Quantum entanglement and temperature fluctuations

Kamel Ourabah[1,*] and Mouloud Tribeche[1,2]
[1]*Theoretical Physics Laboratory, Faculty of Physics, University of Bab-Ezzouar, USTHB, Boîte Postale 32, El Alia, Algiers 16111, Algeria*
[2]*Algerian Academy of Sciences and Technologies, Algiers, Algeria*
(Received 27 September 2016; published 6 April 2017)

In this paper, we consider entanglement in a system out of equilibrium, adopting the viewpoint given by the formalism of superstatistics. Such an approach yields a good effective description for a system in a slowly fluctuating environment within a weak interaction between the system and the environment. For this purpose, we introduce an alternative version of the formalism within a quantum mechanical picture and use it to study entanglement in the Heisenberg XY model, subject to temperature fluctuations. We consider both isotropic and anisotropic cases and explore the effect of different temperature fluctuations (χ^2, log-normal, and F distributions). Our results suggest that particular fluctuations may enhance entanglement and prevent it from vanishing at higher temperatures than those predicted for the same system at thermal equilibrium.

DOI: 10.1103/PhysRevE.95.042111

I. INTRODUCTION

Connections between information theory and statistical mechanics have been recognized over more than 20 years: Information theory provides rigorous justifications for statistical mechanics, and techniques developed for statistical mechanics are used broadly in information theory. Furthermore, both fields share, as a centerpiece, important concepts, such as *entropy*. Interestingly, this connection is also important in quantum information theory where ideas of statistical mechanics are applied in studying quantum information processes and finding new quantum algorithms [1]. It is then quite natural that quantum information research has incorporated some recently introduced formalisms and ongoing developments of statistical mechanics, such as nonextensive statistical mechanics and generalized entropies [2–8]. The relationship between quantum information theory and statistical physics becomes even more inevitable in studying quantum information processes, such as entanglement, in a realistic system at finite temperatures, i.e., *thermal entanglement* [9–11]. For this purpose, entanglement usually is studied between quantum states at thermal equilibrium (*thermal states*). Entanglement of thermal states is rather well understood. In particular, it is well known that entanglement vanishes at a critical temperature, exhibiting therefore a phase transition [10]. However, thermal equilibrium is a property of closed systems, which is not shared by open systems. As a step towards more realism, entanglement in a system out of equilibrium, such as a system coupled to two different temperature heat reservoirs, recently has attracted a great deal of interest [12,13]. In such cases, one has to go beyond equilibrium statistical mechanics and explore statistical mechanics methods inherent to states out of equilibrium. It is then natural to expect another bridge between nonequilibrium quantum information processes and statistical mechanics methods to approach nonequilibrium systems.

Statistical mechanics behind a system at thermal equilibrium is quite simple in the sense that the only parameters required to describe the state of the system are its Hamiltonian and the temperature of its environment. The statistics of such a system is then given by the Boltzmann form $\exp(-\beta E)$—β being the inverse temperature—stating that the probability of a particular energy state decreases exponentially with the energy. When dealing with a system which is not at thermal equilibrium, the task is not that simple. For such a system, *in principle*, the entire past history of perturbations that the system has undergone is required to describe its state. Calculating the steady state of a system driven out of equilibrium is a highly nontrivial and actual matter [14–16], and many approaches have been proposed first for classical systems driven out of equilibrium and later for quantum systems. Among those approaches, renormalization group and density matrix renormalization group methods [17–19], real-time methods [17,18], and field theoretic approaches [20] have been considered. Nevertheless, to date, we have had very few explicit calculations of nonequilibrium properties of open many body quantum systems, and the study of a nonequilibrium open system remains an open field. To overcome the difficulty in dealing with a nonequilibrium system in a steady state, some methods have been introduced that yield an effective description of the system under some assumptions. The goal is to describe a nonequilibrium system provided only a few extra parameters over those required to describe the same system at equilibrium. A possible way to do so is to characterize the system by a superposition of several statistics at different time scales. For instance and for our purposes, to superimpose on the statistics of an equilibrium system a distribution that describes variations of the environment, such as temperature fluctuations (or equivalently, fluctuations of $\beta = 1/T$). Then, the long-term stationary state consists of a superposition of Boltzmann distributions $\exp(-\beta E)$ that are weighted with a probability density $f(\beta)$ to observe a certain β. Such an approach usually is referred to as superstatistics [21] since it consists of a superposition of statistics. It also is related to the concept of hyperensembles introduced by Crooks [22] in which rather than a statistical ensemble, the nonequilibrium system is described by means of a hyperensemble, i.e., *an ensemble of ensembles.* Such an approach yields a good effective description for a system in a slowly fluctuating environment and is a suitable approximation for a continuously varying temperature field that has a temporal correlation length T, much larger than the relaxation time [21].

*Corresponding author: kam.ourabah@gmail.com

So far, the formalism of superstatistics has been developed classically and has known many improvements and mathematical refinements [23,24]. Also, it has been shown to be a useful tool in understanding a range of processes in different nonequilibrium complex systems (see Ref. [25] and references therein). Superstatistics turns out to be an appropriate tool in dealing with systems in a nonequilibrium stationary state thanks to the ability of the formalism to extract analytical results and to generate distributions that exhibit non-Gaussian behavior, such as power laws and stretched exponentials, as observed in numerous systems [21,26,27]. An instructive example is the case of Tsallis distributions that can be generated within χ^2-distributed inverse temperatures [21,26]. Note that such statistics, particularly Tsallis-type statistics, has attracted a great deal of interest in recent literature, being observed in numerous situations, such as distribution of cold atoms in dissipative optical lattices [28], spin-glass relaxation [29], or high energy collisional experiments [30]. Then, quite naturally, some effort has been made to understand the implication of such statistics in quantum information processes as entanglement, their interpretation being based upon nonextensive statistical mechanics, that yields a one-parameter generalization of usual thermal states. However, the physical meaning of this parameter is still somehow obscure although believed to be related to long-ranged interactions, (multi)fractal structures, or nonergodicity. In the present paper, we aim to adopt the alternative point of view given by superstatistics that links this parameter to temperature fluctuations. This approach presents the advantage to link the deformation of the usual thermal states to an empirical fact, namely, temperature fluctuations, which opens the door to an experimental verification. Also, it yields statistics that slightly differ from Tsallis statistics for which experimental evidence can be found in the literature [31].

The aim of this paper is to revisit the formalism of superstatistics within a quantum mechanical picture, having in mind its application to quantum information processes out of equilibrium. In the quantum regime, the quantum probability model takes place in a Hilbert space H of a finite or infinite dimension, and a state is represented by a positive semidefinite linear mapping (a matrix ρ) from this space into itself with a trace of unity. The superstatistical density matrix is then given by a superposition of the usual thermal states weighted with a probability density that describes temperature fluctuations of the environment. It is admitted that, typically, the interaction with the surroundings destroys quantum correlations in the system. However, some exceptions exist: In some situations, interactions with the environment can create extra quantum correlations [32,33]. We are particularly interested here in the effect that may have different environmental temperature fluctuations on entanglement. To ensure the validity of our approach, we assume first that there is sufficient time scale separation between the time scale of the fluctuation and the relaxation time: The temperature changes on a long time scale T, much larger than the relaxation time. We also assume that the interaction between the system and its environment is weak to avoid nontrivial effects, such as the suppression of quantum coherences in the system-bath interaction basis.

The paper is fashioned as follows: In the next section, we will discuss the formalism of superstatistics in a full quantum regime. In Sec. III, we will consider entanglement of thermal superstates, i.e., states that can be described as a superposition of thermal states in the case of the Heisenberg XY model. We will consider both isotropic and anisotropic cases and explore different types of superstatistics. In Sec. IV, we will show, in the particular case of the Heisenberg model, how to map superstatistical states onto a usual thermal state by considering effective temperature-dependent energy states. In the last section, we will discuss our results and present some outlooks.

II. QUANTUM SUPERSTATISTICS

Consider a quantum system in thermal equilibrium within a temperature T (inverse temperature β). The state of such a system is given by

$$\rho = \frac{e^{-\beta H}}{Z} = \frac{1}{Z}\sum_i e^{-\beta E_i}|\psi_i\rangle\langle\psi_i|, \qquad (1)$$

where the partition function Z plays a normalization role and reads as $Z = \sum_i e^{-\beta E_i}$. Such a state is called a thermal state. Note that ρ is a semidefinite matrix from Hilbert space to itself with $\mathrm{Tr}\,\rho = 1$. Let us now go a step further and consider that the temperature, or equivalently its inverse β, fluctuates according to a distribution function $f(\beta)$. Then, one has to superimpose on the thermal state (1) the distribution $f(\beta)$ that describes fluctuation of the surroundings. The state then is given by

$$\tilde{\rho} = \frac{1}{\tilde{Z}}\int_0^\infty d\beta\, f(\beta)e^{-\beta H} = \frac{1}{\tilde{Z}}\sum_i \tilde{B}(E_i)|\psi_i\rangle\langle\psi_i|, \qquad (2)$$

where $\tilde{B}(E_i)$ defines an effective Boltzmann factor that reads

$$\tilde{B}(E) = \int_0^\infty d\beta\, f(\beta)e^{-\beta E}. \qquad (3)$$

$\tilde{Z}$ is a generalized partition function that plays the same normalization role and reads as $\tilde{Z} = \sum_i \tilde{B}(E_i)$. Indeed, we have adopted here superstatistics of type A since the partition function does not appear in the integral (2) but has been added up for normalization purposes (see Ref. [21] for a discussion on type-A and type-B superstatistics). Since for our purposes, we are mainly interested in lattice systems, $f(\beta)$ would be thought of as describing fluctuations in time, i.e., β is uniform through the whole phase space but varies in time. In principle, $f(\beta)$ could be any normalized distribution such that the integral (3) is normalizable. One also can impose on $f(\beta)$ to reduce to the Dirac distribution $\delta(\beta-\beta_0)$ in which case the effective statistics (3) reduces to the usual statistics, and $\tilde{\rho}$ reduces to ρ. Note that, in the quantum regime, we assume that, as the usual thermal state, $\tilde{\rho}$ must be Hermitian ($\tilde{\rho} = \tilde{\rho}^{+}$) and positive semidefinite, i.e, $\forall\, |\psi\rangle \subset H$, $\langle\psi|\tilde{\rho}|\psi\rangle \geqslant 0$.

Among the distributions that satisfy the above properties, the χ^2 distribution (also called the Γ distribution) that appears in many natural systems. The latter reads as

$$f(\beta) = \frac{1}{b\Gamma(c)}\left(\frac{\beta}{b}\right)^{c-1} e^{-\beta/b}, \qquad (4)$$

and leads to an effective Boltzmann factor that coincides with the Tsallis statistics,

$$\tilde{B}(E) = \int_0^\infty d\beta\, f(\beta) e^{-\beta E} = e_q(-\beta E), \tag{5}$$

where

$$e_q(x) = [1 + (1-q)x]^{1/(1-q)}. \tag{6}$$

$\beta_0 = bc$ is the average inverse temperature $\beta_0 = 1/T_0$ and $c \equiv 1/(q-1)$. One also may consider the log-normal distribution,

$$f(\beta) = \frac{1}{\beta s\sqrt{2\pi}} \exp\left\{\frac{-[\log(\beta/m)]^2}{2s^2}\right\}, \tag{7}$$

or the F distribution,

$$f(\beta) = \frac{\Gamma[(v+w)/2]}{\Gamma(v/2)\Gamma(w/2)}\left(\frac{bv}{w}\right)^{v/2} \frac{\beta^{(v/2)-1}}{[1+(vb/w)\beta]^{(v+w)/2}}. \tag{8}$$

The two latter distributions are also ubiquitous in many physical situations. However, in contrast to the χ^2 distributions, the log-normal and F distributions do not lead to analytical results, but the statistics can be obtained numerically or estimated analytically for a sufficiently small energy [21]. As first noticed by Beck [25], by considering a generalized Hamiltonian, one can map superstatistics onto a usual statistics. In fact, suppose that one indexes the inverse temperatures by β_j, one can consider an exotic Hamiltonian $\bar{H}$ such that

$$\beta_j H = \beta \bar{H}. \tag{9}$$

Since ordinary statistical mechanics is valid for arbitrary energy levels, and so for the super-Hamiltonian $\bar{H}$, one may now perform ordinary statistical mechanics for the super-Hamiltonian $\bar{H}$, and the partition function of superstatistics can be written as

$$\tilde{Z} = \sum_j (\mathrm{Tr}\, e^{-\beta \bar{H}^{(j)}}) \simeq \int_0^\infty d\beta\, f(\beta) \mathrm{Tr}\, e^{-\beta H}, \tag{10}$$

which preserves the same form of the usual partition function within indexed energy levels $\bar{H}^{(j)}$. If the temperature is supposed to vary slowly, the sum over the temperature index j can be replaced by an integral, and the superstatistical definition of the partition function is recovered

$$\tilde{Z} \simeq \int_0^\infty d\beta\, f(\beta) \mathrm{Tr}\, e^{-\beta H} = \int_0^\infty d\beta\, f(\beta) Z. \tag{11}$$

III. ENTANGLEMENT OF THERMAL SUPERSTATES

In this section, we will study entanglement of states of the form (2) to shed light on the effect of temperature fluctuations on entanglement. We will adopt the concurrence as a measure of entanglement. Let us consider a pair of qubits 1 and 2 within a density matrix ρ_{12} that can be either pure or mixed. The concurrence corresponding to ρ_{12} reads as

$$C_{12} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{12}$$

where the quantities $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ are the square roots of the eigenvalues of the operator,

$$\varrho_{12} = \rho_{12}(\sigma_y \otimes \sigma_y)\rho_{12^*}(\sigma_y \otimes \sigma_y). \tag{13}$$

The concurrence C_{12} varies from 0 to 1. A zero concurrence corresponds to unentangled qubits, and $C_{12} = 1$ corresponds to maximally entangled qubits. We study here thermal entanglement in the case of the XY model.

A. Isotropic XY model

Let us consider the two-qubit isotropic antiferromagnetic XY model in a constant external magnetic field B, described by the following Hamiltonian:

$$H = \frac{B}{2}\left(\sigma_1^z + \sigma_2^z\right) + J(\sigma_1^+\sigma_2^- + \sigma_2^+\sigma_1^-). \tag{14}$$

The eigenvalues and eigenvectors of H are given by

$$H|00\rangle = -B|00\rangle, \quad H|11\rangle = B|11\rangle, \quad H|\Psi^\pm\rangle = \pm J|\Psi^\pm\rangle, \tag{15}$$

where

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \tag{16}$$

are maximally entangled states. In the standard basis,

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}, \tag{17}$$

the superstatistical state (2) is given by

$$\tilde{\rho} = \frac{1}{\tilde{Z}}\begin{pmatrix} \tilde{B}(B) & 0 & 0 & 0 \\ 0 & \tilde{B}(-J)+\tilde{B}(J) & \tilde{B}(J)-\tilde{B}(-J) & 0 \\ 0 & \tilde{B}(J)-\tilde{B}(-J) & \tilde{B}(-J)+\tilde{B}(J) & 0 \\ 0 & 0 & 0 & \tilde{B}(-B) \end{pmatrix}, \tag{18}$$

where

$$\tilde{Z} = \tilde{B}(-B) + \tilde{B}(B) + \tilde{B}(-J) + \tilde{B}(J) \tag{19}$$

is the superstatistical partition function. From (18), one obtains the following concurrence:

$$\tilde{C} = \max\left\{\frac{\tilde{B}(-J) + \tilde{B}(J) - 2}{\tilde{B}(-B) + \tilde{B}(B) + \tilde{B}(-J) + \tilde{B}(J)}, 0\right\}. \tag{20}$$

The latter is a function of J and B, just as the usual concurrence of the XY model but also is affected by temperature fluctuations since the form of $\tilde{B}(x)$ is determined by the distribution $f(\beta)$ throughout Eq. (3). A different $f(\beta)$ leads to a different concurrence. For instance, if $f(\beta)$ corresponds to a χ^2 distribution, the concurrence has an analytical form that reads as[1]

$$\tilde{C}_q = \max\left\{\frac{\sinh_q(J/T) - 1}{\cosh_q(J/T) + \cosh_q(B/T)}, 0\right\}, \tag{21}$$

[1]Here, we present a nonextensive statistical mechanics approach to the XY model. However, it is trivial that such an approach leads to the concurrence (21) since it uses distributions given by χ^2 superstatistics (see Ref. [35] for such an approach in the case of the two-site Hubbard model).

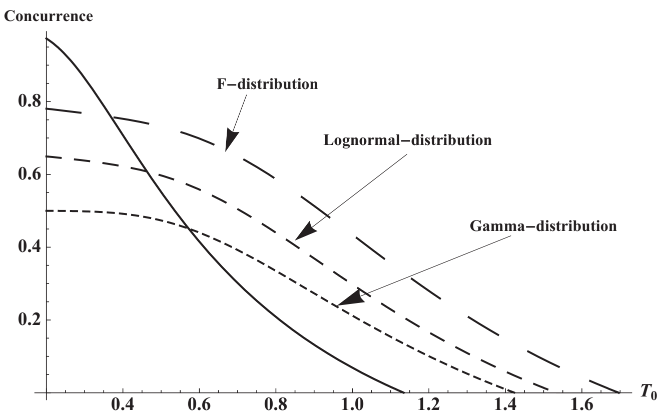

FIG. 1. Plot of the concurrence (20) with the average temperature T_0 for different superstatistics with B and J set equal to 1, and $q = \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2} = 2$ for all superstatistics. One easily can see that the critical temperature is sensitive on the fluctuations.

where

$$\cosh_q(x) = \frac{e_q(x) + e_q(x)}{2} \quad \text{and}$$
$$\sinh_q(x) = \frac{e_q(x) - e_q(-x)}{2} \tag{22}$$

are a generalization of the hyperbolic functions based upon the Tsallis q exponential [34]. Other distributions (particularly the log-normal and F distributions) lead to a concurrence that can be calculated numerically or estimated in a limit of small energies. The concurrence (20) is a generalization of the concurrence of the XY model that takes into account the effect of a fluctuating temperature. In the limit of there are no fluctuations at all, i.e., $f(\beta) = \delta(\beta - \beta_0)$, the concurrence (20) reduces to the usual concurrence of the XY model [10],

$$C = \max\left\{\frac{\sinh(J/T) - 1}{\cosh(J/T) + \cosh(B/T)}, 0\right\}. \tag{23}$$

Figure 1 shows the variation of the concurrence (20) with the average temperature $T_0 = 1/\beta_0$ for different superstatistics with $q = \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2} = 2$. B and J are set equal to 1. The figure shows that the concurrence is very sensitive to the temperature fluctuations. Especially, the critical temperature beyond which entanglement is not allowed (the temperature value for which $\tilde{C} = 0$) is highly sensitive to the type of fluctuation. In the absence of fluctuations, the latter can be estimated at $T_c \simeq 1.13$ (see the figure). For different fluctuations, one has $T_c \simeq 1.42$ (χ^2 distribution), $T_c \simeq 1.52$ (log-normal distribution), and $T_c \simeq 1.70$ (F distribution). In fact, in the absence of fluctuations, the concurrence (23) vanishes if $\sinh(J/T) \leqslant 1$, and the critical temperature is given by

$$T_c = \frac{J}{\text{arcsin } h(1)} \simeq 1.1346J. \tag{24}$$

In the more general case in which fluctuations exist, one has to solve the equation $\tilde{B}(-J) + \tilde{B}(J) = 2$.

B. Anisotropic XY model

Consider the two-qubit anisotropic antiferromagnetic XY model, described by the Hamiltonian,

$$H_a = \frac{J}{2}\left[(1+\gamma)\sigma_1^x\sigma_2^x + (1-\gamma)\sigma_1^y\sigma_2^y\right]$$
$$= J(\sigma_1^+\sigma_2^- + \sigma_2^+\sigma_1^-) + J_\gamma(\sigma_1^+\sigma_2^+ + \sigma_2^-\sigma_1^-), \tag{25}$$

where γ is the anisotropic parameter, i.e, $\gamma = 0$ corresponds to the isotropic XY model without a magnetic field and $\gamma = 1$ corresponds to the Ising model. The eigenvalues and eigenvectors of H_a are given by

$$H_a|\Psi^\pm\rangle = \pm J|\Psi^\pm\rangle, \quad H_a|\Phi^\pm\rangle = \pm J_\gamma|\Phi^\pm\rangle, \tag{26}$$

where

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle). \tag{27}$$

It turns out that the four maximally entangled Bell states are the eigenstates of H_a. The superstatistical state (2) for the Hamiltonian H_a is given by

$$\tilde{\rho} = \frac{1}{\tilde{Z}}\begin{pmatrix} \tilde{B}(-J_\gamma) + \tilde{B}(J_\gamma) & 0 & 0 & \tilde{B}(J_\gamma) - \tilde{B}(-J_\gamma) \\ 0 & \tilde{B}(-J) + \tilde{B}(J) & \tilde{B}(J) - \tilde{B}(-J) & 0 \\ 0 & \tilde{B}(J) - \tilde{B}(-J) & \tilde{B}(-J) + \tilde{B}(J) & 0 \\ 0 & 0 & 0 & \tilde{B}(-J_\gamma) + \tilde{B}(J_\gamma) \end{pmatrix}, \tag{28}$$

where

$$\tilde{Z} = \tilde{B}(-J) + \tilde{B}(J) + \tilde{B}(-J_\gamma) + \tilde{B}(J_\gamma) \tag{29}$$

is the superstatistical partition function. The square roots of the eigenvalues of the operator ϱ_{12} are

$$\tilde{B}(\pm J)/\tilde{Z} \quad \text{and } \tilde{B}(\pm J_\gamma)/\tilde{Z}, \tag{30}$$

and the concurrence (14) reads as

$$\tilde{C} = \max\left\{\frac{\tilde{B}(-J) - \tilde{B}(J) - \tilde{B}(-J_\gamma) - \tilde{B}(J_\gamma)}{\tilde{B}(-J) + \tilde{B}(J) + \tilde{B}(-J_\gamma) + \tilde{B}(J_\gamma)}, 0\right\}. \tag{31}$$

The latter is a generalization of the concurrence obtained by Wang [10] in the case of the anisotropic XY model that takes into account temperature fluctuations. As in the previous case of the isotropic model, the concurrence (31) has an analytic expression in the case of χ^2 superstatistics that reads

$$\tilde{C}_q = \max\left\{\frac{\sinh_q(J/T) - \cosh_q(J_\gamma/T)}{\cosh_q(J/T) + \cosh_q(J_\gamma/T)}, 0\right\}, \tag{32}$$

where $\sinh_q(x)$ and $\cosh_q(x)$ are defined in Eq. (22). In fact, Eq. (31) reduces to the concurrence obtained in the isotropic case (20) for $B = 0$ when $\gamma = 0$. The critical temperature T_c

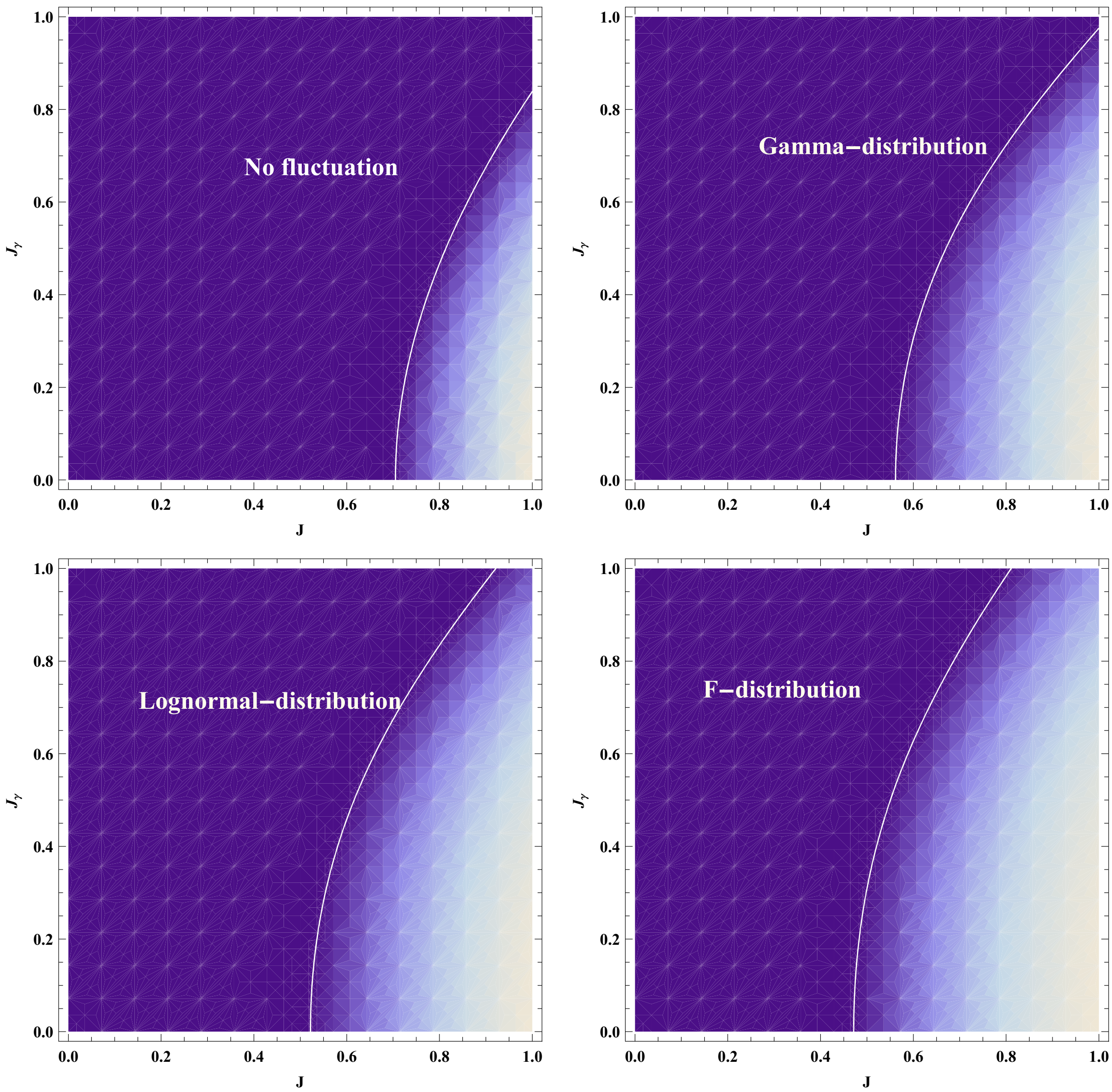

FIG. 2. Plot of the concurrence (31) against J and J_γ with $T_0 = 0.8$ and $q = \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2} = 2$ for all superstatistics. The white region is the region where entanglement is allowed. One easily can see that this region gets bigger for the considered temperature fluctuations.

is determined by the nonlinear equation,

$$\tilde{B}(-J) - \tilde{B}(J) - \tilde{B}(-J_\gamma) - \tilde{B}(J_\gamma) = 0. \tag{33}$$

Figure 2 shows the variation of the concurrence (31) with J and J_γ with an average temperature $T_0 = 0.8$ and $q = \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2} = 2$ for different superstatistics. One easily can see that the region where entanglement is allowed (the white region) gets bigger for all superstatistics within the chosen parameters.

IV. MAPPING SUPERSTATES ONTO A USUAL THERMAL STATE

Let us now illustrate the procedure presented in Sec. II in the particular case of the isotropic XY model to show how the thermal superstate (18) can be mapped onto a usual thermal state. We may formally consider a super-Hamiltonian describing the entire system which at different instants has effective energy levels $\bar{E}_i$ simply by rewriting

$$\beta_0 \bar{H} = \beta_j H, \tag{34}$$

that leads to deformed energy levels which in this case correspond to varying magnetic field B and interaction J,

$$\bar{B}^{(j)} = \frac{\beta_j B}{\beta_0}, \quad \bar{J}^{(j)} = \frac{\beta_j J}{\beta_0}. \tag{35}$$

Then, since in this alternative view the temperature is no longer a stochastic variable, one may perform ordinary statistical mechanics with the deformed eigenvalues (35). The partition function of the superthermal state can be written as

$$\tilde{Z} = \sum_j \{\exp(\beta_0 \bar{B}^{(j)}) + \exp(-\beta_0 \bar{B}^{(j)}) + \exp(\beta_0 \bar{J}^{(j)}) + \exp(-\beta_0 \bar{J}^{(j)})\} \tag{36}$$

$$= \frac{1}{2} \sum_j \{\cosh(\beta_0 \bar{B}^{(j)}) + \cosh(\beta_0 \bar{J}^{(j)})\}, \tag{37}$$

which, abstracting from the sum over j, has the form of the usual partition function [10], and the thermal superstate reads as

$$\tilde{\rho} = \frac{1}{\tilde{Z}}\begin{pmatrix} \sum_j \exp(-\beta_0 \bar{B}^{(j)}) & 0 & 0 & 0 \\ 0 & \sum_j \cosh(\beta_0 \bar{J}^{(j)}) & -\sum_j \sinh(\beta_0 \bar{J}^{(j)}) & 0 \\ 0 & -\sum_j \sinh(\beta_0 \bar{J}^{(j)}) & \sum_j \cosh(\beta_0 \bar{J}^{(j)}) & 0 \\ 0 & 0 & 0 & \sum_j \exp(\beta_0 \bar{B}^{(j)}) \end{pmatrix}, \tag{38}$$

which leads to the concurrence,

$$\tilde{C} = \max\left\{\frac{\sum_j \sinh(\bar{J}^{(j)}/T) - 1}{\sum_j [\cosh(\bar{J}^{(j)}/T) + \cosh(\bar{B}^{(j)}/T)]}, 0\right\}. \tag{39}$$

The latter has the form of the usual concurrence of the XY model [10]. It reduces to the concurrence (20), obtained through the superstatistical approach, when the sum over j is approximated by an integral,

$$\sum_j \cdots \to \int_0^\infty d\beta\, f(\beta) \cdots . \tag{40}$$

V. CONCLUSION

In this paper, we have considered the formalism of superstatistics as a possible tool to study entanglement in a system out of equilibrium. Such an approach is expected to be a suitable approximation for a continuously varying temperature field that has a temporal correlation length, much larger than the relaxation time, providing a weak interaction between the system and the bath. Within these assumptions, the state of the system can be described appropriately by a mixture of different statistics. We have defined thermal superstates as thermal states subject to a slowly varying temperature and studied entanglement of such states in the case of the XY model. Three types of temperature fluctuations, ubiquitous in nature, have been considered, namely, χ^2 (Γ), log-normal, and F distributions. In the case of χ^2 superstatistics, an analytical expression for the concurrence has been derived that coincides with the results of nonextensive statistical mechanics. Note however that the picture given by superstatistics is different since the deformation of the concurrence is based upon an empirical fact, namely, temperature fluctuations. Our results suggest that entanglement (measured here by the concurrence) can be enhanced by particular temperature fluctuations and can be prevented from vanishing at higher temperatures than those predicted for the usual thermal states. This is in agreement with numerous studies dealing with quantum correlations in small multipartite open systems using different approaches, such as non-Markovian master equations [36], path-integral methods [37], or quantum Langevin equations [38–40], which suggest that entanglement can be maintained at higher average temperatures that would be possible if the system were at thermal equilibrium. It suggests therefore that a superstatistical approach is a good approximation and provides a good effective description of such systems, despite its simplicity. In fact, the superstatistical approach is completely different than the aforementioned approaches since the state of the system is not obtained through a microscopic study. Superstatistics gives rather a picture of the nonequilibrium state provided only a few extra parameters over those required to describe the same system at equilibrium. The superstatistical approach to thermal entanglement is used here, and thus, many outlooks can be addressed. Superstatistics, in its quantum picture as presented in this paper, can be useful to study diverse nonequilibrium systems in a full quantum regime, and comparison with different (more sophisticated) approaches or with experimental data can elucidate the complete potential and domain of the validity of this approach. Regarding entanglement, one may consider different models (the Ising model within a magnetic field, the Hubbard model, . . . , etc.) and other measures of entanglement (mutual information, entanglement of formation, . . . , etc.) and explore the effect of temperature fluctuations over those studied in this paper. A superstatistical approach to the process of teleportation, subject to temperature fluctuations, seems also of some interest and will be addressed in a future paper.

[1] *Interface between Quantum Computing and Statistical Physics*, edited by M. Nakahara and S. Tanaka, Kinki University Series on Quantum Computing Vol. 7 (World Scientific, Singapore, 2011).
[2] C. Tsallis, S. Lloyd, and M. Baranger, Phys. Rev. A **63**, 042104 (2001).
[3] J. S. Kim, Phys. Rev. A **81**, 062328 (2010).
[4] P. Sadeghi, S. Khademi, and A. H. Darooneh, Phys. Rev. A **86**, 012119 (2012).
[5] A. K. Rajagopal, Sudha, A. S. Nayak, and A. R. Usha Devi, Phys. Rev. A **89**, 012331 (2014).
[6] K. Ourabah, A. H. Hamici-Bendimerad, and M. Tribeche, Phys. Scr. **90**, 045101 (2015).
[7] K. Ourabah, A. H. Hamici-Bendimerad, and M. Tribeche, Phys. Rev. E **92**, 032114 (2015).
[8] A. E. Rastegin, Phys. Rev. A **93**, 032136 (2016).
[9] M. C. Arnesen, S. Bose, and V. Vedral, Phys. Rev. Lett. **87**, 017901 (2001).
[10] X. G. Wang, Phys. Rev. A **64**, 012313 (2001).
[11] G. L. Kamta and A. F. Starace, Phys. Rev. Lett. **88**, 107901 (2002).
[12] L. Quiroga, F. J. Rodríguez, M. E. Ramírez, and R. París, Phys. Rev. A **75**, 032308 (2007).
[13] N. Pumulo, I. Sinayskiy, and F. Petruccione, Phys. Lett. A **375**, 3157 (2011).
[14] J.-T. Hsiang and B. L. Hu, Ann. Phys. (NY) **362**, 139 (2015).

[15] V. Semin and F. Petruccione, Phys. Rev. A **90**, 052112 (2014).
[16] A. Purkayastha, A. Dhar, and M. Kulkarni, Phys. Rev. A **93**, 062114 (2016).
[17] F. B. Anders and A. Schiller, Phys. Rev. Lett. **95**, 196801 (2005).
[18] N. Shah and A. Rosch, Phys. Rev. B **73**, 081309(R) (2006).
[19] G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003).
[20] A. Mitra, I. Aleiner, and A. J. Millis, Phys. Rev. Lett. **94**, 076404 (2005).
[21] C. Beck and E. G. D. Cohen, Physica A **322**, 267 (2003).
[22] G. E. Crooks, Phys. Rev. E **75**, 041119 (2007).
[23] S. Abe, C. Beck, and E. G. D. Cohen, Phys. Rev. E **76**, 031102 (2007).
[24] D. N. Sob'yanin, Phys. Rev. E **84**, 051128 (2011).
[25] C. Beck, Philos. Trans. R. Soc., A **369**, 453 (2011).
[26] G. Wilk and Z. Włodarczyk, Phys. Rev. Lett. **84**, 2770 (2000).
[27] K. Ourabah, L. Ait Gougam, and M. Tribeche, Phys. Rev. E **91**, 012133 (2015).
[28] P. Douglas, S. Bergamini, and F. Renzoni, Phys. Rev. Lett. **96**, 110601 (2006).
[29] R. M. Pickup, R. Cywinski, C. Pappas, B. Farago, and P. Fouquet, Phys. Rev. Lett. **102**, 097202 (2009).
[30] V. Khachatryan *et al.* (CMS Collaboration), Phys. Rev. Lett. **105**, 022002 (2010).
[31] F. Sattin and L. Salasnich, Phys. Rev. E **65**, 035106(R) (2002).
[32] M. B. Plenio and S. F. Huelga, Phys. Rev. Lett. **88**, 197901 (2002).
[33] S. Diehl *et al.*, Nat. Phys. **4**, 878 (2008).
[34] E. P. Borges, J. Phys. A **31**, 5281 (1998).
[35] H. Hasegawa, Physica A **351**, 273 (2005).
[36] R. Vasile, P. Giorda, S. Olivares, M. G. A. Paris, and S. Maniscalco, Phys. Rev. A **82**, 012313 (2010).
[37] F. Galve, L. A. Pachon, and D. Zueco, Phys. Rev. Lett. **105**, 180501 (2010).
[38] L. A. Correa, A. A. Valido, and D. Alonso, Phys. Rev. A **86**, 012110 (2012).
[39] A. A. Valido, A. Ruiz, and D. Alonso, Phys. Rev. E **91**, 062123 (2015).
[40] M. Ludwig, K. Hammerer, and F. Marquardt, Phys. Rev. A **82**, 012333 (2010).