of the

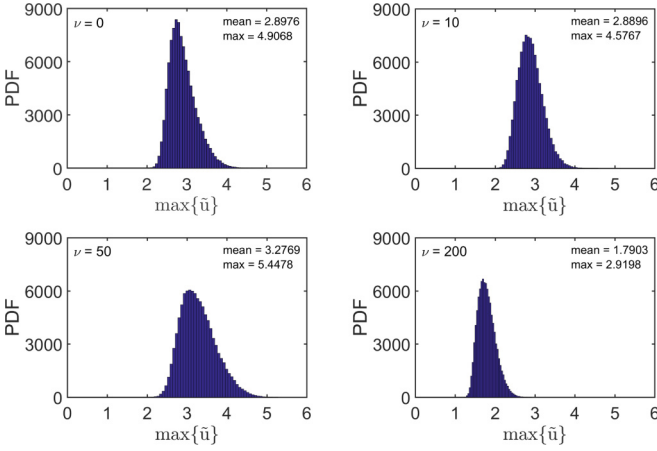


FIG. 2. Probability density functions of the maximum value or $\max\{\tilde{u}\}$ for different values of the nonlocal parameter ν .

form

$$u(x,0) = e^{-x^2/2\sigma^2}, \quad \sigma = 30,$$

perturbed with additional 10% random noise. A wide gaussian with randomness added is a prototype of a set of broad or randomly generated states which can potentially excite more than one wave number; i.e., it can be regarded as a Fourier series of different cw's of different k 's. This is particularly important here as a single cw initial condition may not cause any growth due to the decrease of k_c with ν . For each value of the parameter ν we perform 10^5 trials. In each trial we measure the largest wave amplitude of u over x, z ; we introduce the quantity

$$\tilde{u}(x,z) = \frac{u(x,z)}{\max\{u(x,0)\}},$$

which measures the relative growth in amplitude from an initial state. Here we consider a rogue event as one in which $\tilde{u}(x,z)$ at some value of z is at least three times its maximum initial value. In Fig. 2 we depict PDFs of rogue events for various values of ν .

These PDFs indicate that there is a complex relationship between the occurrence of rogue events and nonlocality. Indeed, starting with $\nu = 10$ the mean of the PDF is approximately comparable to that of the standard NLS equation ($\nu = 0$). With $\nu = 50$ there is a definite shift towards the right indicating that rogue events have increased in both numbers and severity (amplitude). Finally, for $\nu = 200$ there is a sharp decrease of events and their amplitudes. This indicates that there is a nontrivial dependence between the nonlocality and the occurrence of rogue events. The expectation that nonlocality stabilizes the system and thus suppresses extreme phenomena does not hold.

To further investigate the dependence of rogue events with ν , we perform the same analysis for a wider range of the parameter. In Fig. 3 we depict the change of the mean value in the PDFs for the maximum values of \tilde{u} with ν as well as the change of the top 10% of the highest valued events.

Based on this figure, there are three different regions of interest. In the region $0 \leq \nu \lesssim 10$, there is an initial sharp drop from the NLS case ($\nu = 0$) to about $\nu = 0.1$ and then

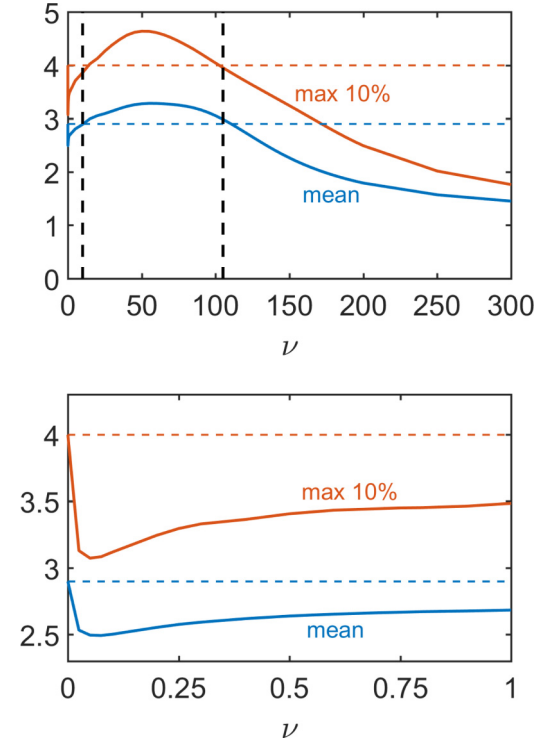


FIG. 3. Top: The mean value of the PDFs and the mean value of the max 10% events with the nonlocal parameter ν . The horizontal dashed lines indicate the relative values for $\nu = 0$ and the vertical dashed lines the values of ν for which these values surpass the NLS system. Bottom: A zoom in around $\nu = 0$.

both curves increase with ν , but still remain below the NLS limit. Near $\nu = 0$ there is a sharp drop, a boundary layer type change, to a local minimum after which both the mean and max curves increase. In the region $10 \lesssim \nu \lesssim 110$, the curves remain well above the NLS limits which translates into the system producing more numerous and more extreme events. Recall, again that for these values the system exhibits very weak growth rates and has a very narrow instability band. Finally, for $\nu > 110$ both curves slowly decay as the nonlocal parameter increases.

B. Universal features of the rogue wave

In the region where rogue waves are large, we turn our attention to the nature of these waves and the mechanism that causes them. In Fig. 4 we show a part of the evolution that contains a rogue wave for different values of the nonlocal parameter ν .

It is clear from these figures that the evolution as ν increases becomes more regular (thus attesting to its stabilizing effect) but the essence of the rogue wave is the same: they appear from nowhere and are relatively short lived.

To our knowledge, there is no analytical description of these waves in this type of nonlocal media. In fact, for the limiting NLS case they are frequently modeled by the so-called Peregrine soliton, a rational solution which reads for Eqs. (1)

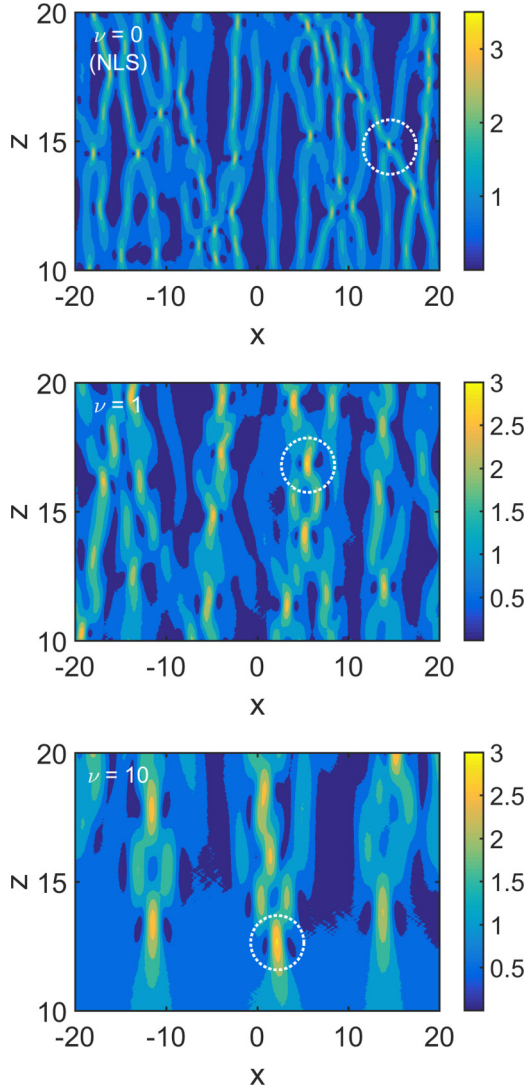


FIG. 4. Snapshots of typical evolutions around the rogue event for different values of the nonlocal parameter. The white circle depicts the rogue event.

(with $\nu = 0$)

$$u_P(x, z) = u_0 \left[1 - \frac{4dq^2 + i(16dgqu_0^2)z}{dq^2 + (4gqu_0^2)x^2 + (16dg^2u_0^4)z^2} \right] \times e^{2igqu_0^2z/q},$$

while the single NLS soliton solution is

$$u_s(x, z) = u_0 \operatorname{sech}(u_0 \sqrt{g/dq} x) e^{iu_0^2 g z / q}.$$

It is counterintuitive (and verified below) to believe that either would be a good candidate to approximate rogue waves in this context as they lack the dependence on the nonlocal parameter ν . Furthermore, the soliton solution of Eqs. (1) is [38]

$$u(x, z) = \frac{3q}{2} \sqrt{\frac{d}{g\nu}} \operatorname{sech}^2(\sqrt{q/2\nu} x) e^{2idq/\nu z},$$

which while it obviously depends on ν , it has fixed amplitude (much like $\chi^{(2)}$ materials [39,40]) which decays with ν . As

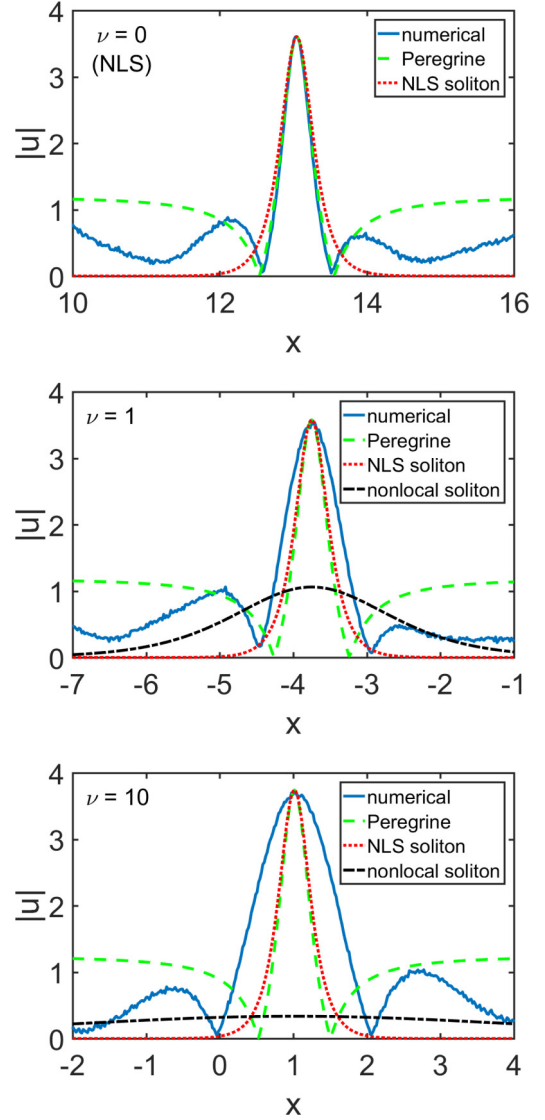


FIG. 5. Comparisons of a (randomly chosen) rogue event of the nonlocal equation with the known soliton and rational solutions.

such, this solution is again not an appropriate candidate to model extreme events (higher nonlocality results in smaller soliton amplitudes). In fact, solutions with a free parameter for this system have been found but only in the defocusing case and under a small amplitude approximation technique [41]. To illustrate we compare all these solutions to an arbitrary rogue event in Fig. 5.

As seen in the figures the two solutions of the regular NLS system ($\nu = 0$) are too narrow to fit the event when ν is well away from zero, while the decaying soliton of the nonlocal system is of small amplitude and wide in width for large ν and appears as a straight (black) line. To further investigate the matter, in Fig. 6, we zoom in around a rogue event for different values of the parameter ν and fit a rational solution around it.

The best amplitude fit is given by the ratio of two fourth order polynomials in x . We notice that the fits become increasingly better as ν increases indicating the profound difference with the integrable NLS system. This is also

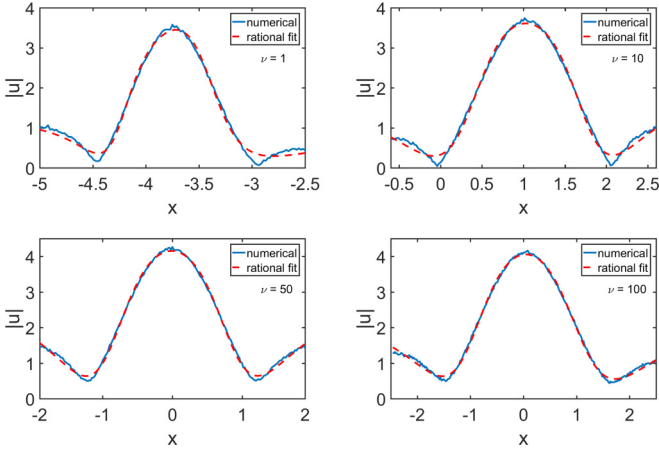


FIG. 6. A zoom in around a rogue event for the different values of the nonlocal parameter ν . A fourth order rational solution has been fitted (red line) in all cases.

consistent with the different soliton solutions. Indeed, the sech-type soliton of the NLS is replaced by the sech² solution of the nonlocal system. This is not the first time that more general (and commonly not known to be integrable) systems give rogue events whose nature differs from that of the typical rational Peregrine soliton. A similar situation was recently observed in deep water waves [23]. We see that in all cases this rational approximation of the rogue event captures the central core with its maximum, the decay to minima, and its increase again. This aspect is similar in all cases.

Next, consider a random or typical event corresponding to different values of the nonlocality as depicted in Fig. 7. We see that there are important common or universal features to

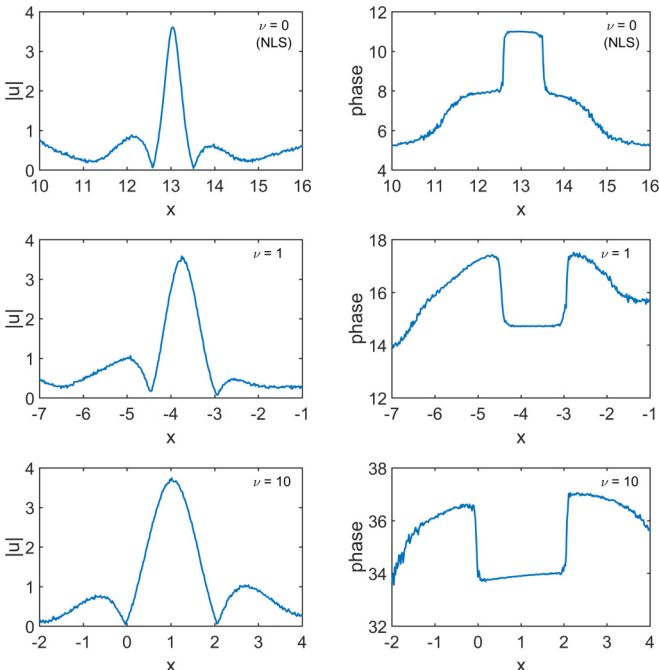


FIG. 7. Left: Amplitude of rogue events for the different values of the nonlocal parameter ν ; right: their corresponding phases.

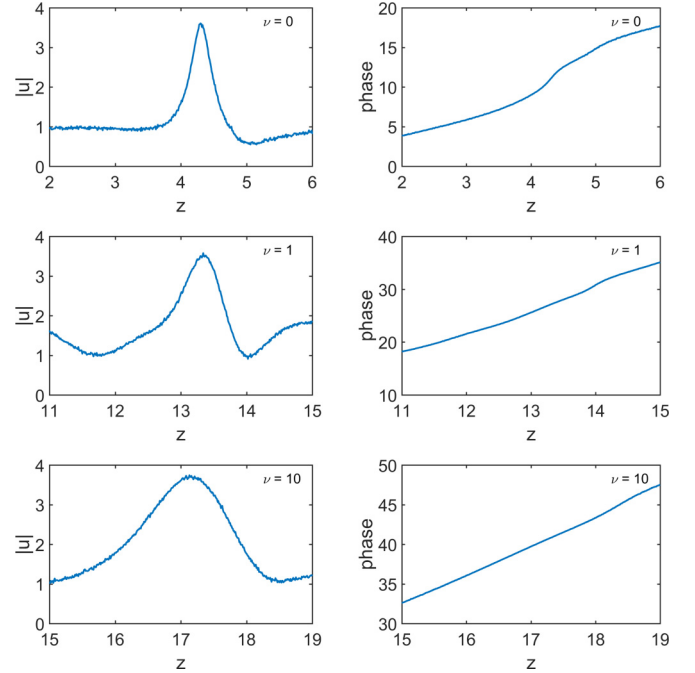


FIG. 8. Left: Amplitude of rogue events for the different values of the nonlocal parameter ν ; right: their corresponding phases.

these rogue events. In their central core they have a large peak with a relatively flat spatial phase. At the edge of the core they decrease to relatively small values and then increase again to a local maximum on each side of the central core. Remarkably, even in the water wave problem [23] these properties are also found at the corresponding rogue events.

The rational fitting, Fig. 6, indicates that for ν sufficiently large the amplitude of the rogue events can be described as rational functions of degree 4; this is contrasted with the Peregrine solution whose amplitude at any value of z is described by a rational function of degree 2.

As can be seen from Figs. 7 and 8, key aspects regarding the structure of these rogue waves have common or universal features. They are short lived; have a π -phase difference between the main core and the accompanying dips; have a spatially flat central phase and smaller maxima on the wings.

We can put the information gleaned from Figs. 7 and 8 together in the neighborhood of the maximum of the rogue wave. Using $u(z, x) = \rho(z, x) \exp(i\phi(z, x))$ in Eqs. (1) and separating real and imaginary parts yields

$$\rho_z + 2\rho_x\phi_x + \rho\phi_{xx} = 0, \quad (3a)$$

$$\rho_{xx} - \rho\phi_x^2 - \rho\phi_z + 2g\theta\rho = 0, \quad (3b)$$

$$v\theta_{xx} - 2q\theta = -2\rho^2. \quad (3c)$$

Figure 7 implies that the x profile of the phase constant $\phi_x = \phi_{xx} = 0$; Fig. 8 indicates that the z profile is linear and as such $\phi_z = \mu$. Then the first of Eqs. (3) is an identity as around the maximum $\rho_z = 0$ and finally the system is reduced to

$$\rho_{xx} - \mu\rho + 2g\theta\rho = 0, \quad (4a)$$

$$v\theta_{xx} - 2q\theta = -2\rho^2. \quad (4b)$$

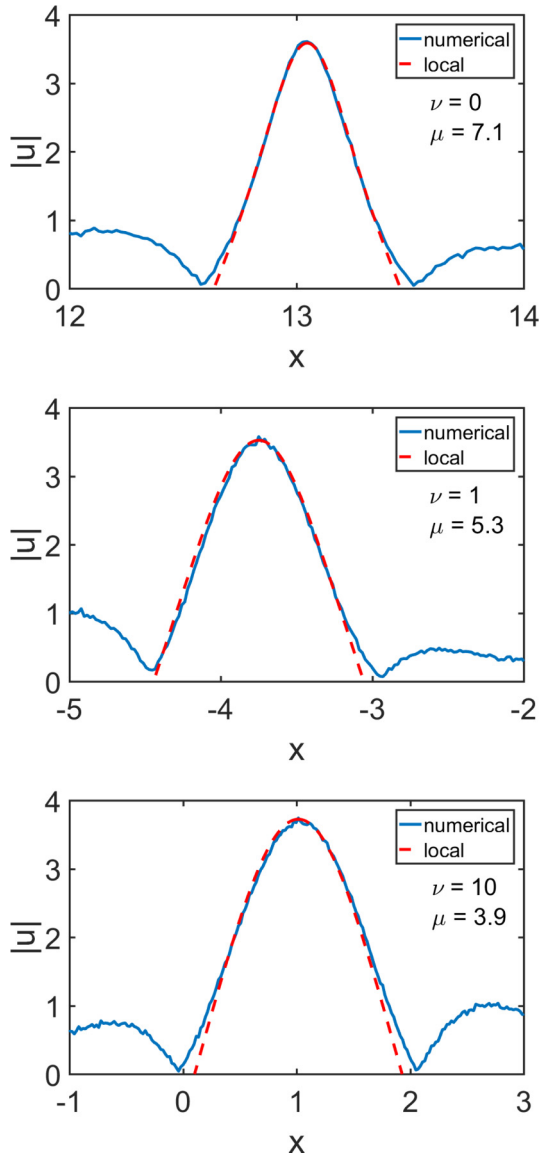


FIG. 9. The local description of rogue waves for different values of the nonlocal parameter ν .

If we assume that the event appears at the location x_{\max} and has amplitude $\rho(x_{\max}) = \rho_0$ [and $\theta(x_{\max}) = \theta_0$], the above

system of differential equations is supplemented by the initial data and value of μ .

$$\rho(x_{\max}) = \rho_0, \quad \rho_x(x_{\max}) = 0, \quad (5)$$

$$\theta(x_{\max}) = \theta_0, \quad \theta_x(x_{\max}) = 0. \quad (6)$$

To test this description, we measure (numerically) the maximum amplitude of the rogue waves for different values of the nonlocal parameter ν (as well as μ) and solve the system to approximate these events. The results are shown in Fig. 9. It is clear that the local description works well.

IV. CONCLUSIONS

To conclude, we have studied rogue wave formation in certain physically significant media described by a nonlocal NLS system. Here the nonlocal term replaces the local cubic nonlinear term in the previously integrable NLS equation. For these systems, as the nonlocality parameter increases, MI is suppressed in both the strength of growth rates and size of instability band. The results of MI alone might suggest the appearance of fewer and smaller, in amplitude, rogue events. Contrary to that we found that for a wide range of values of the nonlocal parameter, the system can produce significantly more events in both size and numbers. To our knowledge this system has not been found to be integrable. The only known solution of the system, a decaying soliton, does not describe these rogue events.

There are universal features that the rogue events found in our nonlocal system exhibit. They appear with relatively large amplitude and then disappear. The amplitude has a main maximum with a spatially flat central phase and two smaller maxima on the wings. The amplitude is well described by rational functions; for $\nu = 0$ we have a ratio of second order polynomials, i.e., the Peregrine solution of the NLS equation; for ν larger than unity the amplitude is described by a rational function of fourth order polynomials.

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