Mechanism for Hall conductance of two-band systems against decoherence

H. Z. Shen,[1,2] Hong Li,[1] Y. F. Peng,[1] and X. X. Yi[1,2,*]

[1] *Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun 130024, China*

[2] *Center for Advanced Optoelectronic Functional Materials Research, and Key Laboratory for UV Light-Emitting Materials and Technology of Ministry of Education, Northeast Normal University, Changchun 130024, China*

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The Kubo formula expresses a linear response of the quantum system to weak classical fields. Previous studies showed that the environment degrades the quantum Hall conductance. By studying the dynamics of dissipative two-band systems, in this paper we find that the formation of system-environment bound states is responsible for the Hall conductance immune to the effect of the environment. The bound states can form only when the system-environment couplings are below a threshold. Our results may be of both theoretical and experimental interest in exploring dissipative topological insulators in realistic situations, and may open new perspectives for designing active quantum Hall devices working in realistic environments.

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I. INTRODUCTION

The quantum Hall effect is manifested by a remarkably precise quantization of the transverse conductance in two-dimensional (2D) electron systems in the presence of a strong perpendicular magnetic field. Its discovery [1,2] has had profound implications for the understanding of matter, and it may find potential applications in quantum information processing [3]. Topological insulators (TIs) [4–7] possess a bulk electronic band gap like an ordinary insulator but protected conducting topological states (edge states) on their surface, which were theoretically predicted to exist and have been experimentally discovered in recent years [8–11]. They have recently attracted considerable attention due to their interesting features and possible applications in quantum computation [12–16]. The connection of the Hall conductance and the topological invariant was first established by Thouless *et al.* [17,18] for closed systems, i.e., the Hall conductance can be represented in terms of the topological invariant (or Chern number) in the linear response theory.

Recently, the studies have been extended to the topological insulator subjected to environments based on the Kubo theory [19]. Topics in this direction include density-matrix Chern insulators caused by thermal noise [20,21], Hall conductance subjected to decoherence [22,23], linear response theory based on the master equation [24–31], the Linbland superoperator [32] and the hierarchical equation of motion [33,34], topological order by dissipation [35–37], the response of quantum systems to single-mode quantized driving [38], optical Hall conductivity [39–42], the Hall response in a quantum quench [43,44], and topological-phase-induced photocurrent [45,46]. While topological materials subjected to environments have become the subject of these studies, the memory effect of the environment on TIs remains unexplored.

From the viewpoint of memory effects, the environment can be divided into two categories: memoryless and memory. The first leads to Markovian dynamics while the second leads to the non-Markovian one. For the quantum system, the absence of dependence of the quantum system on its past time evolution implies a Markovian dynamics, resulting from couplings of the system to a memoryless environment. Nontrivial temporal correlations among different states of the system throughout the dynamics give rise to a non-Markovian quantum process exhibiting memory effects. The presence of memory in the dynamics can help to protect coherence and quantum correlations in open systems at a long time scale. Therefore, understanding the nature of memory effects from various perspectives has become a significant problem for the open quantum system community. These works motivate us to study topological insulators subjected to non-Markovian environments.

For this purpose, we derive a non-Markovian master equation with a general entangled initial state, which goes beyond the master equation based on the uncorrelated initial state in the literature [47–49], then we study the effects of non-Markovianity on the Hall conductance of a two-band system (TBS). Moreover, we give an answer to the question of how the quantum systems respond to an external field in the non-Markovian environment. We find a threshold for the environment effect on the Hall conductance, which originates from the formation of the system-environment bound states [50–52]. With a two-band system (TBS) coupled dissipatively to a non-Markovian environment, we show that the formation of the bound states significantly changes the dynamics of the TBS, and plays an important role in the Hall conductance.

The remainder of the paper is organized as follows. In Sec. II, we introduce a general model to describe the two-band system (TBS) coupled to an environment. A general non-Markovian master equation of the TBS is derived with initial system-environment correlation. In Sec. III, we apply the results to quantum materials and derive the Hall conductance for a topological insulator described by the two-band model in a non-Markovian environment, and discuss the threshold in the influence of the non-Markovian environment on Hall conductance. In Sec. IV, we study the influence of the initial system-environment entanglement on Hall conductance. Section V is devoted to discussion and conclusion.

II. MODEL AND DYNAMICS

A. Model Hamiltonian

In order to study the Hall effect in quantum open system, we first introduce a two-band model to describe quantum Hall

*yixx@nenu.edu.cn

insulators,

$$\hat{H}_S(k) = \vec{d}(k)\vec{\sigma} + \epsilon(k)\mathcal{I}, \tag{1}$$

where $\mathcal{I}$ is the 2×2 identity matrix, $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ are Pauli matrices, and $\epsilon(k)$ and $\vec{d}(k)$ depend on the materials under study and determine its band structure. The two bands may describe different physical degrees of freedom. When the bands are formed by spin-1/2 electrons, $\vec{d}(k)$ stands for the spin-orbit coupling. When they describe the orbital degrees of freedoms, $\vec{d}(k)$ represents the hybridization between bands. The discussion below is completely independent of the physical interpretation of the Hamiltonian Eq. (1), and the conclusion holds for all systems described by the two-band model.

The eigenenergies of the two-band Hamiltonian (1) are $E_1(k) = \varepsilon(k) + d(k)$ and $E_2(k) = \varepsilon(k) - d(k)$, with $d(k) = \sqrt{d_x^2(k) + d_y^2(k) + d_z^2(k)}$. In the Hilbert space spanned by the eigenstates of σ_z, satisfying $\sigma_z|+\rangle = |+\rangle$ and $\sigma_z|-\rangle = -|-\rangle$, the eigenstates of Hamiltonian (1) take

$$\begin{aligned} |\psi_1(k)\rangle &= \begin{pmatrix} \cos\frac{\theta(k)}{2} e^{-i\phi(k)} \\ \sin\frac{\theta(k)}{2} \end{pmatrix}, \\ |\psi_2(k)\rangle &= \begin{pmatrix} \sin\frac{\theta(k)}{2} e^{-i\phi(k)} \\ -\cos\frac{\theta(k)}{2} \end{pmatrix}, \end{aligned} \tag{2}$$

where $\cos\theta(k) = d_z(k)/d(k)$, $\tan\varphi(k) = d_y(k)/d_x(k)$. To calculate the Hall current, i.e., the response of the TBS to the external electric field, we assume the field is along the x direction. Considering the noise in the field, we may decompose the field into a classical part $E_x(t)$ and a quantum fluctuation part $\hat{E}_q(t)$. Namely,

$$\hat{E}(t) = E_x(t) + \hat{E}_q(t), \tag{3}$$

with

$$E_x(t) = E_x \cos\omega t, \tag{4}$$

$$\hat{E}_q(t) = \sum_j E_j(\hat{a}_j^\dagger + \hat{a}_j), \tag{5}$$

where E_x and E_j are classical (with frequency ω) and quantum (with frequency ω_j) electric field, respectively. $\hat{a}_j(\hat{a}_j^\dagger)$ is the annihilation (creation) operator for the quantum fluctuations. In the following discussion, we would treat the quantum fluctuation as an environment. In a space spanned by the eigenstates of $\hat{H}_S(k)$, the coupling between the classical electric field E_x and the TBS can be described by the following Hamiltonian,

$$\hat{H}_c(k) = g_x(k)\tilde{\sigma}_+^k e^{-i\omega t} + g_x^*(k)\tilde{\sigma}_-^k e^{i\omega t} + \frac{e_1(k)}{2}\tilde{\sigma}_z^k, \tag{6}$$

where $g_x(k) = \frac{1}{2}E_x G(k)$ with

$$\begin{aligned} G(k) &= -ie\langle\psi_1(k)|\frac{\partial}{\partial k_x}|\psi_2(k)\rangle, \\ e_1(k) &= 2d(k), \\ \tilde{\sigma}_z^k &= |\psi_1(k)\rangle\langle\psi_1(k)| - |\psi_2(k)\rangle\langle\psi_2(k)|, \end{aligned}$$

$\tilde{\sigma}_+^k = |\psi_1(k)\rangle\langle\psi_2(k)|$, and $\tilde{\sigma}_-^k = |\psi_2(k)\rangle\langle\psi_1(k)|$. Similarly, the Hamiltonian for the couplings between the quantized electric field $\hat{E}_q(t)$ and the TBS reads

$$\hat{H}_q(k) = \sum_j g_j(k)\hat{a}_j\tilde{\sigma}_+^k + g_j^*(k)\tilde{\sigma}_-^k\hat{a}_j^\dagger + \sum_j \hbar\omega_j\hat{a}_j^+\hat{a}_j, \tag{7}$$

with $g_j(k) = E_j G(k)$. In a rotating frame defined by $U(k) = e^{-i\omega\tilde{\sigma}_z^k t/2}$, the total Hamiltonian can be written as

$$\begin{aligned} \hat{H}(k) =& \frac{\Delta_0(k)}{2}\tilde{\sigma}_z^k + \sum_j \hbar\Omega_j\hat{a}_j^+\hat{a}_j + g_x(k)\tilde{\sigma}_+^k + g_x^*(k) \\ &\times\tilde{\sigma}_-^k + \sum_j [g_j(k)\hat{a}_j\tilde{\sigma}_+^k + g_j^*(k)\tilde{\sigma}_-^k\hat{a}_j^+], \end{aligned} \tag{8}$$

where $\Delta_0(k) = e_1(k) - \omega$, and $\Omega_j = \omega_j - \omega$. In this paper, we only consider the dc conductance, therefore we set $\omega \to 0$. Under the weak coupling limit and in the Hilbert space spanned by the eigenstates of Hamiltonian $\frac{\Delta_0(k)}{2}\tilde{\sigma}_z^k + g_x(k)\tilde{\sigma}_+^k + g_x^*(k)\tilde{\sigma}_-^k$, the Hamiltonian in Eq. (8) reduces to

$$\begin{aligned} \hat{H}(k) = \frac{\Delta(k)}{2}\tau_z^k + \sum_j \hbar\omega_j\hat{a}_j^+\hat{a}_j + \sum_j \hbar V_j(k)\tau_+^k\hat{a}_j \\ + V_j^*(k)\tau_-^k\hat{a}_j^+), \end{aligned} \tag{9}$$

where

$$\begin{aligned} V_j(k) &= -\frac{g_j(k)}{\hbar}\cos^2\frac{\mu(k)}{2}e^{-i\nu(k)}, \\ E_{1,2}(k) &= \pm\sqrt{\left|\frac{\Delta_0(k)}{2}\right|^2 + |g_x(k)|^2}, \\ \Delta(k) &= 2E_1(k), \\ |E_1(k)\rangle &= \cos\frac{\mu(k)}{2}e^{i\nu(k)}|\psi_1(k)\rangle + \sin\frac{\mu(k)}{2}|\psi_2(k)\rangle, \\ |E_2(k)\rangle &= \sin\frac{\mu(k)}{2}e^{i\nu(k)}|\psi_1(k)\rangle - \cos\frac{\mu(k)}{2}|\psi_2(k)\rangle, \\ \tau_z^k &= |E_1(k)\rangle\langle E_1(k)| - |E_2(k)\rangle\langle E_2(k)|, \\ \tau_+^k &= |E_1(k)\rangle\langle E_2(k)|, \quad \tau_-^k = |E_2(k)\rangle\langle E_1(k)|, \end{aligned}$$

$\cos\mu(k) = \Delta_0(k)/\sqrt{\Delta_0^2(k) + 4|g_x(k)|^2}$, and $\tan\nu(k) = \mathrm{Im}[g_x(k)]/\mathrm{Re}[g_x(k)]$.

B. Non-Markovian dynamics with entangled initial states

We first derive a non-Markovian master equation to describe the dynamics of the TBS. In contrast with a direct product of the system and the environment state as the initial state, here we choose a system-environment entangled state as the initial state. This choice would contribute to non-Markovian dynamics for the open system,

$$|\psi_k(0)\rangle = \mathcal{Z}_k(0)|E_1(k),\{0_j\}\rangle + \sum_j \beta_j^k(0)|E_2(k),1_j\rangle, \tag{10}$$

which contains *entanglement* between the TBS and the environment (i.e., $|\mathcal{Z}_k(0)| < 1$ and $\beta_j^k(0) \neq 0$). With this initial state (10), the state $|\psi_k(t)\rangle$ at time t can be written as

$$|\psi_k(t)\rangle = \mathcal{Z}_k(t)|E_1(k),\{0_j\}\rangle + \sum_j \beta_j^k(t)|E_2(k),1_j\rangle, \tag{11}$$

with normalized condition, $|\mathcal{Z}_k(t)|^2 + \sum_j |\beta_j^k(t)|^2 = 1$.

Substituting $|\psi_k(t)\rangle$ into the Schrödinger equation, we obtain an integro-differential equation,

$$\dot{\mathcal{Z}}_k(t) + i\omega_c^k \mathcal{Z}_k(t) + \int_0^t \mathcal{Z}_k(\tau) f_k(t-\tau)d\tau = F_k(t), \quad (12)$$

where $\omega_c^k = E_1(k)/\hbar$. The initial value of $\mathcal{Z}_k(t)$ is denoted by $\mathcal{Z}_k(0)$. The effects related to the initial population of the environment are included in

$$F_k(t) = -i\sum_j \beta_j^k(0)V_j(k)e^{-i\Delta_j^k t}. \quad (13)$$

Here $f_k(t) = \sum_j |V_j(k)|^2 e^{-i\Delta_j^k t}$, $\Delta_j^k = \omega_j - \omega_c^k$. The non-Markovian dynamics of the TBS is governed by the master equation by tracing over degrees of freedom of the environment in Eq. (11),

$$\begin{aligned}\dot{\rho}_k =& -iS_k(t)[\tau_+^k\tau_-^k, \rho_k] \\ &+ \left[2\Gamma_k(t) + \Gamma_1^k(t)\right]\left(\tau_-^k\rho_k\tau_+^k - \tfrac{1}{2}\tau_+^k\tau_-^k\rho_k - \tfrac{1}{2}\rho_k\tau_+^k\tau_-^k\right) \\ &+ \Gamma_1^k(t)\left(\tau_+^k\rho_k\tau_-^k - \tfrac{1}{2}\tau_-^k\tau_+^k\rho_k - \tfrac{1}{2}\rho_k\tau_-^k\tau_+^k\right),\end{aligned} \quad (14)$$

where the time-dependent coefficients are given by

$$\begin{aligned}S_k(t) &= -\,\mathrm{Im}[\dot{\mathcal{U}}_k(t)/\mathcal{U}_k(t)], \\ \Gamma_k(t) &= -\,\mathrm{Re}[\dot{\mathcal{U}}_k(t)/\mathcal{U}_k(t)],\end{aligned} \quad (15)$$

and

$$\Gamma_1^k(t) = \frac{|\mathcal{Z}_k(t)|^2\frac{d}{dt}|\mathcal{U}_k(t)|^2 - |\mathcal{U}_k(t)|^2\frac{d}{dt}|\mathcal{Z}_k(t)|^2}{|\mathcal{U}_k(t)|^2[2|\mathcal{Z}_k(t)|^2 - 1]}. \quad (16)$$

Here $\mathcal{U}_k(t)$ satisfies

$$\dot{\mathcal{U}}_k(t) + i\omega_c^k\mathcal{U}_k(t) + \int_0^t \mathcal{U}_k(\tau)f_k(t-\tau)d\tau = 0, \quad (17)$$

with initial value $\mathcal{U}_k(0) = 1$. In the integro-differential equation (17), $\mathcal{U}_k(t)$ plays a role of propagating function for the TBS [53]. Physically, $\mathcal{U}_k(t)$, which is $\beta_j^k(t)$ independent and induced by quantum fluctuations (or environment), contains the effect of dissipation, while $\Gamma_1^k(t)$ depends on the initial population of the environment induced by a thermal-like effect and plays a role of fluctuation in the dynamics. Equation (14) can serve as a nonequilibrium version of the fluctuation-dissipation relation, which ensures the positivity of $\rho_k(t)$ in the time evolution. In addition, $\mathcal{Z}_k(t)$ is, in fact, an operator coefficient and its solution can be obtained analytically from Eq. (12),

$$\mathcal{Z}_k(t) = \mathcal{Z}_k(0)\mathcal{U}_k(t) + \int_0^t \mathcal{U}_k(\tau)F_k(t-\tau)d\tau. \quad (18)$$

We need to point out that the coefficient $\Gamma_1^k(t)$ tends to zero when the initial state is the upper sate of the TBS, i.e., $\mathcal{Z}_k(0) = 1$. The non-Markovian master equation given in Eq. (14) is one of the main results of the paper, which would be used to calculate the response function in the next section.

It is worth addressing that the non-Markovian master equation has been derived in Refs. [47–49], where the authors used the hypothesis that at the initial time the open system and the environment are uncorrelated [$\mathcal{Z}_k(0) = 1$ and $\beta_j^k(0) = 0$], i.e., $\rho_T^k(0) = \rho_S^k \otimes \rho_R$ with $\rho_S^k = |E_1(k)\rangle\langle E_1(k)|$ and $\rho_R = |\{0_j\}\rangle\langle\{0_j\}|$, which leads to $\Gamma_1^k(t) = 0$. In practice, however, the system and its environment might be correlated at the beginning, especially when the system and the environment are strongly coupled [54].

We take the state (10) that describes *entanglement* between TBS and the environment as the initial state [i.e., $|\mathcal{Z}_k(0)| < 1$ and $\beta_j^k(0) \neq 0$] and derive the non-Markovian master equation (14), namely,

$$\rho_T^k(0) = |\psi_k(0)\rangle\langle\psi_k(0)| \neq \rho_S^k \otimes \rho_R, \quad (19)$$

with

$$\begin{aligned}\rho_S^k = \mathrm{Tr}_R\rho_T^k(0) =& |\mathcal{Z}_k(0)|^2|E_1(k)\rangle\langle E_1(k)| \\ &+ [1 - |\mathcal{Z}_k(0)|^2]|E_2(k)\rangle\langle E_2(k)|,\end{aligned} \quad (20)$$

$$\begin{aligned}\rho_R = \mathrm{Tr}_S\rho_T^k(0) =& |\mathcal{Z}_k(0)|^2|\{0_j\}\rangle\langle\{0_j\}| \\ &+ \sum_{jj'}\beta_j^k(0)\beta_{j'}^{k*}(0)|1_j\rangle\langle 1_{j'}|,\end{aligned} \quad (21)$$

with $|\psi_k(0)\rangle$ given by Eq. (10).

With the entangled initial state (10) that satisfies $\rho_T^k(0) \neq \rho_S^k \otimes \rho_R$, we find that the master equation (14) with the time-dependent coefficients in Eqs. (15) and (16) describes a non-Markovian dynamics of TBS. It is easy to find that the quantum entanglement in the initial states can change the fluctuation coefficient $\Gamma_1^k(t)$ but not the damping (dissipation) rate $\Gamma_k(t)$. It is worth addressing that in the framework of the weak coupling limit, our derivation of the master equation (14) goes beyond the master based on the uncorrelated initial state [47–49].

III. HALL CONDUCTANCE IN THE TBS SUBJECTED TO NON-MARKOVIAN ENVIRONMENT

A. Hall conductance for TBS initialized in the entangled state

In this section, we will formulate the Hall conductance for the open system which might characterize the loss of topological degree when the TBS is coupled to a non-Markovian environment. The Hall conductance can be calculated by

$$\Sigma_{xy} = \frac{1}{E_x}\int_{BZ} d^2k\,\mathrm{Tr}\left[\hat{j}_y^k\rho_k^{ss}\right], \quad (22)$$

where the current $\hat{j}_y^k$ in the y direction is generated by a small electric field E_x in the x direction (the 2D system is defined in the xy plane), and ρ_k^{ss} denotes the steady state of the master equation (14).

In the following, we rescale the conductance in units of e^2/h. With the current, $\hat{j}_y^k = (-e)\frac{\partial \hat{H}_S(k)}{\partial k_y}$, we can obtain the Hall conductance for the open system,

$$\begin{aligned}\langle\hat{j}_y\rangle =& \mathrm{Tr}\left[\hat{j}_y\rho_k^{ss}\right] = \left|\mathcal{Z}_k^{ss}\right|^2\langle E_1(k)|\hat{j}_y^k|E_1(k)\rangle \\ &+ \left(1 - \left|\mathcal{Z}_k^{ss}\right|^2\right)\langle E_2(k)|\hat{j}_y^k|E_2(k)\rangle \\ &\approx \left(2\left|\mathcal{Z}_k^{ss}\right|^2 - 1\right)\sin\mu(k)\mathrm{Re}\left[e^{-i\nu(k)}\langle\psi_1(k)|\hat{j}_y^k|\psi_2(k)\rangle\right],\end{aligned} \quad (23)$$

where $\mathcal{Z}_k^{ss}$ is the steady solution of Eq. (18). In the calculation of $\langle E_{1,2}(k)|\hat{j}_y^k|E_{1,2}(k)\rangle$, $|E_{1,2}(k)\rangle$ is represented in the basis $|\pm\rangle$ [substituting Eq. (2) into $|E_{1,2}(k)\rangle$], satisfying $\sigma_z|\pm\rangle = \pm|\pm\rangle$. After straightforward algebras, and expanding Σ_{xy} up to the first order in E_x, we have

$$\Sigma_{xy} = \frac{e^2}{h}\int_{BZ}\frac{idk_xdk_y}{2\pi}\mathcal{N}_k^{ss}\left[\left\langle\frac{\partial\psi_2(k)}{\partial k_x}\middle|\frac{\partial\psi_2(k)}{\partial k_y}\right\rangle - \text{H.c.}\right], \tag{24}$$

where $\mathcal{N}_k^{ss} = 1 - 2|\mathcal{Z}_k^{ss}|^2$, and H.c. denotes Hermitian conjugate. Noticing that the Berry curvature of the lower bare band $|\psi_2(k)\rangle$ is defined by $\eta_{xy}(k) = i[\langle\frac{\partial\psi_2(k)}{\partial k_x}|\frac{\partial\psi_2(k)}{\partial k_y}\rangle - \text{H.c.}]$, we find that Σ_{xy} is simplified as an integral of the Berry curvature weighted by the factor $\mathcal{N}_k^{ss}$. This suggests that the response of the TBS to the external field in the non-Markovian environment may witness the quantum transition points of the system. This is exactly the case as we will show below.

When the two-band system decouples with the non-Markovian environment, $g_j(k) = 0$, we have $\mathcal{Z}_k = 0$. In this case, $\sin\mu(k) \simeq \frac{g_x(k)}{d(k)}$, and Σ_{xy} reduces to the well-known result [55],

$$\Sigma_c = \frac{e^2}{\hbar}\int_{BZ}\frac{dk_xdk_y}{(2\pi)^2}\eta_{xy}(k). \tag{25}$$

We should notice that Σ_c is exactly the conventional Hall conductance, while Σ_{xy} can be understood as the Hall conductance induced by the non-Markovian environment.

B. Exact dynamics for two-band system coupled to a special environment

In this section, via considering explicitly a TBS coupled dissipatively to a bosonic environment, we show that the formation of the bound states significantly changes the dynamics of the TBS, and thus plays a critical role in the non-Markovian dynamics. As a result, a threshold appears below that the environment has no effect on the Hall conductance. This implies that the non-Markovian dynamics can be controlled through engineering the bound state and manipulating the environmental spectrum.

Considering the decoherence suppression induced by the bound state has been observed in the photonic crystal [56–58], in the following discussion, we will take a photonic-crystal-like environment to study the influence of the non-Markovian effect on the Hall conductance.

To get a qualitative understanding of the physics behind the threshold in the non-Markovian effect, we first solve Eq. (17) by Laplace transform $X(s) = \int_0^\infty X(t)e^{-st}dt$ [59–62] that yields

$$\mathcal{U}_k(s) = \frac{1}{s + i\omega_c^k + f_k(s)}, \tag{26}$$

$$f_k(s) = \sum_j |V_j(k)|^2\frac{1}{s + i\Delta_j^k}. \tag{27}$$

Here $g_j(k)$ takes [63]

$$g_j(k) = 2d(k)G(k)\sqrt{\hbar/(2\varepsilon_0\omega_jV)}\vec{e}_j\cdot\vec{u}, \tag{28}$$

where V is the quantization volume, $\vec{e}_j$ are the transverse unit vectors for the environmental modes, and ε_0 is the vacuum dielectric constant. $\vec{u}$ denotes the unit vector of the two-band dipole moment of the transition. For the photonic-crystal-like environment, the dispersion relation takes [64]

$$\omega_j = \omega_e + A\left|\vec{m} - \vec{m}_0^j\right|^2, \tag{29}$$

where ω_e is the cutoff frequency of the band edge. $\vec{m}_0^j$ are the finite collections of symmetry related points, which are associated with the band edge. A is the model-dependent constant. The corresponding state density is $\rho(\omega) \propto \sqrt{(\omega - \omega_e)}\theta(k)(\omega - \omega_e)$ [51,65].

Using the dispersion relation (29), and converting the mode sum over transverse plane waves into an integral and performing the integral (see Appendix A), we have

$$f_k(s) = \frac{-iB\Omega(k)}{\sqrt{\omega_e} + \sqrt{-is - \omega_{1e}^k}}, \tag{30}$$

where $\omega_{1e}^k = \omega_c^k - \omega_e$, $\Omega(k) = d^2(k)\cos^4\frac{\mu(k)}{2}|\langle\psi_1(k)|\frac{\partial\psi_2(k)}{\partial k_x}\rangle|^2$, and

$$B = e^2\sum_{j_1}\sin^2(\theta_{j_1})/(2\pi\hbar\varepsilon_0A^{3/2}).$$

Clearly, parameter B characterizes the strength of system-environment coupling.

The amplitudes $\mathcal{U}_k(t)$ can then be obtained by inverse Laplace transform,

$$\mathcal{U}_k(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathcal{U}_k(s)e^{st}ds, \tag{31}$$

leading to (see Appendix B)

$$\mathcal{U}_k(t) = \sum_n\frac{e^{x_n^{(1)}t}}{G_k'(x_n^{(1)})} + \sum_n\frac{e^{x_n^{(2)}t}}{F_k'(x_n^{(2)})} - \int_0^\infty dy \times\frac{B\Omega(k)\pi^{-1}\sqrt{y}ie^{-yt+i\omega_{1e}^kt}}{B^2\Omega^2(k) - 2B\sqrt{\omega_e}z_k(y)\Omega(k) + (\omega_e - iy)z_k^2(y)}, \tag{32}$$

where $z_k(y) = \omega_c^k + \omega_{1e}^k + iy$, and

$$G_k(s) = s + i\omega_c^k - \frac{iB\Omega(k)}{\sqrt{\omega_e} + \sqrt{-is - \omega_{1e}^k}}, \tag{33}$$

$$F_k(s) = s + i\omega_c^k - \frac{iB\Omega(k)}{\sqrt{\omega_e} - i\sqrt{is + \omega_{1e}^k}}, \tag{34}$$

where $x_n^{(1)}$ are the roots of equation $G_k(s) = 0$ in region [Re$(s) > 0$ or Im$(s) > \omega_{1e}^k$], and $x_n^{(2)}$ are the roots of equation $F_k(s) = 0$ in region [Re$(s) < 0$ and Im$(s) < \omega_{1e}$]. $G_k'(s)$ and $F_k'(s)$ are first derivatives of $G_k(s)$ and $F_k(s)$ with respect to s, respectively. The first term in Eq. (32) corresponds to localized modes with $s = -iE_n(k)$ [$E_n(k)$ are real numbers, which correspond to the energy spectrum of the whole system]. The localized modes exist if and only if the environmental spectral density has band gaps located at the pure imaginary zeros with $G_k(-iE_n(k)) = 0$. These localized modes do not decay, which give dissipationless non-Markovian dynamics. The nonlocalized mode contains two parts: One is the second term in Eq. (32), which is the oscillating damping process due to the complex roots in $F_k(s) = 0$ in the regime of [Re$(s) < 0$ and Im$(s) < \omega_{1e}^k$]. The other is the integral part,

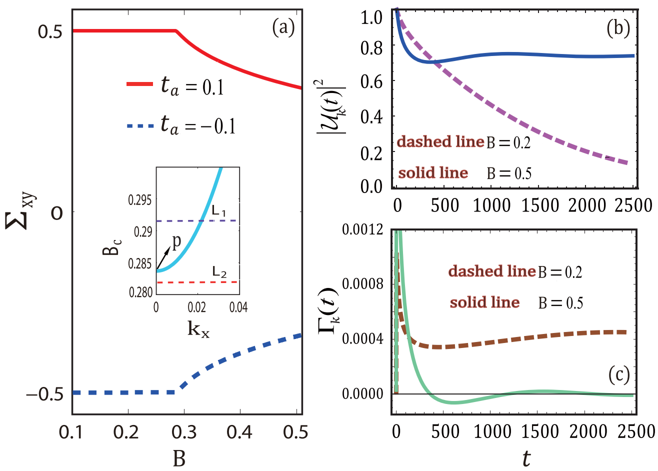

FIG. 1. (a) The Hall conductance Σ_{xy} (in units of e^2/h) for open system as a function of B with different t_a (in units of meV/$\hbar$). The inset shows the critical point B_c, where the line L_1 denotes the regime of the formed bound state. The obtained $|\mathcal{U}_k(t)|^2$ (b) and the corresponding decay rate $\Gamma_k(t)$ (c) with and without bound state. The bound state is formed when $B > B_c = 0.283$ [in units of $(\mathrm{Hz}^{1/2}\hbar^2 m^2)^{-1}$]. The other parameters take $\omega_e = 0.18$ meV/$\hbar$, $k_x = 1.569\ \mathrm{nm}^{-1}$, $E_x = 0.1$ meV/nm, $\eta = 0.1$ meV, $m_0 = p = 1$, $q = 4$.

i.e., the nonexponential parts will oscillate rapidly in time. This rapidly oscillating damping, which originates from the terms containing $e^{i\omega_{1e}^k t}$ in Eq. (32). Therefore the nonlocalized mode parts in Eq. (32) are the contribution of the allowed bands, which usually generate exponential decays. This is the other significance of the non-Markovian dynamics.

C. Threshold in the effect of non-Markovian environment on Hall conductance

Consider electrons in a two-dimensional lattice [66,67] described by the tight-binding model. The Hamiltonian of this system can be written as

$$\hat{H}_s = \sum_k \hat{c}_k^\dagger \hat{H}_s(k)\hat{c}_k \equiv \sum_k \hat{c}_k^\dagger[\vec{d}(k)\cdot\vec{\sigma}]\hat{c}_k, \tag{35}$$

where $\hat{c}_k$ $(\hat{c}_k^\dagger)$ is the fermion annihilation (creation) operator with lattice momentum k, and $\vec{d}(k) = (\eta\cos(k_y l), \eta\sin(k_y l), 2t_a\cos(k_x + 2\pi\frac{p}{q}m_0))$, where energy is taken in units of meV. For fixed $m_0 = 1$, $p = 1$, $q = 4$, $l = 1$, the lower band of $\hat{H}_k$ has Chern number $\mathcal{C} = \frac{t_a}{\sqrt{4t_a^2+\eta^2}}$. Especially, $\mathcal{C} = \frac{1}{2}\mathrm{sgn}(t_a)$ when $\eta \to 0$, which gives the phase transition point for the closed system.

For numerical calculation, we will choose the units of B as $(Hz^{1/2}\hbar^2 m^2)^{-1}$ in the following. At steady state, the Hall conductance for the open quantum system is plotted in Fig. 1(a), and a threshold of $B_c = 0.2837$ can be clearly observed. When $B < B_c$, the Hall conductance stays at $|0.5|e^2/h$. Thus the environment has no effect on the Hall conductance. However, when $B > B_c$, a sharp decrease can be found in the Hall conductance.

This observation suggests that there exists a threshold in the effect of the environment on the Hall conductance, and this threshold may connect closely to the boundary between Markovian and non-Markovian dynamics. The existence of the threshold B_c inspires us to further pursue the physical reason for this phenomenon.

We show that the non-Markovian dynamics of an open system connects closely with the energy-spectrum signatures of the whole system (system plus environment). Therefore the investigation of the energy spectrum may provide us with a meaningful message to understand its dynamics. Since $\hat{N}_k = \tau_+^k\tau_-^k + \sum_j \hat{a}_j^\dagger\hat{a}_j$ is conserved, the Hilbert space splits into independent subspaces with definite $\hat{N}_k$. For our zero-temperature non-Markovian environment, only the subspaces with $\hat{N}_k = 0$ and 1 are involved in the dynamics. Besides the trivial eigenstate $|\phi_0(k)\rangle = |E_2(k),\{0_j\}\rangle$ with $E_2(k)$ for the $N_k = 0$ subspace, we can obtain the eigenstate of the $N_k = 1$ subspace as $|\phi_1(k)\rangle = h_0^k|E_1(k),\{0_j\}\rangle + \sum_j h_j^k|E_2(k),1_j\rangle$ with

$$\left|h_0^k\right|^2 = \frac{1}{1 + \sum_j \frac{\hbar^2|V_j(k)|^2}{\left(\frac{\Delta(k)}{2} - \hbar\omega_j + E(k)\right)^2}},$$

and $E(k)$ decided by

$$\frac{\Delta(k)}{2} + \sum_j \frac{\hbar^2|V_j(k)|^2}{\frac{\Delta(k)}{2} - \hbar\omega_j + E(k)} = E(k), \tag{36}$$

or

$$Y[E(k)] = \omega_c^k - \frac{B\Omega(k)}{\sqrt{\omega_e} + \sqrt{-E(k) - \omega_{1e}^k}} = E(k), \tag{37}$$

where the analytical expression for energy spectrum $E(k)$ is given by Eq. (C3). Since $Y[E(k)]$ decreases monotonically with the increase of $E(k)$ in $E(k) < 0$ and $Y[E(k)]|_{E(k)\to-\infty} = \omega_c^k$, Eq. (37) has an isolated root in the band gap [see Fig. 1(a)] whenever

$$Y\left(-\omega_{1e}^k\right) \leqslant -\omega_{1e}^k \Rightarrow \frac{\left[2\omega_c^k - \omega_e\right]\sqrt{\omega_e}}{\Omega(k)} = B_c, \tag{38}$$

with $\Omega(k)$ given by Eq. (30). We call the eigenstate corresponding to this isolated eigenvalue E_{BS} the bound state. It is remarkable to find from Fig. 1(a) that the critical point for forming the bound state matches well with the threshold found in the Hall conductance [the threshold of Hall conductance equals the minimum value $(k_x = 0)$ of the $B_c(k_x)$ in the two-band system; see the point p of the inset in Fig. 1(a)]. We thus conjecture that the formation of the bound state of the total system is responsible for the quantum Hall conductance immune to environments.

The important role played by the bound state [68] in the quantum Hall conductance can be understood in the following manner. Taking the initial state $|\psi_k(0)\rangle = |E_1(k),\{0_k\}\rangle$, the time-evolution of the whole state can be written as

$$|\psi_k(t)\rangle = p_0^k e^{-iE_{\mathrm{BS}}(k)t}|\phi_{1,BS}(k)\rangle + \sum_{\lambda\in C} p_\lambda^k e^{-iE_\lambda(k)t}|\phi_{1,\lambda}(k)\rangle, \tag{39}$$

where $|\phi_{1,\mathrm{BS}}(k)\rangle$ is the (isolated)bound state with the eigenenergy $E_{\mathrm{BS}}(k)$, $|\phi_{1,\lambda}(k)\rangle$ denotes the eigenstates in the continuous energy band with eigenenergies $E_\lambda(k)$, $p_0^k =$

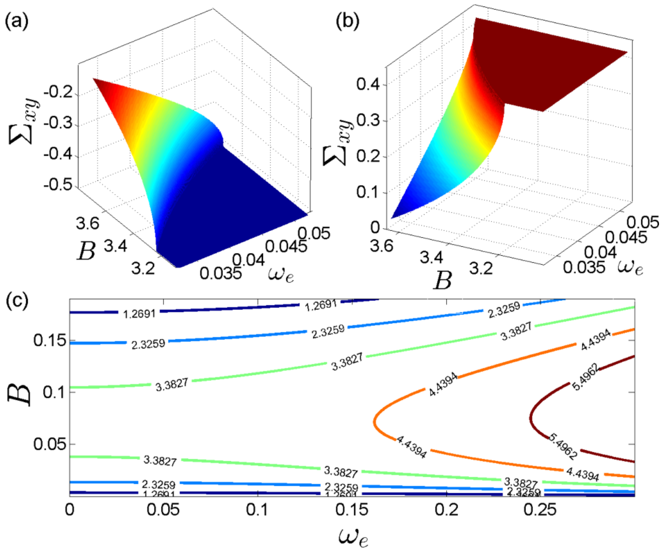

FIG. 2. The plot given by numerically solving the Hall conductance (24) as a function of B_c and ω_e (in units of meV/$\hbar$). The other parameters taken are $E_x = 0.1$ meV/nm, $\eta = 0.1$ meV, $m_0 = p = 1$, $k_x = 1.569$ nm^{-1}, $q = 4$. $t_a = 0.5$ meV/$\hbar$ for (a), $t_a = -0.5$ meV/$\hbar$ for (b). (c) Contour line of the critical point B_c.

$\langle\phi_{1,\mathrm{BS}}(k)|\psi_k(0)\rangle = h_0^k$, and $p_\lambda^k = \langle\phi_{1,\lambda}(k)|\psi_k(0)\rangle$. Due to the out-of-phase interference contributed by the continuous energies $E_\lambda(k)$, all the upper-state population in the components of the summation in Eq. (39) tends to zero and only the one in the bound-state component recovers in the steady state

$$\mathrm{Tr}\left[\tau_+^k\tau_-^k\rho_k^{ss}\right] = \left|\mathcal{U}_k^{ss}\right|^2 = \left|h_0^k\right|^4, \tag{40}$$

where the first identity originates from Eq. (14). If the bound state is absent, then $|\mathcal{U}_k^{ss}|^2$ tends to zero asymptotically. Figures 1(b) and 1(c) plot the upper-state population $|\mathcal{U}_k(t)|^2$ and the decay rate $\Gamma_k(t)$, respectively. We can see that $\Gamma_k(t)$ in the absence of the bound state and $B < B_c$ tends to a positive constant after a short time. The complete positivity of $\Gamma_k(t)$ causes $|\mathcal{U}_k^{ss}|^2$ to decay to zero monotonically. Here the non-Markovian environment has no backaction on the system and thus Hall conductance remains $0.5e^2/h$ as in the closed system. When the bound state is formed in the condition $B > B_c$, the competition between the non-Markovian environment backaction and the dissipation effects on the TBS causes $\Gamma_k(t)$ to transiently take negative value and to asymptotically approach zero [see Fig. 1(c)]. Consequently, after some short-time oscillations, $|\mathcal{U}_k(t)|^2$ approaches a finite value matching well with the result in Eq. (40). This is contrary to the expectation that a larger coupling strength always induces a stronger decoherence. The transient increase of $|\mathcal{U}_k(t)|^2$ causes the increase of the distinguishability and the decrease of the Hall conductance.

The mechanism discussed here in the quantum Hall conductance is expected to be valid for a wide range of parameters. Figure 2 shows Hall conductance with the change of the system-environment coupling strength B and the cutoff frequency ω_e of the band edge. The results confirm that the threshold point of the influence of the non-Markovian environment on Hall conductance matches well with the threshold of the bound state. Our mechanism is applicable to a dissipative system subjected to the general environment, whose dynamics is governed by the same bound-state mechanism as our TBS.

IV. INFLUENCE OF THE INITIAL SYSTEM-ENVIRONMENT ENTANGLEMENT ON THE HALL CONDUCTANCE

In the previous section, we have found a threshold in the Hall conductance of the TBS subjected to a non-Markovian environment, which is a new significant effect of environment on TIs. In this section, we study the influence of the initial system-environment entanglement to the Hall conductance. We will show that the effect of initial entanglement can be fully incorporated into the noise coefficient $\Gamma_1^k(t)$ given in the last sections. And the dissipation coefficients $\Gamma_1^k(t)$ have an effect similar to the temperature in a Markovian master equation with a separate initial state.

We start with the analytical expression of steady state for the non-Markovian master equation (14),

$$\mathcal{Z}_k(t)|_{t\to\infty} = \frac{e^{-iE(k)t/\hbar}}{G'_k[-iE(k)/\hbar]}\left[\mathcal{Z}_k(0) + \frac{\sum_j \beta_j^k(0)V_j(k)}{E(k)/\hbar - \Delta_j^k}\right]. \tag{41}$$

To study the effect aforementioned, we choose the non-Markovian environment state in Eq. (10) with $\beta_j^k(0) = \mathcal{E}V_j(k)/\omega_j$ and $\mathcal{E}$ being the distribution intensity in the non-Markovian environment. This choice is reasonable because the denominator ω_j of $\beta_j^k(0)$ is greater than ω_e [not zero; see Eq. (29)]. With this setting and the use of (28), we have

$$\mathcal{Z}_k(t)|_{t\to\infty} = \frac{e^{-iE(k)t/\hbar}}{G'_k[-iE(k)/\hbar]}\left[\sqrt{1 - \frac{\mathcal{E}^2 B\Omega(k)}{8\omega_e^{3/2}}} + \frac{\mathcal{E}^2 B\Omega(k)\left[E_{1e}^k - \omega_e + 2\sqrt{|E_{1e}^k|\omega_e}\right]}{2\sqrt{\omega_e}\left(E_{1e}^k + \omega_e\right)^2}\right], \tag{42}$$

where $E_{1e}^k = E(k)/\hbar + \omega_{1e}^k$.

To study the influence of the initial system-environment entanglement on Hall conductance of TBS, we calculate Σ_{xy} as a function of $\mathcal{E}$ with various system-environment coupling strength B. The numerical results are shown in Fig. 3. A cutoff frequency of the band edge $\omega_e = 0.2$ meV/$\hbar$ was chosen for the plot. Note from condition (38) that the bound state is formed when $B > 0.2837$; we find that when the bound state is absent, i.e., $B < 0.2837$, the Hall conductance remains unchanged (see solid line at $B = 0.2$ in Fig. 3). However, when the bound state is formed, i.e., $B > 0.2837$, the Hall conductance decreases as the coupling strength B increases, which confirms the prediction based on the bound-state analysis. The difference between Σ_{xy} with different B decreases also as B increases. This can be interpreted as the change of the critical point in the non-Markovian environment, which is determined by the formation of the bound state. In addition, from Fig. 3, we can see that there exists a minimum in Hall conductance at $\mathcal{E} = 0$. This is because for finite value of $\mathcal{E}$, the initial state $\sum_j \beta_j^k(0)|E_2(k),1_j\rangle$ contributes to the ground state $|E_2(k)\rangle$ of the TBS so that the Hall conductance

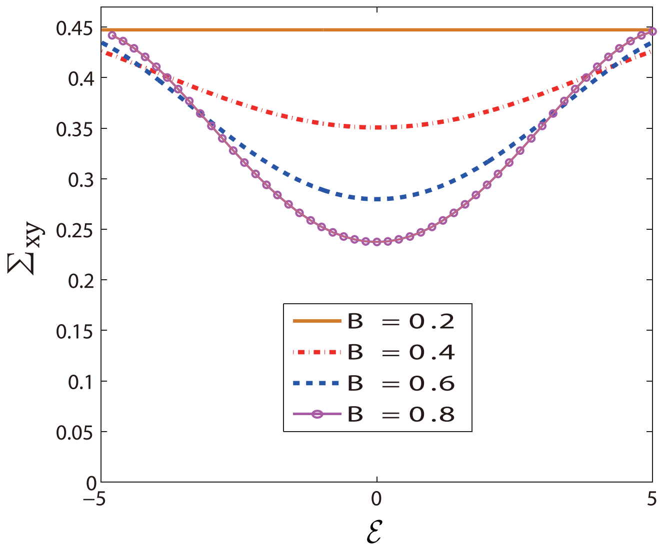

FIG. 3. Influence of the initial system-environment entanglement to Hall conductance with different B. The other parameters take $\omega_e = 0.2$ meV/$\hbar$, $E_x = 0.05$ meV/nm, $\eta = 0.1$ meV, $m_0 = p = 1$, $q = 4$, $t_a = 0.1$ meV/$\hbar$.

increases with the increase of $\mathcal{E}$. This environment-induced reactance describes the energy exchange between the system and the environment, reminiscent of the effect of the initial system-environment entanglement.

V. DISCUSSION AND CONCLUSION

The two-band Hamiltonian used in this paper can describe topological insulators, which can be realized in semiconductor quantum wells [69], alkali-metal cold atom chip [70], and Bose-Einstein condensates [71]. The present model might be realized in a similar way as in [72] with 2D spin-orbit coupling in ultracold 40-K Fermi gases using three lasers, each of them dressing one atomic hyperfine spin state. The non-Markovian system can be simulated by the use of Büttiker's virtual probes [73,74], and the threshold of the influence of the non-Markovian environment on the Hall conductance could be observed by the six-terminal device as employed in Refs. [8,75].

In summary, we have derived a non-Markovian master equation for a two-band sytem with an initially system-environment entanglement. Based on this master equation, we have studied the response of the two-band system to an external field in a non-Markovian environment. We found that the formation of system-environment bound states helps the Hall conductance against the effect of the environment. The formation of the bound state requires the system-environment couplings to be below a threshold. We calculated the threshold and discussed the effect of the initial entanglement on the Hall conductance. Our results of the physical condition on such threshold of Hall conductance might have applications in quantum optics and condensed matter physics.

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APPENDIX A: THE CALCULATION OF $f_k(s)$

With Eq. (28), we can calculate $f_k(s)$ in Eq. (27) as follows:

$$\begin{aligned}
f_k(s) &= \sum_j \frac{|V_j(k)|^2}{s + i\Delta_j^k} \\
&= \frac{[2d(k)G(k)]^2}{2\varepsilon_0 \hbar V} \sum_j \frac{(\vec{e}_j \cdot \vec{u}_d)(\vec{e}_j \cdot \vec{u}_d)}{\Delta_j^k[s + i\Delta_j^k]} \\
&= \frac{[2d(k)G(k)]^2}{2\varepsilon_0 \hbar V} \sum_j \frac{1 - (\vec{m} \cdot \vec{u}_d)^2/j^2}{\Delta_j^k[s + i\Delta_j^k]} \\
&= \frac{[2d(k)G(k)]^2}{16\pi^3 \varepsilon_0 \hbar} \int \frac{[1 - (\vec{m} \cdot \vec{u}_d)^2/k^2] d^3\vec{m}}{\Delta_j^k[s + i\Delta_j^k]},
\end{aligned} \tag{A1}$$

where we have replaced the sum by an integral via $\sum_j \to \frac{V}{(2\pi)^3} \int d^3\vec{m}$ and $(\vec{e}_j \cdot \vec{u}_d)(\vec{e}_j \cdot \vec{u}_d) = 1 - (\vec{m} \cdot \vec{u}_d)(\vec{m} \cdot \vec{u}_d)/m^2$. Near the band edge, the dispersion relation may be expressed approximately by $\omega_j = \omega_e + A|\vec{m} - \vec{m}_0^j|^2$. The angle between the dipole vector of the atom and the jth $\vec{m}_0^j$ is θ_j. The angle between the dipole and $\vec{m}$ near $\vec{m}_0^j$ is replaced approximately by θ_j. We calculate $f_k(s)$ as follows:

$$\begin{aligned}
f_k(s) &= \frac{[2d(k)G(k)]^2}{16\pi^3 \varepsilon_0 \hbar} \int \frac{[1 - (\vec{m} \cdot \vec{u}_d)^2/m^2] d^3\vec{m}}{\Delta_j^k[s + i\Delta_j^k]} \\
&= \frac{[2d(k)G(k)]^2}{16\pi^3 \varepsilon_0 \hbar} \left(\sum_{j_1} \sin^2\theta_{j_1} \right) \\
&\quad \times \int \frac{d^3 m}{[Am^2 - \omega_{1e}^k][s + i(Am^2 - \omega_{1e}^k)]} \\
&= \frac{[d(k)G(k)]^2}{\pi^2 \varepsilon_0 \hbar} \left(\sum_{j_1} \sin^2\theta_{j_1} \right) \\
&\quad \times \int_0^\infty \frac{m^2 dm}{[Am^2 - \omega_{1e}^k][s + i(Am^2 - \omega_{1e}^k)]}.
\end{aligned} \tag{A2}$$

Consequently, integrating the last line in the above equation, we obtain Eq. (30).

APPENDIX B: THE CALCULATION OF THE FUNCTION $\mathcal{U}_k(t)$ IN EQ. (32)

The amplitude $\mathcal{U}_k(t)$ can be obtained by means of the inverse Laplace transform,

$$\begin{aligned}
\mathcal{U}_k(t) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{U}_k(s) e^{st} ds \\
&= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds e^{st} \frac{1}{s + i\omega_c^k - \frac{iB\Omega(k)}{\sqrt{\omega_e} + \sqrt{-is - \omega_{1e}^k}}}.
\end{aligned} \tag{B1}$$

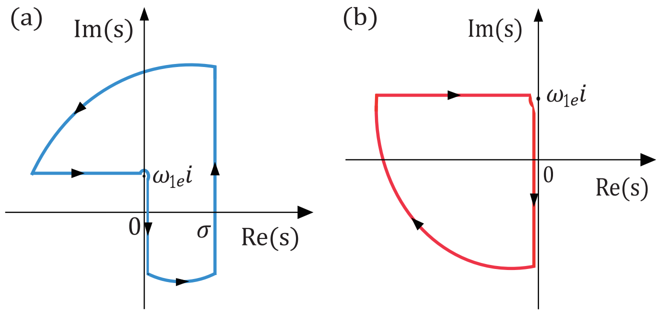

FIG. 4. The integration contours for Eq. (B2).

With the integration contours as shown in Fig. 4(a), we have

$$\mathcal{U}_k(t) = \sum_n \frac{e^{x_n^{(1)}t}}{G_k'(x_n^{(1)})} - \frac{1}{2\pi i}\left[\int_{-i\omega_e-\infty}^{-i\omega_e+0} + \int_{-i\omega_e+0}^{-i\infty+0} ds e^{st} \times \frac{1}{s + i\omega_c^k - \frac{iB\Omega(k)}{\sqrt{\omega_e}+\sqrt{-is-\omega_{1e}^k}}}\right], \tag{B2}$$

where the function $G_k(s)$ is given by Eq. (33), and $x_n^{(1)}$ is the root of $G_k(s) = 0$ in the region [Re$(s) > 0$ or Im$(s) > \omega_{1e}^k$], the real number σ and the real number $s = \sigma$ lie to the right of all the singularities $x_n^{(1)}$. The last term can be calculated with the integration contours as shown in Fig. 4(b):

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\omega_e+0}^{-i\infty+0} ds e^{st}\frac{1}{s + i\omega_c^k + f_k(s)} \\
&= \frac{1}{2\pi i}\int_{-i\omega_e}^{-i\infty} ds e^{st}\frac{1}{s + i\omega_c^k + \beta_k(s)} \\
&= -\sum_n \frac{e^{x_n^{(2)}t}}{F_k'(x_n^{(2)})} - \frac{1}{2\pi i}\left[\int_{-i\omega_e-\infty}^{-i\omega_e+0} ds e^{st}\frac{1}{s + i\omega_c^k + \beta_k(s)}\right],
\end{aligned} \tag{B3}$$

where $F_k(s)$ is given by Eq. (34), and $\beta_k(s) = -\frac{iB\Omega(k)}{\sqrt{\omega_e} - i\sqrt{is+\omega_{1e}^k}}$, and $x_n^{(2)}$ is the root of $F_k(s) = 0$ in the region [Re$(s) < 0$ and Im$(s) < \omega_{1e}^k$].

From Eqs. (B1)–(B3), we can obtain the Green function (32) by setting $s = -y + i\omega_{1e}^k$.

APPENDIX C: THE ANALYTICAL EXPRESS FOR ENERGY SPECTRUM

The quantity p_k satisfies a cubic equation,

$$p_k^3 + bp_k^2 + c_k p_k + d_k = 0, \tag{C1}$$

with $b = -2\sqrt{\omega_e}$, $c_k = \omega_{1e}^k + \omega_c^k + \omega_e$, $d_k = -B\Omega(k)$. Its solutions can be found in any handbook of mathematics. If $|B_1^k|^2 - 4A_1^kC_1^k < 0$ with $A_1^k = b^2 - 3c_k$, $B_1^k = bc_k - 9d_k$, $C_1^k = c_k^2 - 3bd_k$. There are three different real roots,

$$\begin{aligned}
p_1^k &= \frac{-b - 2\sqrt{A_1^k}\cos\theta_1(k)}{3}, \\
p_2^k &= \frac{-b + \sqrt{A_1^k}[\cos\theta_1(k) - \sqrt{3}\sin\theta_1(k)]}{3}, \\
p_3^k &= \frac{-b + \sqrt{A_1^k}[\cos\theta_1(k) + \sqrt{3}\sin\theta_1(k)]}{3},
\end{aligned} \tag{C2}$$

where $\theta_1(k) = \frac{1}{3}\arccos\left(\frac{2A_1^kb - 3B_1^k}{2\sqrt{|A_1^k|^3}}\right)$. Finally, based on Eq. (C2), we can obtain the eigenspectrum,

$$E(k) = -\left[|p^k|^2 + \omega_{1e}^k - 2p^k\sqrt{\omega_e} + \omega_e\right], \tag{C3}$$

where $p^k = p_1^k, p_2^k, p_3^k$.

[1] K. V. Klitzing, G. Dorda, and M. Pepper, New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance, Phys. Rev. Lett. **45**, 494 (1980).
[2] D. C. Tsui, H. L. Störmer, and A. C. Gossard, Two-Dimensional Magnetotransport in the Extreme Quantum Limit, Phys. Rev. Lett. **48**, 1559 (1982).
[3] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. **303**, 2 (2003); A. Stern and N. H. Lindner, Topological quantum computation from basic concepts to first experiments, Science **339**, 1179 (2013).
[4] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. **82**, 3045 (2010).
[5] X. L. Qi and S. C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. **83**, 1057 (2011).
[6] X.-L. Qi and S.-C. Zhang, The quantum spin Hall effect and topological insulators, Phys. Today **63**(1), 33 (2010).
[7] J. E. Moore, The birth of topological insulators, Nature (London) **464**, 194 (2010).
[8] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X. L. Qi, and S. C. Zhang, Quantum spin Hall insulator state in HgTe quantum wells, Science **318**, 766 (2007).
[9] B. A. Bernevig, T. L. Hughes, and S. C. Zhang, Quantum spin Hall effect and topological phase transition in HgTe quantum wells, Science **314**, 1757 (2006).
[10] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, A topological Dirac insulator in a quantum spin Hall phase, Nature (London) **452**, 970 (2008).
[11] A. Roth, C. Brüne, H. Buhmann, L. W. Molenkamp, J. Maciejko, X. L. Qi, and S. C. Zhang, Nonlocal transport in the quantum spin Hall state, Science **325**, 294 (2009).
[12] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. **80**, 1083 (2008).
[13] A. Hutter, J. R. Wootton, and D. Loss, Parafermions in a Kagome Lattice of Qubits for Topological Quantum Computation, Phys. Rev. X **5**, 041040 (2015).

[14] A. Miyake, Quantum Computation on the edge of a Symmetry-Protected Topological Order, Phys. Rev. Lett. **105**, 040501 (2010).
[15] Z. Y. Xue, M. Gong, J. Liu, Y. Hu, S. L. Zhu, and Z. D. Wang, Robust interface between flying and topological qubits, Sci. Rep. **5**, 12233 (2015).
[16] Z. Y. Xue, L. B. Shao, Y. Hu, S. L. Zhu, and Z. D. Wang, Tunable interfaces for realizing universal quantum computation with topological qubits, Phys. Rev. A **88**, 024303 (2013).
[17] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential, Phys. Rev. Lett. **49**, 405 (1982).
[18] M. Kohmoto, Topological Invariant and the Quantization of the Hall Conductance, Ann. Phys. **160**, 343 (1985).
[19] R. Kubo, Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems, J. Phys. Soc. Jpn. **12**, 570 (1957).
[20] A. Rivas, O. Viyuela, and M. A. Martin-Delgado, Density-matrix Chern insulators: Finite-temperature generalization of topological insulators, Phys. Rev. B **88**, 155141 (2013).
[21] O. Viyuela, A. Rivas, and M. A. Martin-Delgado, Thermal instability of protected end states in a one-dimensional topological insulator, Phys. Rev. B **86**, 155140 (2012).
[22] H. Z. Shen, W. Wang, and X. X. Yi, Hall conductance and topological invariant for open systems, Sci. Rep. **4**, 6455 (2014).
[23] H. Z. Shen, M. Qin, X. Q. Shao, and X. X. Yi, General response formula and application to topological insulator in quantum open system, Phys. Rev. E **92**, 052122 (2015).
[24] J. E. Avron and Z. Kons, Quantum response at finite fields and breakdown of Chern numbers, J. Phys. A: Math. Gen. **32**, 6097 (1999).
[25] J. E. Avron, M. Fraas, G. M. Graf, and O. Kenneth, Quantum response of dephasing open systems, New J. Phys. **13**, 053042 (2011).
[26] J. E. Avron, M. Fraas, and G. M. Graf, Adiabatic response for Lindblad dynamics, J. Stat. Phys. **148**, 800 (2012).
[27] C. Uchiyama, M. Aihara, M. Saeki, and S. Miyashita, Master equation approach to line shape in dissipative systems, Phys. Rev. E **80**, 021128 (2009).
[28] M. Saeki, C. Uchiyama, T. Mori, and S. Miyashita, Comparison among various expressions of complex admittance for quantum systems in contact with a heat reservoir, Phys. Rev. E **81**, 031131 (2010).
[29] A. R. Kolovsky, Hall conductivity beyond the linear response regime, Europhys. Lett. **96**, 50002 (2011).
[30] O. Narayan, Linear response formula for open systems, Phys. Rev. E **83**, 061110 (2011).
[31] L. C. Venuti and P. Zanardi, Dynamical response theory for driven-dissipative quantum systems, Phys. Rev. A **93**, 032101 (2016).
[32] R. Chetrite and K. Mallick, Quantum fluctuation relations for the lindblad master equation, J. Stat. Phys. **148**, 480 (2012).
[33] J. S. Jin, X. Zheng, and Y. J. Yan, Exact dynamics of dissipative electronic systems and quantum transport: Hierarchical equations of motion approach, J. Chem. Phys. **128**, 234703 (2008).
[34] J. H. Wei and Y. J. Yan, Linear response theory for quantum open systems, arXiv:1108.5955.
[35] C. E. Bardyn, M. A. Baranov, C. V. Kraus, E. Rico, A. İmamoğlu, P. Zoller, and S. Diehl, Topology by dissipation, New J. Phys. **15**, 085001 (2013).
[36] J. C. Budich, P. Zoller, and S. Diehl, Dissipative preparation of Chern insulators, Phys. Rev. A **91**, 042117 (2015).
[37] Y. Hu, P. Zoller, and J. C. Budich, Dynamical Buildup of a Quantized Hall Response from Non-Topological States, Phys. Rev. Lett. **117**, 126803 (2016).
[38] Z. C. Shi, H. Z. Shen, W. Wang, and X. X. Yi, Response of two-band systems to a single-mode quantized field, Phys. Rev. E **93**, 032120 (2016).
[39] T. Morimoto, Y. Hatsugai, and H. Aoki, Optical Hall Conductivity in Ordinary and Graphene Quantum Hall Systems, Phys. Rev. Lett. **103**, 116803 (2009).
[40] J. G. Pedersen, M. H. Brynildsen, H. D. Cornean, and T. G. Pedersen, Optical Hall conductivity in bulk and nanostructured graphene beyond the Dirac approximation, Phys. Rev. B **86**, 235438 (2012).
[41] H. Dehghani and A. Mitra, Optical Hall conductivity of a Floquet topological insulator, Phys. Rev. B **92**, 165111 (2015).
[42] T. Morimoto, Y. Hatsugai, and H. Aoki, Optical Hall conductivity in 2DEG and graphene QHE systems, Physica E **42**, 751 (2010).
[43] J. H. Wilson, J. C. W. Song, and G. Refael, Remnant Geometric Hall Response in a Quantum Quench, Phys. Rev. Lett. **117**, 235302 (2016).
[44] F. N. Ünal, E. J. Mueller, and M. Ö. Oktel, Nonequilibrium fractional Hall response after a topological quench, Phys. Rev. A **94**, 053604 (2016).
[45] S. Vajna, B. Horovitz, B. Dóra, and G. Zaránd, Floquet topological phases coupled to environments and the induced photocurrent, Phys. Rev. B **94**, 115145 (2016).
[46] B. Gulácsi and B. Dóra, From Floquet to Dicke: Quantum Spin Hall Insulator Interacting with Quantum Light, Phys. Rev. Lett. **115**, 160402 (2015).
[47] B. Vacchini and H. P. Breuer, Exact master equations for the non-Markovian decay of a qubit, Phys. Rev. A **81**, 042103 (2010).
[48] H. P. Breuer, E. M. Laine, J. Piilo, and B. Vacchini, Colloquium: Non-Markovian dynamics in open quantum systems, Rev. Mod. Phys. **88**, 021002 (2016).
[49] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[50] E. Yablonovitch, Inhibited Spontaneous Emission in Solid-State Physics and Electronics, Phys. Rev. Lett. **58**, 2059 (1987).
[51] S. John and J. Wang, Quantum Electrodynamics Near a Photonic Band Gap: Photon Bound States and Dressed Atoms, Phys. Rev. Lett. **64**, 2418 (1990).
[52] S. Y. Zhu, Y. Yang, H. Chen, H. Zheng, and M. S. Zubairy, Spontaneous Radiation and Lamb Shift in Three-Dimensional Photonic Crystals, Phys. Rev. Lett. **84**, 2136 (2000).
[53] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).
[54] A. Royer, Reduced Dynamics with Initial Correlations, and Time-Dependent Environment and Hamiltonians, Phys. Rev. Lett. **77**, 3272 (1996).
[55] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu, and J. Zwanzige, *The Geometric Phase in Quantum Systems* (Springer-Verlag, Berlin, 2003).
[56] P. Lodahl, A. F. van Driel, I. S. Nikolaev, A. Irman, K. Overgaag, D. Vanmaekelbergh, and W. L. Vos, Controlling the dynamics of spontaneous emission from quantum dots by photonic crystals, Nature (London) **430**, 654 (2004).

[57] S. Noda, M. Fujita, and T. Asano, Spontaneous-emission control by photonic crystals and nanocavities, Nat. Photonics **1**, 449 (2007).

[58] F. Dreisow, A. Szameit, M. Heinrich, T. Pertsch, S. Nolte, A. Tünnermann, and S. Longhi, Decay Control via Discrete-to-Continuum Coupling Modulation in an Optical Waveguide System, Phys. Rev. Lett. **101**, 143602 (2008).

[59] P. Ullersma, An exactly solvable model for Brownian motion: I. Derivation of the Langevin equation, Physica **32**, 27 (1966).

[60] P. Ullersma, An exactly solvable model for Brownian motion: II. Derivation of the Fokker-Planck equation and the master equation, Physica **32**, 56 (1966).

[61] P. Ullersma, An exactly solvable model for Brownian motion: III. Motion of a heavy mass in a linear chain, Physica **32**, 74 (1966).

[62] P. Ullersma, An exactly solvable model for Brownian motion: IV. Susceptibility and Nyquist's theorem, Physica **32**, 90 (1966).

[63] S. John and T. Quang, Spontaneous emission near the edge of a photonic band gap, Phys. Rev. A **50**, 1764 (1994).

[64] S. John, Strong Localization of Photons in Certain Disordered Dielectric Superlattices, Phys. Rev. Lett. **58**, 2486 (1987).

[65] S. John and J. Wang, Quantum optics of localized light in a photonic band gap, Phys. Rev. B **43**, 12772 (1991).

[66] M. Kohmoto, Zero modes and the quantized Hall conductance of the two-dimensional lattice in a magnetic field, Phys. Rev. B **39**, 11943 (1989).

[67] L. Wang, A. A. Soluyanov, and M. Troyer, Proposal for Direct Measurement of Topological Invariants in Optical Lattices, Phys. Rev. Lett. **110**, 166802 (2013).

[68] H. B. Liu, J. H. An, C. Chen, Q. J. Tong, H. G. Luo, and C. H. Oh, Anomalous decoherence in a dissipative two-level system, Phys. Rev. A **87**, 052139 (2013); Q. J. Tong, J. H. An, H. G. Luo, and C. H. Oh, Quantum phase transition in the delocalized regime of the spin-boson model, Phys. Rev. B **84**, 174301 (2011); H. B. Liu, W. L. Yang, J. H. An, and Z. Y. Xu, Mechanism for quantum speedup in open quantum systems, Phys. Rev. A **93**, 020105(R) (2016); P. Zhang, B. You, and L. X. Cen, Long-lived quantum coherence of two-level spontaneous emission models within structured environments, Opt. Lett. **38**, 3650 (2013).

[69] N. H. Lindner, G. Refael, and V. Galitski, Floquet topological insulator in semiconductor quantum wells, Nat. Phys. **7**, 490 (2011).

[70] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Majorana Fermions in Equilibrium and in Driven Cold-Atom Quantum Wires, Phys. Rev. Lett. **106**, 220402 (2011); N. Goldman, I. Satija, P. Nikolic, A. Bermudez, M. A. Martin-Delgado, M. Lewenstein, and I. B. Spielman, Realistic Time-Reversal Invariant Topological Insulators with Neutral Atoms, *ibid.* **105**, 255302 (2010).

[71] Z. Wu, L. Zhang, W. Sun, X. T. Xu, B. Z. Wang, S. C. Ji, Y. Deng, S. Chen, X. J. Liu, and J. W. Pan, Realization of two-dimensional spin-orbit coupling for Bose-Einstein condensates, Science **354**, 83 (2016).

[72] L. Huang, Z. Meng, P. Wang, P. Peng, S. L. Zhang, L. Chen, D. Li, Q. Zhou, and J. Zhang, Experimental realization of two-dimensional synthetic spin-orbit coupling in ultracold Fermi gases, Nat. Phys. **12**, 540 (2016).

[73] M. Büttiker, Role of quantum coherence in series resistors, Phys. Rev. B **33**, 3020 (1986).

[74] J. R. Shi and X. C. Xie, Dephasing and the metal-insulator transition, Phys. Rev. B **63**, 045123 (2001).

[75] H. Jiang, S. G. Cheng, Q. F. Sun, and X. C. Xie, Topological Insulator: A New Quantized Spin Hall Resistance Robust to Dephasing, Phys. Rev. Lett. **103**, 036803 (2009).