Effect of node attributes on the temporal dynamics of network structure

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Many natural and social networks evolve in time and their structures are dynamic. In most networks, nodes are heterogeneous, and their roles in the evolution of structure differ. This paper focuses on the role of individual attributes on the temporal dynamics of network structure. We focus on a basic model for growing networks that incorporates node attributes (which we call "quality"), and we focus on the problem of forecasting the structural properties of the network in arbitrary times for an arbitrary initial network. That is, we address the following question: If we are given a certain initial network with given arbitrary structure and known node attributes, then how does the structure change in time as new nodes with given distribution of attributes join the network? We solve the model analytically and obtain the quality-degree joint distribution and degree correlations. We characterize the role of individual attributes in the position of individual nodes in the hierarchy of connections. We confirm the theoretical findings with Monte Carlo simulations.

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I. INTRODUCTION

Temporal dynamics of network structure is a flourishing new direction in network science studies. Most studies on temporal networks consider the microdynamics of node and link activation during interactions between nodes and seek to characterize these dynamics and extract their effects on the aggregated properties of networks [1]. These studies are temporally refined, in the sense that their temporal resolution is high enough to capture interaction patterns (bursty dynamics, etc.). Perhaps an opposite approach to this would be to decrease temporal resolution to the maximum and focus on the steady-state properties of networks. That is, acknowledging that most real networks evolve and grow, one can devise models to characterize the growth process but then analyze the model to extract the properties of the network in the long run, that is, the limit as $t \to \infty$. This approach was initiated by the seminal Barabási-Albert (hereinafter BA) model [2,3], reviving the model proposed in [4]. Our approach is more complete in the sense that we consider arbitrary times and arbitrary initial conditions in the network-growth process, but we consider an intermediate time resolution in the sense that we neglect the bursty nature of individual links. In other words, the networks we consider are relational networks (such as networks of citations, scientific collaborations, website links, etc.), rather than networks of interaction (such as conversation, exchange, etc.).

Several growth models were proposed subsequent to the BA model. The commonality of these models is that they ascribe a measurable macro attribute to underlying micromechanisms that are hypothesized to drive the growth process, for example, power-law degree distributions, reported in studies on diverse networks [5–11] (also see [12–15] which follow the maximum likelihood approach prescribed in [16]). The BA model was an example of the growth models devised in order to emulate this trait in the long-time limit. This model hypothesizes that every existing node has a probability of receiving links from subsequent nodes that are proportional to its degree. Examples of other growth models include models with edge growth [11,17,18], aging effects [19–22], node deletion [23–26], accelerated growth [5,27–31], and copying [32–35]. These models are all purely structural; that is, the connectivity of new nodes is determined by factors that depend on time or structural measures of the network.

In most real networks, nodes possess attributes other than degree that influence their connectivity. For example, in social networks, people have heterogeneous levels of clout and social skills. In online social networks, users share different qualities of content. In citation networks, some papers are more novel that others. In all these examples, there are qualities that individual nodes possess which determine their further connectivity. To emulate this heterogeneity, there exist models that endow nodes with additional attributes that contribute to the likelihood of link reception of nodes. The Bianconi-Barabási model [36] was the first example, which envisages a multiplicative *fitness* for each node, that is, the probability that each existing node receiving a link from subsequent nodes is proportional to the product of its degree and its quality, which is drawn from a given distribution. The possibility of node deletion in quality-based network growth is envisaged in [37,38]. In [39], a purely quality-based model is devised, where the link reception probability of a node depends solely on its quality values and not on degree. The advantage of models that do incorporate node attributes is that they are capable of capturing the phenomenon that does occur in reality: newcomers do occasionally attract many links and become hubs. This means that, although popularity does matter in receiving links (e.g., in the scientific collaboration network or paper citations), quality also has a contribution that can make up for low initial degree. The disadvantage these studies share with the purely structural ones is the restriction to the long-time limits, where the effects of the initial network has already vanished. For example, we would not be able to ask how quickly does a certain newcomer with certain high quality (compared to existing nodes) catch up with them, or how high

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must the quality of a given node be in order to acquire a certain place in the hierarchy of connections in a certain time frame?

In the present paper, we consider a attribute-based network-growth model. We call these attributes "quality," rather than "fitness" (which has evolutionary-dynamics connotations). We endow nodes with intrinsic quality drawn from a given probability distribution over the possible quality values. The quality has context-dependent interpretation. In citation networks it reflects the quality and ubiquity of the idea behind a paper and its novelty. In scientific collaboration networks it can reflect intelligence, versatility, and collegiality of a scholar. In online social networks it can model the quality of the content that a user shares. In social networks it can model personal charisma and clout. In collaboration networks of film actors it can emulate the skill, versatility, and lucrativeness of an actor. On the web, it can model the quality of content and essentiality of provided service.

We consider additive quality as the most basic scenario amenable to analytical treatment. The shortcoming associated with multiplicative quality (that is, when the quality value of a node is multiplied by its degree to yield its chance of link reception) is that nodes with degree zero will not have any chance of link reception. Hence, they will remain isolated. Example indications of such property are as follows: A newcomer in a community will remain secluded no matter how sociable and gregarious, a paper will never get citations regardless of its novelty level, a scholar will never attract any collaborator no matter how collegial and smart, and a web page will never receive a link regardless of content. With additive quality, the values of the quality is added to the degree. This means that, for example, in citation networks, a newcomer can compensate its lack of citations with quality and novelty. Or in a social network, a person that is new to a group and has zero acquaintances will have a chance of befriending people that is determined by the person's sociability.

We assume that an arbitrary initial network (also called the substrate hereinafter) is given in which the degree and quality of every node is known. We find the degree distribution as a function of time. This is the main methodological difference between our approach and those of the above-mentioned network-growth models, which are all analyzed solely in the limit as $t \to \infty$ (also called the steady state, or the thermodynamic limit). In this limit, transient effects have vanished. This would function as a sole approximation for all networks (since no real network has an infinite number of nodes), but this regime would be particularly unrealistic in rapidly growing networks (such as citation networks or scholarly collaboration networks) for which the transients clearly exist. Focusing on the dynamics for arbitrary times and arbitrary initial conditions would enable one to examine these situations more accurately.

We then use the arbitrary-time solution to investigate the steady-state behavior of the model and find that, in this limit, the asymptotic degree distribution is a power law whose exponent depends only on the mean of the quality distribution and on no other property of this distribution. So the model generates scale-free networks with tunable exponent with two degrees of freedom: the number of links that incoming nodes establish and the mean of the quality distribution.

We also show how quality and degree correlate. We find the average degree as a function of quality, and show that it is an increasing function of quality. In other words, when transient effects vanish, the expected degree of a node is an increasing function of its quality.

We then characterize the nearest-neighbor correlations. That is, given the degree and quality of a node, we obtain the joint degree-quality distribution of its neighbors. This illuminates the assortativity properties of the network.

Throughout the paper, results are obtained for general quality distributions. Calculations are generic and no particular quality distribution is assumed.

The rest of the paper is organized as follows. First we introduce the model and delineate model specificities in Sec. II. Then in Sec. III we find the joint degree-quality distribution as a function of time. This solution yields the fraction of nodes that have quality θ and degree k at arbitrary time t. Then in Sec. IV, we focus on the steady-state behavior of the model and show that in the asymptotic, a scale-free network with tunable exponent emerges. Then in Sec. V we demonstrate that the growth mechanism constitutes a hierarchy among nodes, where nodes with higher quality values have higher expected degrees. In Sec. VII we confirm our theoretical predictions with simulations.

II. MODEL

We endow every node in the network with a quality, drawn from a distribution $\rho(\theta)$. The growth process starts at time $t = 0$. At this time, an initial network is given, and the degrees and quality values of all of its nodes are known. New nodes are added to the network successively at rate α. So it takes $\Delta t = \frac{1}{\alpha}$ for each node to be appended to the network. Each node is endowed with its quality value upon birth, drawn from $\rho(\theta)$, and its quality does not change thereafter. Each incoming node attaches to β existing nodes. The probability that an existing node with degree k and quality θ receives a link is proportional to $k + \theta$.

Let us find the attachment probabilities at time t. We have asserted that the probability that a degree-k node whose quality is θ receives a link is proportional to $k + \theta$. Thus, we need to divide $k + \theta$ by the sum of this value over all existing nodes of the network to attain a probability. Let k_x be the degree of node x, and let θ_x denote its quality. The normalization constant is

$$D \overset{\text{def}}{=} \sum_x (k_x + \theta_x) = \sum_x k_x + \sum_x \theta_x. \tag{1}$$

The first term is the some over all the degrees, which is twice the number of links. Let $L(0)$, $\overline{k}$, and $N(0)$ denote the number of links, average degree, and number of nodes in the initial network. Then the number of links in the network at time t is $L(0) + \alpha\beta t$, and the sum of the degrees is $2L(0) + 2\alpha\beta t$, or, equivalently, $N(0)\overline{k} + 2\alpha\beta t$. For the second sum in (1), we have to separate the initial nodes and the subsequent nodes. Let $\overline{\theta}$ be the average of the quality of the nodes that constitute the initial network; thus, $N(0)\overline{\theta}$ will be the sum of their quality values. For the subsequent nodes, since we are considering only expected values, we will take the expected value of the sum of the quality values. Let μ denote the mean of the quality distribution $\rho(\theta)$, so the expected sum of the quality values of the subsequent nodes up to time t will be μt, and the total quality value for all the nodes in the network will be $N(0)\overline{k} + N(0)\overline{\theta} + (2\beta + \mu)\alpha t$. Let us define

$$\lambda \stackrel{\text{def}}{=} N(0)\overline{k} + N(0)\overline{\theta}, \tag{2}$$

$$\nu \stackrel{\text{def}}{=} \beta + \frac{\mu}{2}. \tag{3}$$

Using these definitions, the probability that node x whose degree at time t is degree k_x receives a link from the incoming node is equal to $\frac{k_x}{\lambda+2\nu\alpha t}$. That is, for each of the links that the newcomer establishes, this is the probability that node x will receive that link. If we consider all β links that the incoming node establishes, then the expected number of links that node x receives equals $\frac{\beta k_x}{\lambda+2\nu\alpha t}$.

III. THE JOINT DISTRIBUTION OF QUALITY AND DEGREE

Let $N_t(k,\theta)$ denote the number of nodes with quality θ and degree k at time t. This quantity can change under certain conditions upon insertion of a new node to the network. If a node whose degree at time $t-1$ is $k-1$ and whose quality is θ receives a link, its degree increments, and it becomes a node of degree k; hence, $N_t(k,\theta)$ increments. If a node whose degree at time $t-1$ is k and whose quality is θ receives a link, its degree increments, and it becomes a node of degree $k+1$, hence, $N_t(k,\theta)$ decrements. Finally, in the special case of $k=\beta$, at each time step one new node is being born with degree β, so the expected growth of $N_t(\beta,\theta)$ equals $\rho(\theta)$. Let us denote $\frac{1}{\alpha}$ by Δt, which is the time it takes for one node to be appended to the network. The following rate equation combines all these possible events upon insertion of a new node, accounting for pertinent probabilities:

$$\begin{aligned} N_{t+\Delta t}(k,\theta) - N_t(k,\theta) \\ = \rho(\theta)\delta_{k,\beta} + \frac{\beta(k-1+\theta)N_t(k-1,\theta)}{\zeta + 2\nu\alpha t} \\ - \frac{\beta(k+\theta)N_t(k,\theta)}{\zeta+2\nu\alpha t}. \end{aligned} \tag{4}$$

This can be equivalently expressed as follows:

$$\begin{aligned} \frac{N_{t+\Delta t}(k,\theta) - N_t(k,\theta)}{\Delta t} \\ = \rho(\theta)\alpha\delta_{k,\beta} + \frac{\alpha\beta(k-1+\theta)N_t(k-1,\theta)}{\zeta + 2\nu\alpha t} \\ - \frac{\alpha\beta(k+\theta)N_t(k,\theta)}{\zeta+2\nu\alpha t}. \end{aligned} \tag{5}$$

To proceed, we make a time-continuous approximation: We replace the left-hand side of (5) with a time derivative, transforming this equation into a differential-difference equation. Note that approximating the left-hand side of (5) introduces a relative error that is proportional to $\frac{1}{(\lambda+2\nu\alpha t)^2}$ (this follows readily from Taylor expanding the left-hand side up to second order). For long times, the error vanishes. For short times, the error is controlled by $\frac{1}{\lambda^2}$. Note that λ is twice the number of links in the initial network. We assume that the initial network is large, so that this error is negligible. In numerical simulations we have verified the validity of this approximation: For initial networks with 10^2 links the continuous-time approximation works remarkably accurately (error smaller than 1%). Note that 100 links is a very conservative requirement when that assuming typical real systems that are studies in the complex-networks literature (such as the web, collaboration networks, citation networks, social networks, online social media) have millions, if not billions, of links.

So we focus on a differential equation analog of (5) in the following form:

$$\begin{aligned} \frac{\partial N_t(k,\theta)}{\partial t} = \frac{\alpha\beta(k-1+\theta)N_t(k-1,\theta)}{\zeta+2\nu\alpha t} \\ - \frac{\alpha\beta(k+\theta)N_t(k,\theta)}{\zeta+2\nu\alpha t} + \rho(\theta)\alpha\delta_{k,\beta}. \end{aligned} \tag{6}$$

Hereafter, we assume that node quality can take only non-negative integer values. Now let us define the generating function:

$$\begin{aligned} \psi(z,y,t) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty}\sum_{\theta=0}^{\infty} N_t(k,\theta) z^{-k} y^{-\theta}, \\ R(y) \stackrel{\text{def}}{=} \sum_{\theta=0}^{\infty} \rho(\theta) y^{-\theta}. \end{aligned} \tag{7}$$

Note that the restriction on θ taking integer values does not diminish the generality of the results, and every step we undertakes throughout the analysis would hold, except all the sums over θ should be converted to integrals. So the second generating function in (7), which is a Z transform, would change to a Laplace transform. However, in the present paper, we restrict ourselves to the discrete case, and all the simulations are also conducted for discrete quality distributions.

Multiplying both sides of (6) and summing over k and θ, we arrive at

$$\begin{aligned} \frac{\partial\psi(z,y,t)}{\partial t} = \frac{\alpha\beta(z-1)}{\zeta+2\nu\alpha t}\frac{\partial\psi(z,y,t)}{\partial z} \\ - \frac{\alpha\beta(z^{-1}-1)y}{\zeta+2\nu\alpha t}\frac{\partial\psi(z,y,t)}{\partial y} + \alpha z^{-\beta}R(y). \end{aligned} \tag{8}$$

To proceed, we use the method of characteristics to solve this partial differential equation. The reader is referred to [40] for detailed discussions about this method or to Appendix D in [41] for a concise introduction through an example. In Appendix A we solve (8). The answer reads

$$\begin{aligned} \psi(z,y,t) = \psi\left(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c},0\right) + \frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta} \\ \times\left[F(z,y) - F\left(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c}\right)\right], \end{aligned} \tag{9}$$

where c and $F(z,y)$ are defined as

$$c \stackrel{\text{def}}{=} 1 - \left(\frac{\lambda}{\lambda+2\nu\alpha t}\right)^{\frac{1}{2+\frac{\mu}{\beta}}}, \tag{10}$$

$$F(z,y) \stackrel{\text{def}}{=} \sum_{m=0}^{\infty}\sum_{\theta=0}^{\infty}\frac{\Gamma\left(3+\frac{\mu}{\beta}+m\right)z^{-m-2-\beta-\frac{\mu}{\beta}}y^{-\theta}}{\Gamma\left(3+\frac{\mu}{\beta}\right)m!\left(m+2+\beta+\theta+\frac{\mu}{\beta}\right)}. \tag{11}$$

Note that when $t = 0$, the value of c is zero and $\frac{z-c}{1-c}$ becomes z, so the right-hand side of (9) yields $\psi(z,y,0)$, which is correct.

We need to invert this expression term by term in order to write it in the k,θ domain, instead of z,y domain.

In Appendix C we take the inverse transform of $(z-1)^{2+\frac{\mu}{\beta}}F(z,y)$, which enables us to write $(z-1)^{2+\frac{\mu}{\beta}}F(z,y)$ in the two dimensions as $\sum_{k,\theta} f_{k,\theta} z^{-k} y^{-\theta}$. The result is

$$(z-1)^{2+\frac{\mu}{\beta}}F(z,y) \xrightarrow{z^{-1},y^{-1}} \rho(\theta)\frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}u(k-\beta). \tag{12}$$

This yields the inverse of the second term on the right-hand side of (9). However, the first and third terms on the right-hand side of (9) remain to be inverted. This is done in Appendix D; the full inverse transform is found. The result is

$$\begin{aligned}N_t(k,\theta) = &(1-c)^{\theta}c^k\sum_{m=0}^{k}N(0,m,\theta)\left(\frac{1-c}{c}\right)^m\binom{k+\theta-1}{m+\theta-1}\\ &+\frac{(\lambda+2\nu\alpha t)}{\beta}\rho(\theta)\frac{(k+\theta-1)!}{(\beta+\theta-1)!}\\ &\times\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}u(k-\beta)\\ &-\frac{\lambda}{\beta}\rho(\theta)(1-c)^{\theta}c^k\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{(\beta+\theta-1)!}\\ &\times\sum_{m=\beta}^{k}\frac{(m+\theta-1)!}{\Gamma(m+3+\theta+\frac{\mu}{\beta})}\left(\frac{1-c}{c}\right)^m\binom{k+\theta-1}{m+\theta-1}.\end{aligned} \tag{13}$$

Dividing this by the total number of nodes at time t, which is $N = N(0) + \alpha t$, the degree-quality distribution of the network at time t is obtained:

$$\begin{aligned}P_t(k,\theta) = &\frac{(1-c)^{\theta}c^kN(0)}{N(0)+\alpha t}\sum_{m=0}^{k}P(0,m,\theta)\left(\frac{1-c}{c}\right)^m\\ &\times\binom{k+\theta-1}{m+\theta-1}+\frac{(\lambda+2\nu\alpha t)}{\beta(N(0)+\alpha t)}\rho(\theta)\frac{\Gamma(k+\theta)}{\Gamma(\beta+\theta)}\\ &\times\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma\left(k+3+\theta+\frac{\mu}{\beta}\right)}u(k-\beta)\\ &-\frac{\lambda}{\beta}\rho(\theta)\frac{(1-c)^{\theta}c^k}{N(0)+\alpha t}\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{(\beta+\theta-1)!}\\ &\times\sum_{m=\beta}^{k}\frac{(m+\theta-1)!}{\Gamma\left(m+3+\theta+\frac{\mu}{\beta}\right)}\left(\frac{1-c}{c}\right)^m\binom{k+\theta-1}{m+\theta-1}.\end{aligned} \tag{14}$$

The first term on the right-hand side is the effect of the initial nodes in the network. The second and third terms are due to subsequent nodes. The second term, as we shall clarify, is the dominant term in long times and does not vanish in the long-time limit. The last term, however, is transient and decays in time, vanishing in the limit as $t \to \infty$.

Note that this model allows the possibility of having negative values of θ, with some restrictions that we shall discuss. Negative quality values can model hostile interactions in social networks, for example. That is, in addition to different levels of popularity of a given node, which drives its chances of receiving links, we can allow for negative quality values to model hostility, so that the chance of a particular node for receiving links can be actually less than its preferential component, due to bad reputation, etc. The only restriction would be that attachment probabilities should remain positive. That is, the magnitude of a negative quality should not exceed that of the degree for any node. Since quality values are drawn from a distribution, this means that to ensure positivity of link reception probabilities, we should restrict the range of quality values so that the minimum degree is greater than the absolute value of the minimum quality value. The incoming nodes have minimum degree of at least β at any time; the minimum degree of the initial network must also be taken into account. Denoting the minimum quality value by $\theta_{\min}$, we should have $\min\{\beta,k_{\min}\} + \theta_{\min} > 0$. In Sec. VII we present an example case with negative quality along with the results.

IV. STEADY STATE

We can now look at the behavior of the system in the limit as $t \to \infty$. Note that c is zero when $t = 0$, and in the limit as $t \to \infty$, we have $c = 1$. The powers of $(\frac{1-c}{c})$ in the summands of the first and third terms on the right-hand side of (14) are all positive, because the effective range of the summation index m that is permitted by the binomial coefficients is $2 \leqslant m \leqslant k$. Since $(1-c)$ is zero in the limit as $t \to \infty$, the first and third terms on the right-hand side of (14) vanish in this limit, and only the second term survives. Now note that

$$\lim_{t\to\infty}\frac{\lambda+2\nu\alpha t}{N(0)+\alpha t} = 2\nu = 2\beta+\mu. \tag{15}$$

So the steady-state degree-quality distribution is

$$P(k,\theta) = \left(2+\frac{\mu}{\beta}\right)\rho(\theta)\frac{\Gamma(k+\theta)}{\Gamma(\beta+\theta)}\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma\left(k+3+\theta+\frac{\mu}{\beta}\right)}u(k-\beta). \tag{16}$$

The subscript stands for steady state.

To find the degree distribution, we would have to sum over all values of θ; that is, we would have to perform the summation $P(k) = \sum_{\theta} P(k,\theta)$. This can be done only for specified quality distribution $\rho(\theta)$.

The asymptotic behavior of the joint distribution for large degrees is investigated in Appendix F. We find that

$$P(k,\theta) \sim k^{-3-\frac{\mu}{\beta}},\ \forall\,\theta. \tag{17}$$

This means that the degree distribution $P(k)$, which is obtained by summing $P(k,\theta)$ over all values of θ, has the same asymptotic behavior, since all the terms in the summand would have identical asymptotic behavior and only the prefactors would depend on θ. So we arrive at

$$P(k) \sim k^{-3-\frac{\mu}{\beta}}. \tag{18}$$

The asymptotic degree distribution in the steady state depends only on the mean of the quality distribution, no other statistic.

The conditional degree distribution can be obtained by dividing the right-hand side of (16) by $\rho(\theta)$. The result is

$$P(k|\theta) = \left(2+\frac{\mu}{\beta}\right)\frac{\Gamma(k+\theta)}{\Gamma(\beta+\theta)}\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma\left(k+3+\theta+\frac{\mu}{\beta}\right)}u(k-\beta). \tag{19}$$

Note that in the steady state, the conditional degree distribution does not depend on the shape of the quality distribution, but solely on its mean. For general times, however, that is not true.

In the special case where $\rho(\theta)=\delta_{\theta,\theta_0}$, the model is tantamount to the shifted-linear preferential attachment, discussed, for example, in [42–44]. In this case, all the nodes have identical quality, and we can eliminate θ from $P(k,\theta)$, that is, the degree-quality distribution reduces to the degree distribution. Also in this case μ and θ_0 coincide, because there is a single permitted value for θ for every node, and this makes the average value for the quality of all nodes be equal to θ_0. Let us denote the degree distribution in this case by $P_{sh}(k)$, where the sh subscript denotes "shifted." From (16) we obtain

$$P_{sh}(k) = \left(2+\frac{\theta_0}{\beta}\right)\frac{\Gamma(k+\theta_0)}{\Gamma(\beta+\theta_0)}\frac{\Gamma\left(\beta+2+\theta_0+\frac{\theta_0}{\beta}\right)}{\Gamma\left(k+3+\theta_0+\frac{\theta_0}{\beta}\right)}u(k-\beta). \tag{20}$$

This is exactly what one would get in the degree distribution of shifted-linear kernels given, for example, in Eq. 9 in [42], and in Eq. D.9 in [44].

Finally, setting θ equal to zero, all nodes will have zero quality, and attachments will be purely degree proportional, synonymous to the conventional preferential-attachment model proposed initially in [2]. From (20) with $\theta=\mu=0$ we obtain

$$P_{BA}(k) = \frac{2\beta(\beta+1)}{k(k+1)(k+2)}u(k-\beta). \tag{21}$$

This is identical to the degree distribution of the conventional BA network, obtained, for example, in [42,43,45].

V. FORMATION OF HIERARCHY

Now let us focus on the effect of quality on the degree in the steady state. We find the average degree of the nodes with given quality and investigate how this average degree depends on the value of quality. Using the conditional degree distribution obtained in (19), we have

$$\begin{aligned}\langle k\rangle_\theta &= \sum_k kP(k|\theta) = \left(2+\frac{\mu}{\beta}\right)\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma(\beta+\theta)}\\ &\quad\times\sum_{k=\beta}^{\infty}\frac{k\Gamma(k+\theta)}{\Gamma\left(k+3+\theta+\frac{\mu}{\beta}\right)}.\end{aligned} \tag{22}$$

The following identity is proved in Appendix H :

$$\sum_{k=\beta}^{\infty}\frac{k\Gamma(k+\theta)}{\Gamma\left(k+3+\theta+\frac{\mu}{\beta}\right)} = \frac{\beta\left(2+\frac{\mu}{\beta}+\frac{\theta}{\beta}\right)\Gamma(\beta+\theta)}{\left(1+\frac{\mu}{\beta}\right)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(2+\beta+\theta+\frac{\mu}{\beta}\right)}. \tag{23}$$

Using this identity, we can perform the summation on the right-hand side of (22). The result is

$$\begin{aligned}\langle k\rangle_\theta &= \left(2+\frac{\mu}{\beta}\right)\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma(\beta+\theta)}\\ &\quad\times\frac{\beta\left(2+\frac{\mu}{\beta}+\frac{\theta}{\beta}\right)\Gamma(\beta+\theta)}{\left(1+\frac{\mu}{\beta}\right)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(2+\beta+\theta+\frac{\mu}{\beta}\right)}\\ &= \beta\frac{\left(2+\frac{\mu}{\beta}+\frac{\theta}{\beta}\right)}{\left(1+\frac{\mu}{\beta}\right)} = \beta\frac{2\beta+\mu+\theta}{\beta+\mu}.\end{aligned} \tag{24}$$

The average degree grows linearly with quality. The slope of this relation decreases as μ increases. Heuristically, larger μ means that other nodes are of higher quality, and one would require more increase in quality to elevate one's degree centrality as μ becomes larger. We can immediately proceed to find the mean degree:

$$\begin{aligned}\bar{k} &= \sum_\theta \rho(\theta)\langle k\rangle_\theta\\ &= \beta\frac{\left(2+\frac{\mu}{\beta}\right)}{1+\frac{\mu}{\beta}}\sum_\theta\rho(\theta) + \frac{1}{1+\frac{\mu}{\beta}}\sum_\theta\theta\rho(\theta)\\ &= \beta\frac{\left(2+\frac{\mu}{\beta}\right)}{1+\frac{\mu}{\beta}} + \frac{\mu}{1+\frac{\mu}{\beta}} = \frac{2\beta+2\mu}{1+\frac{\mu}{\beta}} = 2\beta.\end{aligned} \tag{25}$$

VI. NEAREST-NEIGHBOR QUALITY-DEGREE DISTRIBUTION

So far we have looked at how the quality of a node affects with its own degree. However, there are other interesting questions we can ask: How does the quality of a node situate it within the network? That is, if a node has high quality, will its neighbors turn out to be mostly high-quality nodes? How about degree: How does the degree of a node relate to the degrees of the neighbors of that node? Do high-degree nodes tend to connect to other high-degree nodes? All these questions can be answered if we could answer the following question. Suppose we have a node with degree k and quality θ. If we select one of its neighbors uniformly at random, what is the probability of that neighbor having degree ℓ and quality ϕ? We address these questions only in the steady state, due to theoretical convenience.

We aim at finding $P(\ell,\phi|k,\theta)$. It is the fraction of neighbors of a node of degree k and quality θ, who have degree ℓ and quality ϕ. What this means is the following. Let us identify all the nodes in the network who have degree k and quality θ. Let us put all the neighbors of these nodes in a set, and let us call this set V. We should also count for multiplicities: If a node is the neighbor of, for example, three nodes with degree k and quality θ, this node should be included three times in V. After constituting V, then $P(\ell,\phi|k,\theta)$ equals the fraction of its members whose degrees are ℓ and whose quality is ϕ. We will refer to $P(\ell,\phi|k,\theta)$ as NNQDD hereinafter for brevity, which stands for nearest-neighbor quality-degree distribution. Note that an equivalent conceptualization of NNQDD is as a joint quality-degree distribution of the neighbors of a node with degree k and degree θ.

We quantify the expected change of $N_t(k,\theta,\ell,\phi)$ upon the introduction of a single new node by writing the rate equation. Throughout, we will call the incoming node the child of the existing nodes that it attaches to, and they will be its parents. There are two events that could decrement $N_t(k,\theta,\ell,\phi)$ when a new node is added: If the new node connects to a node of degree ℓ and quality ϕ who is attached to a parent of degree k and quality θ, or if the new node attaches to the parent node in such a pair of nodes. There are two ways that $N(k,\theta,\ell,\phi)$ could increment: if there is a child of degree $\ell-1$ and quality ϕ who is connected to a parent of degree k and quality θ and the child receives a link, or if there is a parent node of degree $k-1$ and quality θ with a child of degree ℓ and quality ϕ and the parent receives a link. Finally, with probability $\rho(\phi)$, the new node has quality ϕ, and if the new node forms a link to an existing node of degree $k-1$ and quality θ, then $N_t(k,\theta,\ell,\phi)$ increments. The following rate equation subsumes all these cases with their respective probabilities:

$$N_{t+1}(k,\theta,\ell,\phi) = N_t(k,\theta,\ell,\phi) + \beta\left[\frac{(\ell-1+\phi)N_t(k,\theta,\ell-1,\phi) - (\ell+\phi)N_t(k,\theta,\ell,\phi)}{\zeta+(2\beta+\mu)t}\right] + \beta\left[\frac{(k-1+\theta)N_t(k-1,\theta,\ell,\phi) - (k+\theta)N_t(k,\theta,\ell,\phi)}{\zeta+(2\beta+\mu)t}\right] + \rho(\phi)\delta_{\ell,\beta}\frac{\beta(k-1+\theta)N_t(k-1,\theta)}{\zeta+(2\beta+\mu)t}. \tag{26}$$

Writing this equation in terms of $n_t(k,\theta,\ell,\phi)$, this can expressed as follows:

$$[N(0)+t+1]n_{t+1}(k,\theta,\ell,\phi) - [N(0)+t]n_t(k,\theta,\ell,\phi) = \beta\left[\frac{(\ell-1+\phi)n_t(k,\theta,\ell-1,\phi) - (\ell+\phi)n_t(k,\theta,\ell,\phi)}{[\zeta+(2\beta+\mu)t]/[N(0)+t]}\right] + \beta\left[\frac{(k-1+\theta)n_t(k-1,\theta,\ell,\phi) - (k+\theta)n_t(k,\theta,\ell,\phi)}{[\zeta+(2\beta+\mu)t]/[N(0)+t]}\right] + \rho(\phi)\delta_{\ell,\beta}\frac{\beta(k-1+\theta)n_t(k-1,\theta)}{[\zeta+(2\beta+\mu)t]/[N(0)+t]}. \tag{27}$$

Now let us look at the steady state, that is, the limit as $t\to\infty$. In this limit, the values of $n_t(k,\theta,\ell,\phi)$ reach horizontal asymptotes; hence, we drop the t subscripts. Also note that we have

$$\lim_{t\to\infty}\frac{\beta}{[\zeta+(2\beta+\mu)t]/[N(0)+t]} = \frac{1}{2+\frac{\mu}{\beta}}. \tag{28}$$

Thus, we can rewrite (27) equivalently as follows:

$$n(k,\theta,\ell,\phi) = \left[\frac{(\ell-1+\phi)n(k,\theta,\ell-1,\phi) - (\ell+\phi)n(k,\theta,\ell,\phi)}{2+\frac{\mu}{\beta}}\right] + \left[\frac{(k-1+\theta)n(k-1,\theta,\ell,\phi) - (k+\theta)n(k,\theta,\ell,\phi)}{2+\frac{\mu}{\beta}}\right] + \rho(\phi)\delta_{\ell,\beta}\frac{(k-1+\theta)P(k-1,\theta)}{2+\frac{\mu}{\beta}}. \tag{29}$$

We multiply both sides of this equation by $2+\frac{\mu}{\beta}$ and rearrange the terms to obtain the following recursive equation:

$$n(k,\theta,\ell,\phi) = \frac{(\ell-1+\phi)n(k,\theta,\ell-1,\phi)}{2+\frac{\mu}{\beta}+k+\ell+\theta+\phi} + \frac{(k-1+\theta)n(k-1,\theta,\ell,\phi)}{2+\frac{\mu}{\beta}+k+\ell+\theta+\phi} + \rho(\phi)\delta_{\ell,\beta}\frac{(k-1+\theta)P(k-1,\theta)}{2+\frac{\mu}{\beta}+k+\ell+\theta+\phi}. \tag{30}$$

In Appendix I we solve this difference equation. The solution is

$$n(k,\theta,\ell,\phi) = \rho(\phi)\rho(\theta)\left(2+\frac{\mu}{\beta}\right)\frac{(k-1+\theta)!(\ell-1+\phi)!}{(\beta-1+\theta)!(\beta-1+\phi)!}\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma\left(3+\frac{\mu}{\beta}+k+\ell+\theta+\phi\right)} \times \sum_{j=\beta+1}^{k}\frac{\Gamma\left(2+\frac{\mu}{\beta}+j+\beta+\theta+\phi\right)}{\Gamma\left(j+2+\theta+\frac{\mu}{\beta}\right)}\binom{k-j+\ell-\beta}{\ell-\beta}. \tag{31}$$

Now we can obtain the NNQDD. We need to find the total number of neighbors who are neighbors of nodes with degree k and quality ϕ. This equals $[n(k,\theta,\ell,\phi)+n(\ell,\phi,k,\theta)]N$, because being a neighbor of a node with degree k and quality θ means being either its child or its parent, so we need to add up the two numbers. We need to divide this by the total number of neighbors of nodes with degree k and quality θ, which equals $kNP(k,\theta)$, because there are $NP(k,\theta)$ nodes with degree k and quality θ and each of them has k neighbors. Using (31) together with the expression for $P(k,\theta)$ given in (16), we arrive at

$$P(\ell,\phi|k,\theta) = \frac{\rho(\phi)}{k}\frac{(\ell-1+\phi)!}{(\beta-1+\phi)!}\frac{\Gamma\left(k+\theta+3+\frac{\mu}{\beta}\right)\Gamma\left(\beta+2+\phi+\frac{\mu}{\beta}\right)}{\Gamma\left(k+\theta+3+\frac{\mu}{\beta}+\ell+\phi\right)} \times\left[\sum_{j=\beta+1}^{k}\frac{\Gamma\left(j+\theta+2+\frac{\mu}{\beta}+\beta+\phi\right)\binom{k-j+\ell-\beta}{\ell-\beta}}{\Gamma\left(\beta+2+\phi+\frac{\mu}{\beta}\right)\Gamma\left(j+\theta+2+\frac{\mu}{\beta}\right)}+\sum_{j=\beta+1}^{\ell}\frac{\Gamma\left(j+\theta+2+\frac{\mu}{\beta}+\beta+\phi\right)\binom{\ell-j+k-\beta}{k-\beta}}{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)\Gamma\left(j+\phi+2+\frac{\mu}{\beta}\right)}\right]. \quad (32)$$

This conditional distribution stores enough information to extract several quantities of interest. For example, for the nearest-neighbor quality distribution $P(\phi|\theta)$ we can calculate

$$p(\phi|\theta) = \sum_{\ell}\sum_{k} P(k)P(\ell,\phi|k,\theta). \quad (33)$$

Similarly, the nearest-neighbor degree distribution is obtained via

$$p(\ell|k) = \sum_{\phi}\sum_{\theta} \rho(\theta)P(\ell,\phi|k,\theta). \quad (34)$$

VII. SIMULATION RESULTS AND DISCUSSIONS

For simulation purposes we use $\alpha = 1$ in all the simulations. In Fig. 1, the simulation results and the theoretical prediction for $N_t(k,\theta)$ is depicted. The example values of $k = 12$ and $\theta = 2$ are selected for expository purposes. The initial network is a 4-regular ring of 100 nodes. We construct such network by situating the nodes on a ring and then connecting each node to its second-nearest neighbor. The quality distribution is exponential in the set $\theta \in \{1,2,3,4,5\}$, with decay factor $q = 0.8$. That is, the probability of a given θ is proportional to q^{θ}. So higher-quality values are less probable, as one would intuitively expect in real settings. The value of β is 2, which means that each incoming node attaches to 2 existing nodes. Note that in all depictions, $N_t(k,\theta)$ can take noninteger values at any time step, because it is an expected value and need not be an integer.

Figure 2 illustrates theoretical predictions along with simulation results for a ring of 200 nodes, exponential quality distribution with decay factor $q = 0.8$, and $\theta \in \{1,2,\ldots,8\}$. The corresponding degree-quality distribution, $P_t(k,\theta)$, is depicted in Fig. 3.

To the demonstrate the validity of the theoretical predictions for short times, we use a ring of 200 nodes with exponential quality distribution with decay factor 0.7 and quality values in the set $\theta \in \{1,2,3,4,5\}$ and run the simulations up to $t = 20$. As can be seen in Fig. 4, short times do not adversely affect the accuracy of the theoretical predictions.

In Fig. 5, a power-law quality distribution with exponent -1 is considered. That is, the probability of quality θ is proportional to θ^{-1}. The theoretical predictions and simulation results for $N_t(14,1)$ are depicted. Example values of $k = 14$ and $\theta = 1$ are selected. The value of β is 5, and the initial network is a 6-regular ring. In Fig. 6, the joint distribution $P_t(14,1)$ is depicted for the same setting.

Let us also verify the theoretical predictions in the steady state. Figure 7 depicts theoretical predictions and simulation results for the conditional degree distribution $P(k|\theta)$. The quality distribution is defined in the set $\theta \in \{1,10,20\}$. The value of β is 5. As can be seen in the figure, for small values of

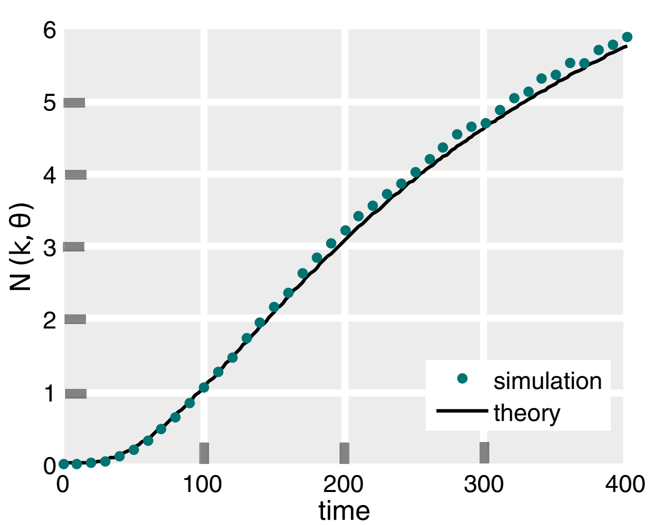

FIG. 1. Simulation results along with theoretical predictions as given in (13). The initial network is a 4-regular ring of 100 nodes. The values of θ are in the set $\{1,2,3,4,5\}$. The value of β is 2, k is 12, and θ is 2. The quality distribution is given by $\rho(\theta) = \frac{(\mathbf{0.8})^{\theta}}{\sum_{\theta}(\mathbf{0.8})^{\theta}}$. The results are averaged over 200 Monte Carlo trials.

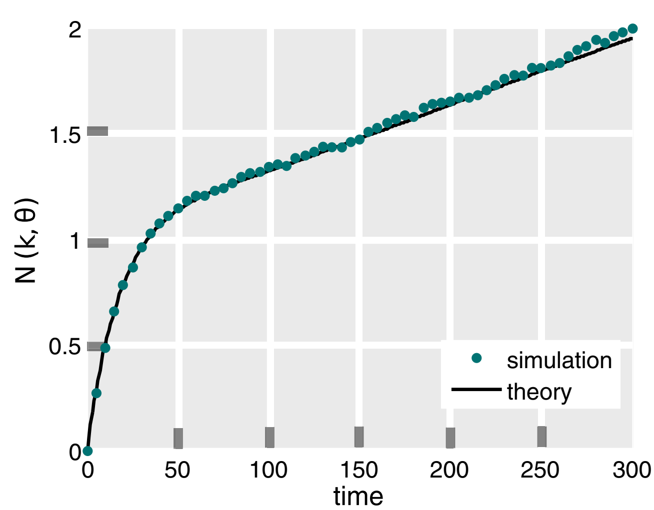

FIG. 2. Simulation results along with theoretical predictions as given in (13). The initial network is a ring of 200 nodes. The values of θ are in the set $\{1,2,\ldots,8\}$. The value of β is 4, k is 8, and θ is 8. The quality distribution is given by $\rho(\theta) = \frac{(\mathbf{0.8})^{\theta}}{\sum_{\theta}(\mathbf{0.8})^{\theta}}$. The results are averaged over 200 Monte Carlo trials.

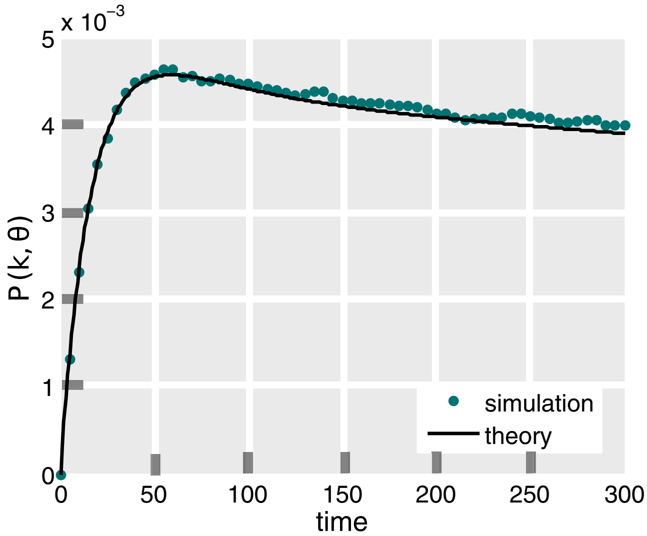

FIG. 3. $\mathbf{P_t(k,}\theta\mathbf{)}$ for $\mathbf{k = 8}$ and $\theta = \mathbf{8}$. The values of β is 4. The quality distribution is exponential with exponent 0.8, and the quality values are in the set $\theta \in \{\mathbf{1,2,3},\ldots,\mathbf{8}\}$. The initial network is a ring of 200 nodes. The results are averaged over 200 Monte Carlo trials.

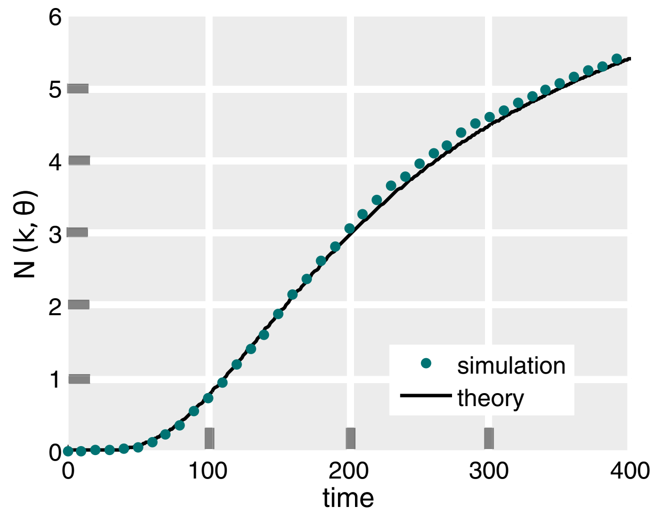

FIG. 5. Simulation results for power-law quality distribution with exponent -1, that is, along with theoretical predictions as given in (13). We have $N_t(k,\theta)$ for $k = 14$, $\theta = 1$. The quality distribution is given by $\rho(\theta) = \frac{\theta^{-1}}{\sum_\theta \theta^{-1}}$. The initial network is a 6-regular ring of 100 nodes. The value of β is 5. Quality values are in the set $\theta \in \{1,2,3,4,5\}$.

k, the curve for $\theta = 1$ is above the other two, which means that nodes with quality equal to 1 are more likely than the other two groups to have low degrees. On the other hand, nodes with quality 20 dominate the other two in high-degree regions, as shown in the inset. The curve for $\theta = 20$ is between the other two for both small degrees and large degrees.

Now we investigate the validity of (24) through simulations. Figure 8 depicts the simulation results and the theoretical predictions for $\langle k\rangle_\theta$. As (24) predicts, the slope of the curve is determined solely by β and μ and is equal to $\frac{\beta}{\beta+\mu}$. We set $\beta = 4$ and run the simulations for two different quality distributions. The first one is uniform in the set $\theta \in \{1,2,\ldots,10\}$, and the second one is uniform in the set $\theta \in \{4,5,6,7\}$. Both these distributions have $\mu = 5.5$, so we expect their curves for $\langle k\rangle_\theta$ to coincide. It is visible from Fig. 8 that they do. We have also depicted the simulation results for $\beta = 1$ and $\beta = 2$ to illustrate the effect of β on the slope and intercept of the average conditional degree curve.

As mentioned in Sec. III, the model reduces to preferential attachment with "initial attractiveness" for the case of constant quality, that is, $\rho(\theta) = \delta(\theta,\theta_0)$. Figure 9 shows the simulation

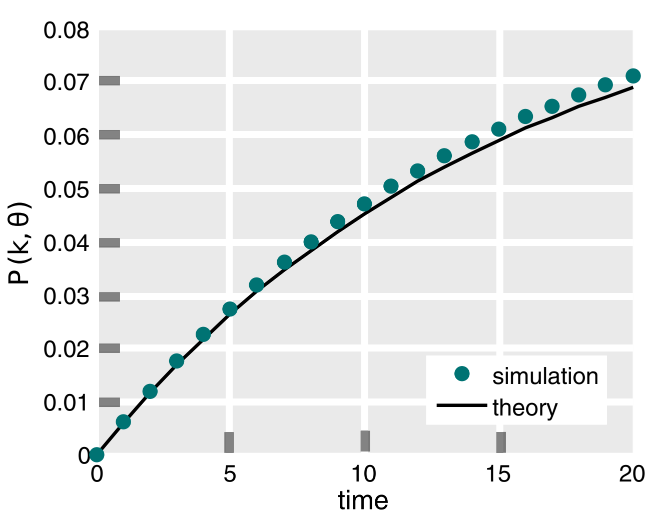

FIG. 4. Accuracy of predictions in short times: $\mathbf{P_t(3,1)}$ for exponential quality distribution with decay factor 0.7. The initial network is a ring of 200 nodes. The value of β is 5. Quality values are in the set $\theta \in \{\mathbf{1,2,3,4,5}\}$. The results are averaged over 200 Monte Carlo trials.

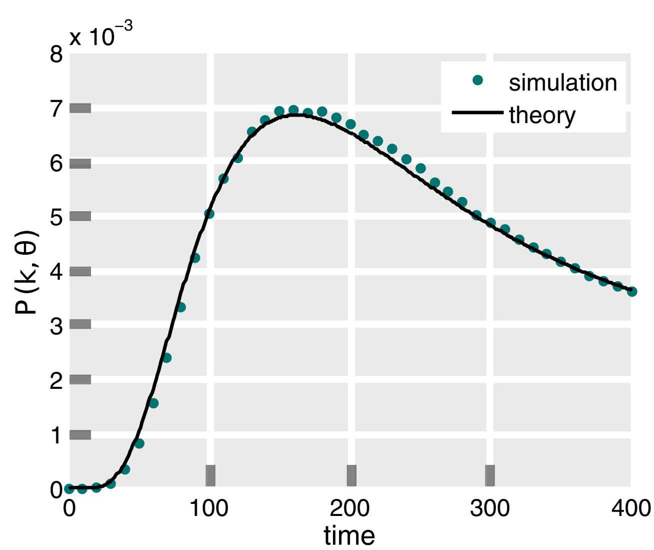

FIG. 6. Simulation results for power-law quality distribution with exponent -1, that is, along with theoretical predictions as given in (13). We have set $P_t(k,\theta)$ for $k = 14$, $\theta = 1$. The quality distribution and initial conditions are identical to those of Fig. 5.

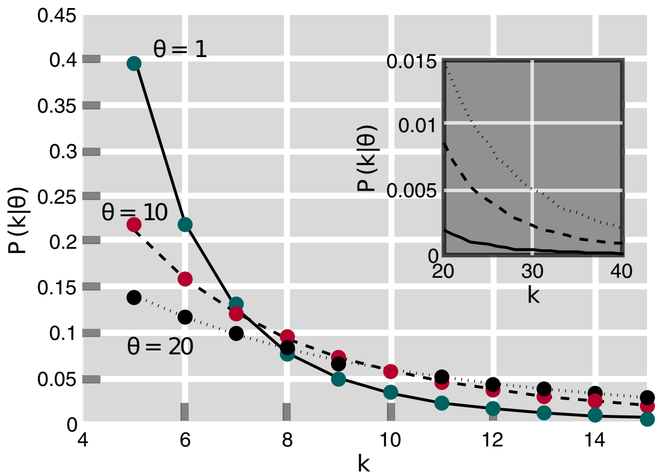

FIG. 7. The conditional degree distribution $P(k|\theta)$ in the steady state for three different values of θ. The markers depict simulation results, and curves depict theoretical predictions. The value of β is 5. Three possible values for quality exist: $\theta \in \{1,10,20\}$. The three values have equal probabilities. The results are averaged over 100 Monte Carlo trials. It is visible that for large values of k, higher θ yields a higher probability; that is, nodes with higher quality values are more likely to have high degrees.

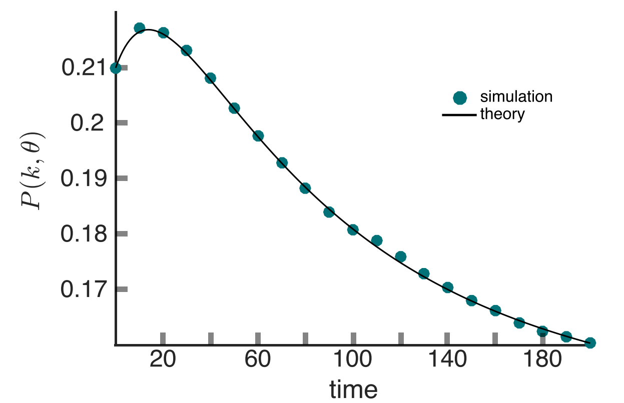

FIG. 9. Simulation results for a ring of 100 nodes with $p(\theta) = \delta_{\theta,50}$

results for a ring of 100 nodes with quality distribution $\rho(\theta) = \delta_{\theta,50}$.

As mentioned in Sec. III, the model also allows for negative quality values with certain restrictions discussed above. Figure 10 shows the simulation results for a ring of 100 nodes with quality distribution $\rho(\theta) = \delta_{\theta,-1}$. The match between theoretical predictions and simulation results confirms that negative quality values are fine in the model as long as the said restrictions are applied.

We point out that if the quality distribution is in such that the quality values are small and negligible as compared to the degrees, then the process reduces to the conventional preferential attachment. On the other hand, if the quality values are much larger than the degrees, then quality alone will control the growth process. For a given initial network, one can choose the quality distribution such that the shares of degree and quality in the growth dynamics are balanced. To this end, one must simply take into account the range of degrees in the initial network, as well as the value of β. The said parameters and the quality values must have comparable magnitudes to get a balanced growth process.

It is of note that in the steady state, the additive quality model always generates power-law degree distributions, as discussed in Sec. IV. Also, the conditional degree distribution of nodes for any given quality value is also power law. This is in contrast with the multiplicative fitness model [36], where the steady-state degree distribution is only a power law for constant fitness, that is, for $\rho(\theta) = \delta_{\theta,\theta_0}$. For uniform fitness distribution, the multiplicative model produces a degree distribution which is power-law multiplied by an inverse

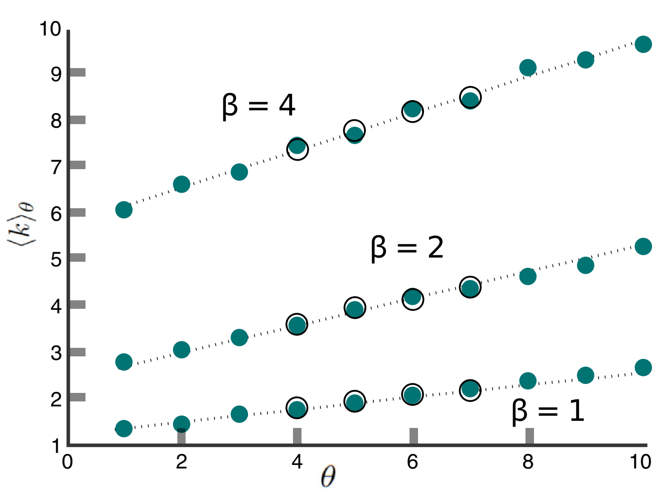

FIG. 8. Average conditional degree, as predicted by Eq. (24). The above curve pertains to the case of $\beta = 4$. The solid markers pertain to uniform quality distribution over values $\theta \in \{1,2,\ldots\}$, and open markers correspond to a uniform quality distribution over values $\theta \in \{4,5,6,7\}$. Both simulations have $\mu = 5.5$, and their average conditional degree curves coincide. The middle curve pertains to $\beta = 2$, and the lower curve pertains to $\beta = 1$. It is visible that for the same μ, higher β yields higher slope.

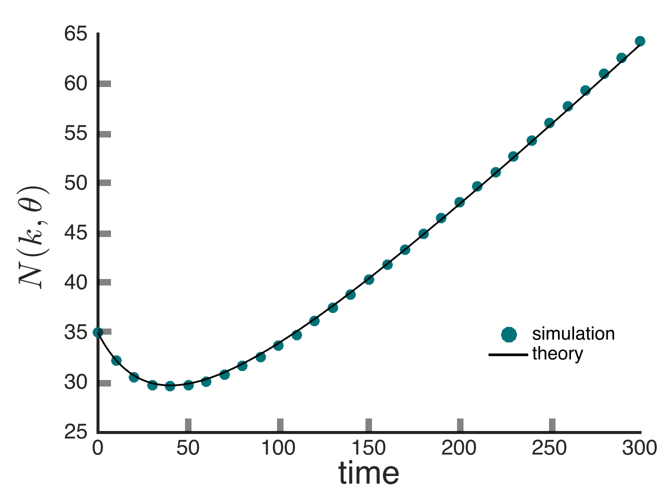

FIG. 10. Simulation results for a ring of 100 nodes with $p(\theta) = \delta_{\theta,-1}$.

logarithm factor. The additive quality generates power laws with tunable exponent and hierarchical structure. So the steady-state behavior of networks grown under multiplicative fitness are different than those grown under additive quality.

VIII. SUMMARY AND FUTURE WORK

The dominant focus on the steady state, which springs from an emphasis on the extraction of asymptotic power laws, has become a convention in the network-growth literature. The transient dynamics of most models are conventionally disregarded and the steady-state behavior is analyzed. This paper focuses on the dynamics of the network for arbitrary times. An additive quality-based network-growth model is proposed, and the rationale behind it is discussed. We found the joint quality-degree distribution as a function of time for arbitrary initial networks. We extracted the asymptotic steady-state behavior of the model, which yields a power-law degree distribution with a tunable exponent that is contingent on the quality distribution and on β, the initial degree of incoming nodes. We observed that, other than the mean, the exponent does not depend on any other property of the quality distribution. We also focused on nearest-neighbor statistics and found the conditional degree-quality distribution. Throughout the paper, we corroborated theoretical findings with Monte Carlo simulations.

An interesting extension to this work would be to focus on specific quality distributions, with one or two parameters, and derive maximum likelihood estimations that would enable us to fit such a model to actual temporal network data. This would enable us in turn to see what quality distribution matches a system better. For example, in the case of citation networks, we intuitively expect that the quality distribution would be a decreasing function of quality; that is, most of the papers do not possess a high quality and a small number of them have high values of quality. However quality is not easy to quantify. It would be interesting to be able to answer a question of the following form: "Among these hypothesized quality distributions, which one matches the observed temporal dynamics of the degree distribution the best?" To infer the quality distribution without prior assumptions is another way forward. That is, we can pursue nonparametrically to estimate quality values for individual nodes from the observed temporal dynamics of the degrees. This is work in progress. Since the naive maximum likelihood faces immense computational cost, alternative methods must be employed.

Another extension to this work would be to add edge growth, where existing nodes also are able to establish links between one another. This scenario is not applicable to citation networks but it is suitable, for example, for online social media, collaboration networks, and the web.

Finally, let us spell out a plausible generic problem pertaining to network-growth processes of random nature to shed light on the necessary future steps for obtaining rigorous statistical recipes to test network-growth models. Consider any network-growth model with random nature (preferential and uniform attachment are two examples), and, for the sake of simplicity, suppose that the focus of the study is the degree distribution of the network as a function of time (the arguments apply to the NNQDD, or any other time-dependent quantity). Now consider, for example, $p_7(t)$, the fraction of nodes in the network that have degree 7 at time t. If we simulate the growth process for a very large number of trials and average the values of $p_7(t)$ for all the trials, we get the expected value of $p_7(t)$. This quantity is theoretically found in [41]. However, what about individual simulation trials? An individual growth process will have its own $p_7(t)$, which might be close to or far from the expected value. In reality, when we posit a model to capture the essence of the growth process of an empirically observable network, we face a single growth trajectory, and we need to have estimates of what paths are possible, and what is the probability of occurrence for each of them. So at time t, we need a distribution for $p_7(t)$, which would indicate what values are possible for p_7 at time t, and each with what probability. So we need a distribution over all the possible degree distributions. Such distribution would be particularly key if one attempts fitting the hypothesized growth model to empirically observed temporal data. The hypothesis would yield a distribution at each instant in time, and the empirical observation would give a single value at each instant whose position and corresponding occurrence probability can be assessed having a distribution of possibilities at hand. Then we can undertake more rigorous steps such as hypothesis testing or constituting confidence intervals or p values. Such rigor would prove constructive in the literature of network growth and evolution, whose primary emphasis has been the steady state heretofore. It would enable us to study the temporal evolution of networks and grasp the underlying driving mechanisms.

APPENDIX A: SOLVING EQ. (8)

The equation we want to solve is

$$\frac{\partial\psi(z,y,t)}{\partial t} = \frac{\beta(z-1)}{\zeta+2\nu\alpha t}\frac{\partial\psi(z,y,t)}{\partial z} - \frac{\beta(z^{-1}-1)y}{\zeta+2\nu\alpha t}\frac{\partial\psi(z,y,t)}{\partial y} + z^{-\beta}R(y). \tag{A1}$$

This can be rearranged and expressed as follows:

$$\frac{\partial\psi(z,y,t)}{\partial t} - \frac{\beta(z-1)}{\zeta+2\nu\alpha t}\frac{\partial\psi(z,y,t)}{\partial z} + \frac{\beta(z^{-1}-1)y}{\zeta+2\nu\alpha t}\frac{\partial\psi(z,y,t)}{\partial y} = z^{-\beta}R(y). \tag{A2}$$

The method of characteristics yields the following system of equations:

$$\frac{dt}{1} = \frac{dz}{-\frac{\beta(z-1)}{\zeta+2\nu\alpha t}} = \frac{dy}{\frac{\beta(z^{-1}-1)y}{\zeta+2\nu\alpha t}} = \frac{d\psi}{z^{-\beta}R(y)}. \tag{A3}$$

First, consider the following:

$$\frac{dt}{1} = \frac{dz}{-\frac{\beta(z-1)}{\zeta+2\nu\alpha t}} \Longrightarrow \frac{dt}{\zeta+2\nu\alpha t} + \frac{dz}{\beta(z-1)} = 0. \tag{A4}$$

Rearranging and integrating this equation, we arrive at

$$(z-1)^{\frac{2\nu}{\beta}}(\zeta+2\nu\alpha t) = C_1 \Longrightarrow (z-1)^{2+\frac{\mu}{\beta}}(\zeta+2\nu\alpha t) = C_1, \tag{A5}$$

where C_1 is constant. Then, from (A3) we select the second equation:

$$\frac{dz}{\left[-\frac{\beta(z-1)}{\zeta+2\nu\alpha t}\right]}=\frac{dy}{\frac{\beta(z^{-1}-1)y}{\zeta+2\nu\alpha t}}. \tag{A6}$$

Integrating this, we obtain

$$\frac{y}{z}=C_2, \tag{A7}$$

where C_2 is constant. Finally, from (A3) we select

$$\frac{dz}{\left[-\frac{\beta(z-1)}{\zeta+2\nu\alpha t}\right]}=\frac{d\psi}{z^{-\beta}R(y)}. \tag{A8}$$

Replacing y with $C_2 z$ and replacing $(\zeta+2\nu\alpha t)$ with $\frac{C_1}{(z-1)^{2+\frac{\mu}{\beta}}}$, this transforms into

$$d\psi=-\frac{C_1}{\beta}\frac{z^{-\beta}R(C_2 z)}{(z-1)^{3+\frac{\mu}{\beta}}}dz. \tag{A9}$$

To proceed, we need to integrate both sides of this equation to obtain ψ. It is straightforward to check the following identity through Taylor expansion:

$$\frac{1}{(z-1)^{3+\frac{\mu}{\beta}}}=z^{-3-\frac{\mu}{\beta}}\sum_{m=0}^{\infty}\frac{\Gamma\left(3+\frac{\mu}{\beta}+m\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)m!}z^{-m}. \tag{A10}$$

We can rewrite Eq. (A9) as follows:

$$d\psi=-\frac{C_1}{\beta}R(C_2 z)\sum_{m=0}^{\infty}\frac{\Gamma\left(3+\frac{\mu}{\beta}+m\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)m!}z^{-m-3-\beta-\frac{\mu}{\beta}}dz. \tag{A11}$$

To proceed, let us assume that the quality distribution $\rho(\theta)$, where θ can take integer values as discussed in the text, has the following generating function:

$$R(y)\stackrel{\text{def}}{=}\sum_{\theta=0}^{\infty}\rho(\theta)y^{-\theta}. \tag{A12}$$

We can plug this expression into (A11), replacing y with $C_2 z$, to obtain

$$d\psi=-\frac{C_1}{\beta}\sum_{m=0}^{\infty}\sum_{\theta=0}^{\infty}\rho(\theta)\frac{\Gamma\left(3+\frac{\mu}{\beta}+m-\theta\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)(m-\theta)!}\times C_2^{-\theta}z^{-m-3-\beta-\frac{\mu}{\beta}}dz, \tag{A13}$$

from which we arrive at

$$\psi(z,y,t)=\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}\times\sum_{m=0}^{\infty}\sum_{\theta=0}^{\infty}\rho(\theta)\frac{\Gamma\left(3+\frac{\mu}{\beta}+m-\theta\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)(m-\theta)!}\times C_2^{-\theta}\frac{z^{-m-2-\beta-\frac{\mu}{\beta}}}{m+2+\beta+\frac{\mu}{\beta}}+\Phi(C_1,C_2)$$

$$=\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}\times\sum_{m=0}^{\infty}\sum_{\theta=0}^{\infty}\rho(\theta)\frac{\Gamma\left(3+\frac{\mu}{\beta}+m-\theta\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)(m-\theta)!}\times y^{-\theta}\frac{z^{-m+\theta-2-\beta-\frac{\mu}{\beta}}}{m+2+\beta+\frac{\mu}{\beta}}+\Phi(C_1,C_2). \tag{A14}$$

With no initial conditions, the function $\Phi(C_1,C_2)$ is arbitrary; that is, for any differentiable $\Phi(\cdot)$ we use in (A11), the $\psi(z,y,t)$ it yields will satisfy (8). However, for a given initial condition, it is uniquely determined. In this paper we assume the initial condition; that is, the initial graph on top of which the growth process takes place, is known. So $\psi(z,y,0)$ denotes the generating function for the initial graph. Let us define

$$F(z,y,t)\stackrel{\text{def}}{=}\sum_{m=0}^{\infty}\sum_{\theta=0}^{\infty}\rho(\theta)\frac{\Gamma\left(3+\frac{\mu}{\beta}+m-\theta\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)(m-\theta)!}y^{-\theta}\times\frac{z^{-m+\theta-2-\beta-\frac{\mu}{\beta}}}{m+2+\beta+\frac{\mu}{\beta}}. \tag{A15}$$

So (A14) can be expressed in the following compact form:

$$\psi(z,y,t)=\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}F(z,y,t)+\Phi(C_1,C_2). \tag{A16}$$

In Appendix B we determine the form of $\Phi(\cdot)$ uniquely, and (A14) transforms into

$$\psi(z,y,t)=\psi\left(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c},0\right)+\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}\times\left[F(z,y)-F\left(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c}\right)\right]. \tag{A17}$$

APPENDIX B: DETERMINING $\Phi(\cdot,\cdot)$ IN (A14) FOR GIVEN INITIAL CONDITIONS

From (A14) and using the explicit form of C_1 given in (A5), we have

$$\psi(z,y,t)=\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}F(z,y)+\Phi((z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t),C_2). \tag{B1}$$

Putting $t=0$ in this equation, we get

$$\Phi((z-1)^{2+\frac{\mu}{\beta}}\lambda,C_2)=\psi(z,y,0)-\frac{\lambda(z-1)^{2+\frac{\mu}{\beta}}}{\beta}F(z,y). \tag{B2}$$

Let us denote $\lambda(z-1)^{2+\frac{\mu}{\beta}}$ by X. Note that the following holds:

$$z=\left(\frac{X}{\lambda}\right)^{1/(2+\frac{\mu}{\beta})}+1$$

$$y=C_2\left[\left(\frac{X}{\lambda}\right)^{1/(2+\frac{\mu}{\beta})}+1\right]. \tag{B3}$$

Using these, we can express (B2) equivalently as

$$\Phi(X,C_2) = \psi\left(\left(\frac{X}{\lambda}\right)^{1/(2+\frac{\mu}{\beta})} + 1, C_2\left[\left(\frac{X}{\lambda}\right)^{1/(2+\frac{\mu}{\beta})} + 1\right], 0\right) - \frac{X}{\beta}F\left(\left(\frac{X}{\lambda}\right)^{1/(2+\frac{\mu}{\beta})} + 1, C_2\left[\left(\frac{X}{\lambda}\right)^{1/(2+\frac{\mu}{\beta})} + 1\right]\right). \quad \text{(B4)}$$

In Eq. (B2) the first argument of Φ is $(z-1)^{2+\frac{\mu}{\beta}}(\lambda + 2\nu t)$, which sit in place of X as given in the form of Eq. (A4). Inserting $(z-1)^{2+\frac{\mu}{\beta}}(\lambda + 2\nu t)$ as X into Eq. (B4), we arrive at the following result:

$$\begin{aligned}
&\Phi((z-1)^{2+\frac{\mu}{\beta}}(\lambda + 2\nu\alpha t), C_2) \\
&= \psi\left((z-1)\left(\frac{\lambda+2\nu t}{\lambda}\right)^{\frac{1}{2+\frac{\mu}{\beta}}} + 1, C_2\left[(z-1)\left(\frac{\lambda+2\nu t}{\lambda}\right)^{\frac{1}{2+\frac{\mu}{\beta}}} + 1\right], 0\right) \\
&\quad - \frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}F\left((z-1)\left(\frac{\lambda+2\nu t}{\lambda}\right)^{\frac{1}{2+\frac{\mu}{\beta}}} + 1, C_2\left[(z-1)\left(\frac{\lambda+2\nu t}{\lambda}\right)^{\frac{1}{2+\frac{\mu}{\beta}}} + 1\right]\right).
\end{aligned} \quad \text{(B5)}$$

Now using the definition of c, which is $1 - (\frac{\lambda}{\lambda+2\nu\alpha t})^{\frac{1}{2+\frac{\mu}{\beta}}}$, we have

$$(z-1)\left(\frac{\lambda+2\nu t}{\lambda}\right)^{\frac{1}{2+\frac{\mu}{\beta}}} + 1 = \frac{z-c}{1-c}. \quad \text{(B6)}$$

Inserting this into (B5), we get

$$\Phi((z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t), C_2) = \psi\left(\frac{z-c}{1-c}, \frac{y}{z}\frac{z-c}{1-c}, 0\right) - \frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}F\left(\frac{z-c}{1-c}, \frac{y}{z}\frac{z-c}{1-c}\right). \quad \text{(B7)}$$

This is to be inserted into (A17).

APPENDIX C: TAKING THE INVERSE TRANSFORM OF $(z-1)^{2+\frac{\mu}{\beta}}F(z,y)$

From (A15) we have

$$(z-1)^{2+\frac{\mu}{\beta}}F(z,y) = (z-1)^{2+\frac{\mu}{\beta}}\sum_{m=0}^{\infty}\sum_{\theta=0}^{\infty}\rho(\theta)\frac{\Gamma\left(3+\frac{\mu}{\beta}+m-\theta\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)(m-\theta)!}y^{-\theta}\frac{z^{-m+\theta-2-\beta-\frac{\mu}{\beta}}}{m+2+\beta+\frac{\mu}{\beta}}. \quad \text{(C1)}$$

We need to take the inverse transform of this function in the k domain. That is, we need to write it in the form

$$F(z,y) = \sum_{k=0}^{\infty} f_k(y)z^{-k}, \quad \text{(C2)}$$

which implies that the coefficients will depend on y. The Taylor expansion of $(z-1)^{2+\frac{\mu}{\beta}}$ is given by

$$(z-1)^{2+\frac{\mu}{\beta}} = z^{2+\frac{\mu}{\beta}}(1-z^{-1})^{2+\frac{\mu}{\beta}} = z^{2+\frac{\mu}{\beta}}\sum_{r=0}^{\infty}\frac{\Gamma\left(3+\frac{\mu}{\beta}\right)}{r!\,\Gamma\left(3+\frac{\mu}{\beta}-r\right)}(-1)^r z^{-r}. \quad \text{(C3)}$$

Plugging this into (C1), we get

$$(z-1)^{2+\frac{\mu}{\beta}}F(z,y) = \sum_{m=0}^{\infty}\sum_{\theta=0}^{\infty}\sum_{r=0}^{\infty}\left[\rho(\theta)y^{-\theta}\frac{\Gamma\left(3+\frac{\mu}{\beta}+m-r-\theta\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)(m-r-\theta)!}\frac{\Gamma\left(3+\frac{\mu}{\beta}\right)(-1)^r}{r!\,\Gamma\left(3+\frac{\mu}{\beta}-r\right)}\frac{z^{-m-\beta+\theta}}{m-r+2+\beta+\frac{\mu}{\beta}}\right]. \quad \text{(C4)}$$

First note that the generating function for a discrete δ such as $\delta[k-q]$ for some integer q has the form z^{-q}. So for the inverse transform in the k domain we have

$$(z-1)^{2+\frac{\mu}{\beta}}F(z,y) \xrightarrow{\mathcal{Z}^{-1}} \sum_{m=0}^{\infty}\sum_{\theta=0}^{\infty}\sum_{r=0}^{\infty}\rho(\theta)y^{-\theta}\frac{\Gamma\left(3+\frac{\mu}{\beta}+m-r-\theta\right)}{(m-r-\theta)!}\frac{(-1)^r}{r!\,\Gamma\left(3+\frac{\mu}{\beta}-r\right)}\frac{\delta[k+\theta-m-\beta]}{m-r+2+\beta+\frac{\mu}{\beta}}. \quad \text{(C5)}$$

Using the properties of the δ function, we can perform the summation over m. Only on term from the sum over m will survive, which is the term that renders the argument of the δ equal to zero. Thus, we drop the sum over m and replace every m with

$k+\theta-\beta$. We get

$$(z-1)^{2+\frac{\mu}{\beta}}F(z,y)\xrightarrow{\mathcal{Z}^{-1}}\sum_{\theta=0}^{\infty}\sum_{r=0}^{k-\beta}\rho(\theta)y^{-\theta}\frac{\Gamma\left(3+\frac{\mu}{\beta}+k-\beta-r\right)}{(k-\beta-r)!}\frac{(-1)^r}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)}\frac{1}{k-r+2+\theta+\frac{\mu}{\beta}}. \tag{C6}$$

Now we use the following identity:

$$\sum_{r=0}^{k-\beta}\frac{\Gamma\left(3+\frac{\mu}{\beta}+k-r-\beta\right)(-1)^2}{(k-r-\beta)!r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)}\frac{1}{k+2-r+\theta+\frac{\mu}{\beta}}=\frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}. \tag{C7}$$

We have proved this identity in Appendix G . We use this identity to perform the summation over r in (C6) and transform (C6) into

$$(z-1)^{2+\frac{\mu}{\beta}}F(z,y)\xrightarrow{\mathcal{Z}^{-1}}\sum_{\theta=0}^{\infty}\rho(\theta)y^{-\theta}\frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}u(k-\beta). \tag{C8}$$

Now we can immediately take the inverse transform in the y domain. The sum on the right-hand side of (C8) is already in the form of $\sum_\theta b_\theta y^{-\theta}$, whose inverse transform (by definition) is b_θ. So we have

$$(z-1)^{2+\frac{\mu}{\beta}}F(z,y)\xrightarrow{\mathcal{Z}^{-1},y^{-1}}\rho(\theta)\frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}u(k-\beta). \tag{C9}$$

In the text, we have referred to the right-hand side of (C9) as $f_{k\theta}$. In other words, $(z-1)^{2+\frac{\mu}{\beta}}F(z,y)$ is the two-dimensional generating function of the coefficients $f_{k\theta}$.

APPENDIX D: TAKING THE INVERSE TRANSFORM OF (9)

We need to take the inverse transform of

$$\psi(z,y,t)=\psi\left(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c},0\right)+\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}\times\left[F(z,y)-F\left(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c}\right)\right], \tag{D1}$$

where, as given in (12), we know

$$(z-1)^{2+\frac{\mu}{\beta}}F(z,y)\xrightarrow{\mathcal{Z}^{-1}y^{-1}}\rho(\theta)\frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}u(k-\beta). \tag{D2}$$

First, let us define

$$z'\stackrel{\text{def}}{=}\frac{z-c}{1-c},\qquad y'\stackrel{\text{def}}{=}\frac{y}{z}\frac{z-c}{1-c}. \tag{D3}$$

Using these definitions, we can rewrite (D1) in the following form:

$$\psi(z,y,t)=\psi(z',y',0)+\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}F(z,y)-\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}F(z',y'). \tag{D4}$$

Note that we have

$$z'-1=\frac{z-c}{1-c}-1=\frac{z-1}{1-c}, \tag{D5}$$

from which we obtain

$$(z-1)^{2+\frac{\mu}{\beta}}=(z'-1)^{2+\frac{\mu}{\beta}}(1-c)^{2+\frac{\mu}{\beta}}=(z'-1)^{2+\frac{\mu}{\beta}}\left(\frac{\lambda}{\lambda+2\nu\alpha t}\right). \tag{D6}$$

Also note that $\frac{y}{z}=\frac{y'}{z'}$. Using this fact and inserting (D6) in the last term on the right-hand side of (D7), we get

$$\psi(z,y,t)=\psi(z',y',0)+\frac{(z-1)^{2+\frac{\mu}{\beta}}(\lambda+2\nu\alpha t)}{\beta}F(z,y)-\frac{(z'-1)^{2+\frac{\mu}{\beta}}\lambda}{\beta}F(z',y'). \tag{D7}$$

Now define the following function for brevity:

$$G(z,y)\stackrel{\text{def}}{=}(z-1)^{2+\frac{\mu}{\beta}}F(z,y). \tag{D8}$$

So (D7) can be expressed equivalently in the following compact form:

$$\psi(z,y,t)=\psi(z',y',0)+\frac{(\lambda+2\nu\alpha t)}{\beta}G(z,y)-\frac{\lambda}{\beta}G(z',y'). \tag{D9}$$

Now we focus on taking the inverse transform of this expression term by term.

Suppose there is a sequence $a_{k\theta}$ whose generating function is $\xi(z,y)$, which is defined by $\sum a_{k,\theta}z^{-k}y^{-\theta}$. What is the sequence $b_{k\theta}$, whose generating function is $\xi(z',y')$? This question is answered in Appendix E . The answer is

$$b_{k\theta}=(1-c)^\theta c^k\sum_{m=0}^{k}a_{m\theta}\left(\frac{1-c}{c}\right)^m\binom{k+\theta-1}{m+\theta-1}. \tag{D10}$$

Applying this to the first term in (D9) gives the inverse transform in terms of $N(0,k,\theta)$ readily. For the second term of

(D9), using the inverse transform of $G(z,y)$ given in (12), we have

$$\frac{(\lambda+2\nu\alpha t)}{\beta}G(z,y)\xrightarrow{z^{-1},y^{-1}}\frac{(\lambda+2\nu\alpha t)}{\beta}\rho(\theta)\frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}u(k-\beta). \tag{D11}$$

Finally, the inverse transform of the third term on the right-hand side of (D9) can be obtained by combining (D10) and (D11), that is, using the identity (D10), and using the right-hand side of (D11) as the coefficients $a_{m\theta}$. Putting the inverse transforms of all terms of the right-hand side of (D9) together, we obtain

$$\begin{aligned}N_t(k,\theta)=(1-c)^\theta c^k\sum_{m=0}^{k}N(0,m,\theta)\left(\frac{1-c}{c}\right)^m\binom{k+\theta-1}{m+\theta-1}+\frac{(\lambda+2\nu\alpha t)}{\beta}\rho(\theta)\frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}u(k-\beta)\\ -\frac{\lambda}{\beta}\rho(\theta)(1-c)^\theta c^k\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{(\beta+\theta-1)!}\sum_{m=\beta}^{k}\frac{(m+\theta-1)!}{\Gamma\left(m+3+\theta+\frac{\mu}{\beta}\right)}\left(\frac{1-c}{c}\right)^m\binom{k+\theta-1}{m+\theta-1}.\end{aligned} \tag{D12}$$

APPENDIX E: EXTRACTING THE INVERSE TRANSFORM OF $\xi(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c})$ FROM THAT of $\xi(z,y)$

Suppose we know the sequence $a_{k\theta}$, which satisfies

$$\begin{aligned}\xi(z,y)&=\sum_m\sum_n a_{mn}z^{-m}y^{-n},\\ a_{k\theta}&=\frac{1}{2\pi i}\oint_z\oint_y\xi(z,y)z^{k-1}y^{\theta-1}.\end{aligned} \tag{E1}$$

Replacing z and y with $\frac{z-c}{1-c}$ and $\frac{y}{z}\frac{z-c}{1-c}$, respectively, this transforms into

$$\xi\left(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c}\right)=\sum_m\sum_n a_{mn}\left(\frac{z-c}{1-c}\right)^{-m}\left(\frac{y}{z}\frac{z-c}{1-c}\right)^{-n}=\sum_m\sum_n a_{mn}(1-c)^{m+n}\frac{z^n y^{-n}}{(z-c)^{m+n}}. \tag{E2}$$

We seek $b_{k\theta}$ that satisfies

$$\xi\left(\frac{z-c}{1-c},\frac{y}{z}\frac{z-c}{1-c}\right)=\sum_k\sum_\theta b_{k\theta}z^{-k}y^{-\theta}. \tag{E3}$$

To take the inverse transform, we multiply (E2) by $z^{k-1}y^{\theta-1}$ and perform the contour integration encircling the origin and the totality of the unit disk to account for the pole at $z=c$ (noting that c is no larger than unity at all instances of time) and dividing by $\frac{1}{2\pi i}$. We have

$$b_{k\theta}=\frac{1}{(2\pi i)^2}\sum_m\sum_n a_{mn}(1-c)^{m+n}\oint_z\oint_y\frac{z^{n+k-1}y^{\theta-n-1}}{(z-c)^{m+n}}. \tag{E4}$$

The integral over y is readily evaluated, giving $2\pi i\delta[\theta-n]$, simplifying (E4) into

$$b_{k\theta}=\frac{1}{(2\pi i)}\sum_m a_{m\theta}(1-c)^{m+\theta}\oint_z\frac{z^{\theta+k-1}}{(z-c)^{m+\theta}}. \tag{E5}$$

To find the residue of $\frac{z^{\theta+k-1}}{(z-c)^{m+\theta}}$, we need to differentiate $z^{\theta+k-1}$ in the numerator $m+\theta-1$ times, evaluate the result at the pole, that is, $z=c$, and divide by $(m+\theta-1)!$. This is zero if $k<m-1$, and nonzero when $k\geqslant m$. Differentiating $z^{\theta+k-1}$ for $m+\theta-1$ times yields $z^{k-m}\prod_{j=0}^{j=m+\theta-2}(\theta+k-1-j)$, or, equivalently, $z^{k-m}\frac{(k+\theta-1)!}{(k-m)!}$. Evaluating this at $z=c$ and dividing by $(m+\theta-1)!$ simplifies (E5) into

$$b_{k\theta}=\sum_m a_{m\theta}(1-c)^{m+\theta}c^{k-m}\binom{k+\theta-1}{m+\theta-1}, \tag{E6}$$

which is what we sought.

APPENDIX F: ASYMPTOTIC ANALYSIS OF EQ. (16)

Let us repeat the equation for easy reference:

$$P(k,\theta)=\left(2+\frac{\mu}{\beta}\right)\rho(\theta)\frac{\Gamma(k+\theta)}{\Gamma(\beta+\theta)}\frac{\Gamma(\beta+2+\theta+\frac{\mu}{\beta})}{\Gamma(k+3+\theta+\frac{\mu}{\beta})}u(k-\beta). \tag{F1}$$

We seek the behavior of the joint distribution for large degrees. The leading asymptotic term for $\Gamma(x)$ for large x is $x^{x-\frac{1}{2}}\exp(-x)$, as can be found, for example, in [46]. Using this approximation for $\Gamma(k+\theta)$ in the numerator as well as for $\Gamma(k+3+\theta+\frac{\mu}{\beta})$ in the denominator, we get

$$\frac{\Gamma(k+\theta)}{\Gamma\left(k+\theta+3+\frac{\mu}{\beta}\right)} \sim \frac{(k+\theta)^{(k+\theta)}}{\left(k+\theta+3+\frac{\mu}{\beta}\right)^{(k+\theta+3+\frac{\mu}{\beta})}}. \tag{F2}$$

Now let us define

$$\mathcal{H}_k \stackrel{\text{def}}{=} \frac{(k+\theta)^{(k+\theta)}}{\left(k+\theta+3+\frac{\mu}{\beta}\right)^{(k+\theta+3+\frac{\mu}{\beta})}}. \tag{F3}$$

Taking the logarithm of both sides, we get

$$\begin{aligned}\ln\mathcal{H}_k &= (k+\theta)\ln(k+\theta) - \left(k+\theta+3+\frac{\mu}{\beta}\right)\ln\left(k+\theta+3+\frac{\mu}{\beta}\right)\\ &= (k+\theta)\ln(k+\theta) - \left(k+\theta+3+\frac{\mu}{\beta}\right)\left[\ln(k+\theta)+\ln\left(1+\frac{3+\frac{\mu}{\beta}}{k+\theta}\right)\right].\end{aligned} \tag{F4}$$

Using the Taylor expansion of the last logarithm, this transforms into

$$\ln\mathcal{H}_k \simeq -\left(3+\frac{\mu}{\beta}\right)\ln(k+\theta) - \left(k+\theta+3+\frac{\mu}{\beta}\right)\frac{3+\frac{\mu}{\beta}}{k+\theta} \tag{F5}$$

From this we obtain

$$\ln\mathcal{H}_k \sim -\left(3+\frac{\mu}{\beta}\right)\ln k, \tag{F6}$$

or, equivalently,

$$\mathcal{H}_k \sim k^{-(3+\frac{\mu}{\beta})}. \tag{F7}$$

Using this, the asymptotic form of (F1) for large degrees becomes

$$P(k,\theta) \sim \left[\left(2+\frac{\mu}{\beta}\right)\rho(\theta)\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma(\beta+\theta)}\right]k^{-(3+\frac{\mu}{\beta})}. \tag{F8}$$

Only the prefactor depends on θ, and the asymptotic dependence on k is the same for all values of θ. So we can write

$$P(k,\theta) \sim k^{-(3+\frac{\mu}{\beta})},\ \forall\,\theta. \tag{F9}$$

APPENDIX G: PROOF THE THE IDENTITY GIVEN IN EQ. (C7)

Let us repeat the identity for easy reference:

$$\sum_{r=0}^{k-\beta}\frac{\Gamma\left(3+\frac{\mu}{\beta}+k-r-\beta\right)(-1)^r}{(k-r-\beta)!\,r!\,\Gamma\left(3+\frac{\mu}{\beta}-r\right)}\frac{1}{k+2-r+\theta+\frac{\mu}{\beta}} = \frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}. \tag{G1}$$

We will denote the left-hand side of this equality by h_k. Define

$$\mathcal{H}(x) \stackrel{\text{def}}{=} \sum_k h_k x^k. \tag{G2}$$

Also, the following identities can be immediately proved through elementary Taylor expansions:

$$\sum_{j=0}^{\infty}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha)j!}x^j = (1-x)^{-\alpha}, \tag{G3}$$

$$\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-j)j!}x^j = (1+x)^{\alpha}. \tag{G4}$$

We will begin from the left-hand side of (G1). We have

$$\begin{aligned}
\mathcal{H}(x) &= \sum_k \sum_r \frac{\Gamma\left(3+\frac{\mu}{\beta}+k-r-\beta\right)}{(k-r-\beta)!} \frac{(-1)^r}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)} \frac{1}{k+2-r+\theta+\frac{\mu}{\beta}} x^k \\
&= \sum_r \frac{(-1)^r}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)} \sum_k \frac{\Gamma\left(3+\frac{\mu}{\beta}+k-r-\beta\right)}{(k-r-\beta)!} \frac{x^k}{k+2-r+\theta+\frac{\mu}{\beta}} \\
&= \sum_r \frac{(-1)^r}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)} x^{-2+r-\theta-\frac{\mu}{\beta}} \sum_k \frac{\Gamma\left(3+\frac{\mu}{\beta}+k-r-\beta\right)}{(k-r-\beta)!} \frac{x^{k+2-r+\theta+\frac{\mu}{\beta}}}{k+2-r+\theta+\frac{\mu}{\beta}} \\
&= \sum_r \frac{(-1)^r}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)} x^{-2+r-\theta-\frac{\mu}{\beta}} \sum_k \frac{\Gamma\left(3+\frac{\mu}{\beta}+k-r-\beta\right)}{(k-r-\beta)!} \int^x x^{k+1-r+\theta+\frac{\mu}{\beta}} dx \\
&= \sum_r \frac{(-1)^r\Gamma\left(3+\frac{\mu}{\beta}\right)}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)} x^{-2+r-\theta-\frac{\mu}{\beta}} \int^x \sum_k \frac{\Gamma\left(3+\frac{\mu}{\beta}+k-r-\beta\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)(k-r-\beta)!} x^{k+1-r+\theta+\frac{\mu}{\beta}} dx \\
&= \sum_r \frac{(-1)^r\Gamma\left(3+\frac{\mu}{\beta}\right)}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)} x^{-2+r-\theta-\frac{\mu}{\beta}} \int^x x^{1+\theta+\beta+\frac{\mu}{\beta}} \sum_k \frac{\Gamma\left(3+\frac{\mu}{\beta}+k-r-\beta\right)}{\Gamma\left(3+\frac{\mu}{\beta}\right)(k-r-\beta)!} x^{k-r-\beta} dx \\
&\overset{\text{(G3)}}{=} \sum_r \frac{(-1)^r\Gamma\left(3+\frac{\mu}{\beta}\right)}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)} x^{-2+r-\theta-\frac{\mu}{\beta}} \int^x x^{1+\theta+\beta+\frac{\mu}{\beta}} (1-x)^{-3-\frac{\mu}{\beta}} dx \\
&= \left[\sum_r \frac{(-x)^r\Gamma\left(3+\frac{\mu}{\beta}\right)}{r!\Gamma\left(3+\frac{\mu}{\beta}-r\right)} x^{-2-\theta-\frac{\mu}{\beta}}\right] \int^x x^{1+\theta+\beta+\frac{\mu}{\beta}} (1-x)^{-3-\frac{\mu}{\beta}} dx \\
&\overset{\text{(G4)}}{=} (1-x)^{2+\frac{\mu}{\beta}} x^{-2-\theta-\frac{\mu}{\beta}} \int^x x^{1+\theta+\beta+\frac{\mu}{\beta}} (1-x)^{-3-\frac{\mu}{\beta}} dx.
\end{aligned} \tag{G5}$$

Let us define

$$\begin{aligned}
f_1(x) &\overset{\text{def}}{=} (1-x)^{2+\frac{\mu}{\beta}} x^{-2-\theta-\frac{\mu}{\beta}}, \\
f_2(x) &\overset{\text{def}}{=} \int^x x^{1+\theta+\beta+\frac{\mu}{\beta}} (1-x)^{-3-\frac{\mu}{\beta}} dx.
\end{aligned} \tag{G6}$$

Then we can rewrite (G5) in the following compact form:

$$\mathcal{H}(x) = f_1(x) f_2(x). \tag{G7}$$

Taking the derivative of both sides, we get

$$\mathcal{H}'(x) = f_1(x) f_2'(x) + f_1'(x) f_2(x) = f_1(x) f_2'(x) + f_1'(x) \frac{\mathcal{H}(x)}{f_1(x)} = f_1(x) f_2'(x) + \frac{f_1'(x)}{f_1(x)} \mathcal{H}(x). \tag{G8}$$

From the definition of $f_1(x)$ and $f_2(x)$, the following can be reached through elementary algebraic steps:

$$\begin{aligned}
f_1(x) f_2'(x) &= (1-x)^{-1} x^{\beta-1}, \\
\frac{f_1'(x)}{f_1(x)} &= \frac{-\left(2+\theta+\frac{\mu}{\beta}\right)}{x(1-x)} + \frac{\theta}{1-x}.
\end{aligned} \tag{G9}$$

Using Taylor expansion, the following holds:

$$\frac{-\left(2+\theta+\frac{\mu}{\beta}\right)}{x(1-x)} + \frac{\theta}{1-x} = \frac{-\theta}{x} - \left(2+\frac{\mu}{\beta}\right) \sum_{k=-1}^{\infty} x^k. \tag{G10}$$

Plugging the expansion form of $\mathcal{H}(x)$ given in (G2) into (G8) and using (G10) and (G9), we get

$$\sum_k (k+1) h_{k+1} x^k = (1-x)^{-1} x^{\beta-1} + \left[-\frac{\theta}{x} - \left(2+\frac{\mu}{\beta}\right) \sum_{k=-1}^{\infty} x^k\right] \sum_k h_k x^k. \tag{G11}$$

Equating the coefficients of x^k on both sides, we get

$$(k+1)h_{k+1} = u(k+1-\beta) - \theta h_{k+1} - \left(2+\frac{\mu}{\beta}\right)\sum_{j=0}^{k+1} h_j. \tag{G12}$$

Let us write the same equation for k rather than $k+1$:

$$(k)h_k = u(k-\beta) - \theta h_k - \left(2+\frac{\mu}{\beta}\right)\sum_{j=0}^{k} h_j. \tag{G13}$$

Subtracting (G13) from (G12), we get

$$(k+1)h_{k+1} - kh_k = \delta[k+1-\beta] - \theta h_{k+1} + \theta h_k - \left(2+\frac{\mu}{\beta}\right)h_{k+1}. \tag{G14}$$

This can be expressed equivalently as follows:

$$h_{k+1} = \frac{(\theta+k)}{\left(k+3+\theta+\frac{\mu}{\beta}\right)}h_k + \frac{1}{\left(k+3+\theta+\frac{\mu}{\beta}\right)}\delta[k+1-\beta]. \tag{G15}$$

This first-order linear inhomogeneous difference equation has closed-form solutions that can be found for example in [47]. Here we find the solution by writing the first few terms and speculating the pattern. For $k=\beta-1$ we have

$$h_\beta = \frac{1}{\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}. \tag{G16}$$

For $k=\beta$ we have

$$h_{\beta+1} = \frac{(\beta+\theta)}{\left(\beta+2+\theta+\frac{\mu}{\beta}\right)\left(\beta+3+\theta+\frac{\mu}{\beta}\right)}. \tag{G17}$$

For $k=\beta+1$ we have

$$h_{\beta+2} = \frac{(\beta+\theta)(\beta+\theta+1)}{\left(\beta+2+\theta+\frac{\mu}{\beta}\right)\left(\beta+3+\theta+\frac{\mu}{\beta}\right)\left(\beta+4+\theta+\frac{\mu}{\beta}\right)}. \tag{G18}$$

The pattern is apparent. For general k we have

$$h_k = \frac{\prod_{j=0}^{k-\beta-1}(\beta+\theta+j)}{\prod_{j=0}^{k-\beta}\left(\beta+2+\theta+\frac{\mu}{\beta}+j\right)}. \tag{G19}$$

The numerator equals $\frac{(k+\theta-1)!}{\beta+\theta-1)!}$. Using the properties of the Γ function, namely the fact that $\Gamma(x+1) = x\Gamma(x)$, the denominator can be written as $\frac{\Gamma(k+2+\theta+\frac{\mu}{\beta})}{\Gamma(\beta+3+\theta+\frac{\mu}{\beta})}$. Plugging these two expressions into (G19), we arrive at

$$h_k = \frac{(k+\theta-1)!}{(\beta+\theta-1)!}\frac{\Gamma\left(\beta+2+\frac{\mu}{\beta}+\theta\right)}{\Gamma\left(k+3+\frac{\mu}{\beta}+\theta\right)}. \tag{G20}$$

This is identical to the right-hand side of (G1); hence, the proof is concluded.

APPENDIX H: PROOF OF THE IDENTITY GIVEN IN EQ. (23)

We repeat the identity for easy reference:

$$\sum_{k=\beta}^{\infty}\frac{k\Gamma(k+\theta)}{\Gamma\left(k+3+\theta+\frac{\mu}{\beta}\right)} = \frac{\beta\left(2+\frac{\mu}{\beta}+\frac{\theta}{\beta}\right)\Gamma(\beta+\theta)}{\left(1+\frac{\mu}{\beta}\right)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(2+\beta+\theta+\frac{\mu}{\beta}\right)}. \tag{H1}$$

First, we will prove the following identity for general positive integers n,m and arbitrary real positive number s:

$$\sum_{j=n}^{n+m}\frac{\Gamma(j)}{\Gamma(j+s)} = \frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{\Gamma(n+m+1)}{\Gamma(n+m+s)}}{(s-1)}. \tag{H2}$$

We prove this identity by mathematical induction on m. The initial step is to show the validity of (H2) for the case of $m = 0$. In this case, the left-hand side of (H2) has only one term, which is $\frac{\Gamma(n)}{\Gamma(n+s)}$. The right-hand side is

$$\frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{\Gamma(n+1)}{\Gamma(n+s)}}{(s-1)} = \frac{\frac{\Gamma(n)(n-1+s)}{\Gamma(n+s)} - \frac{n\Gamma(n)}{\Gamma(n+s)}}{(s-1)}, \tag{H3}$$

which simplifies to $\frac{\Gamma(n)}{\Gamma(n+s)}$. For the inductive step, assuming that (H2) holds for m, we write it for $m+1$:

$$\sum_{j=n}^{n+m+1} \frac{\Gamma(j)}{\Gamma(j+s)} = \sum_{j=n}^{n+m} \frac{\Gamma(j)}{\Gamma(j+s)} + \frac{\Gamma(n+m+1)}{\Gamma(n+m+1+s)}. \tag{H4}$$

The first term on the right-hand side can be substituted by the induction hypothesis, which is given in (H2). We get

$$\sum_{j=n}^{n+m+1} \frac{\Gamma(j)}{\Gamma(j+s)} = \frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{\Gamma(n+m+1)}{\Gamma(n+m+s)}}{(s-1)} + \frac{\Gamma(n+m+1)}{\Gamma(n+m+1+s)}. \tag{H5}$$

The right-hand side can be simplified as follows:

$$\begin{aligned}
&\frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{\Gamma(n+m+1)}{\Gamma(n+m+s)}}{(s-1)} + \frac{\Gamma(n+m+1)}{\Gamma(n+m+1+s)} \\
&= \frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{\Gamma(n+m+1)}{\Gamma(n+m+s)}}{(s-1)} + \frac{(s-1)\Gamma(n+m+2)}{(s-1)(n+m+1)\Gamma(n+m+1+s)} = \frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{\Gamma(n+m+1)}{\Gamma(n+m+s)} + \frac{(s-1)\Gamma(n+m+2)}{(n+m+1)\Gamma(n+m+1+s)}}{s-1} \\
&= \frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{(n+m+s)\Gamma(n+m+2)}{(n+m+1)\Gamma(n+m+1+s)} + \frac{(s-1)\Gamma(n+m+2)}{(n+m+1)\Gamma(n+m+1+s)}}{s-1} = \frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{(n+m+1)\Gamma(n+m+2)}{(n+m+1)\Gamma(n+m+1+s)}}{s-1} = \frac{\frac{\Gamma(n)}{\Gamma(n-1+s)} - \frac{\Gamma(n+m+2)}{\Gamma(n+m+1+s)}}{s-1}.
\end{aligned} \tag{H6}$$

This is the statement of induction for $m+1$, so the proof is complete.

Note that in the limit as $m \to \infty$, the second term in the denominator of the right-hand side of (H2) vanishes. So we arrive at the following identity:

$$\sum_{j=n}^{\infty} \frac{\Gamma(j)}{\Gamma(j+s)} = \frac{\Gamma(n)}{(s-1)\Gamma(n-1+s)}. \tag{H7}$$

Now let us use identity (H7) with $n = \beta + \theta$ and $s = 3 + \frac{\mu}{\beta}$. The sum becomes $\sum_{j=\beta+\theta}^{\infty} \frac{\Gamma(j)}{\Gamma(j+3+\frac{\mu}{\beta})}$. This can be equivalently expressed as $\sum_{k=\beta}^{\infty} \frac{\Gamma(k+\theta)}{\Gamma(k+\theta+3+\frac{\mu}{\beta})}$. From (H7) we get

$$\sum_{k=\beta}^{\infty} \frac{\Gamma(k+\theta)}{\Gamma\left(k+\theta+3+\frac{\mu}{\beta}\right)} = \frac{\Gamma(\beta+\theta)}{\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)}. \tag{H8}$$

Now let us use (H7) for $n = \beta + \theta + 1$ and $s = 2 + \frac{\mu}{\beta}$. We have

$$\sum_{j=\beta+\theta+1}^{\infty} \frac{\Gamma(j)}{\Gamma\left(j+2+\frac{\mu}{\beta}\right)} = \sum_{k=\beta}^{\infty} \frac{\Gamma(k+\theta+1)}{\Gamma\left(k+\beta+1+2+\frac{\mu}{\beta}\right)} = \sum_{k=\beta}^{\infty} \frac{(k+\theta)\Gamma(k+\theta)}{\Gamma\left(k+\beta+3+\frac{\mu}{\beta}\right)} \overset{(H7)}{=} \frac{\Gamma(\beta+\theta+1)}{\left(1+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)}. \tag{H9}$$

Let us multiply (H8) by θ and write the result together with (H9):

$$\sum_{k=\beta}^{\infty} \frac{\theta\Gamma(k+\theta)}{\Gamma\left(k+\theta+3+\frac{\mu}{\beta}\right)} = \frac{\theta\Gamma(\beta+\theta)}{\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)}, \quad \sum_{k=\beta}^{\infty} \frac{(k+\theta)\Gamma(k+\theta)}{\Gamma\left(k+\beta+3+\frac{\mu}{\beta}\right)} = \frac{\Gamma(\beta+\theta+1)}{\left(1+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)}. \tag{H10}$$

We subtract the first equation from the second one. The left-hand side will be the same as the left-hand side of (H1). The right-hand side is

$$\begin{aligned}
&\frac{\Gamma(\beta+\theta+1)}{\left(1+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)} - \frac{\theta\Gamma(\beta+\theta)}{\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)} = \frac{\Gamma(\beta+\theta+1)\left(2+\frac{\mu}{\beta}\right) - \theta\Gamma(\beta+\theta)\left(1+\frac{\mu}{\beta}\right)}{\left(1+\frac{\mu}{\beta}\right)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)} \\
&= \frac{(\beta+\theta)\Gamma(\beta+\theta)\left(2+\frac{\mu}{\beta}\right) - \theta\Gamma(\beta+\theta)\left(1+\frac{\mu}{\beta}\right)}{\left(1+\frac{\mu}{\beta}\right)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)} = \frac{\Gamma(\beta+\theta)\left[(\beta+\theta)\left(2+\frac{\mu}{\beta}\right) - \theta\left(1+\frac{\mu}{\beta}\right)\right]}{\left(1+\frac{\mu}{\beta}\right)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)} \\
&= \frac{\Gamma(\beta+\theta)[\theta+\mu+2\beta]}{\left(1+\frac{\mu}{\beta}\right)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)} = \frac{\beta\left[2+\frac{\theta}{\beta}+\frac{\mu}{\beta}\right]\Gamma(\beta+\theta)}{\left(1+\frac{\mu}{\beta}\right)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+\theta+2+\frac{\mu}{\beta}\right)}.
\end{aligned} \tag{H11}$$

This is identical to the right-hand side of (H1). Hence, the proof is concluded.

APPENDIX I: SOLVING DIFFERENCE EQ. (30)

Let us repeat the difference equation for easy reference:

$$n(k,\theta,\ell,\phi) = \frac{(\ell-1+\phi)n(k,\theta,\ell-1,\phi)}{2+\frac{\mu}{\beta}+k+\ell+\theta+\phi} + \frac{(k-1+\theta)n(k-1,\theta,\ell,\phi)}{2+\frac{\mu}{\beta}+k+\ell+\theta+\phi} + \rho(\phi)\delta_{\ell,\beta}\frac{(k-1+\theta)P(k-1,\theta)}{2+\frac{\mu}{\beta}+k+\ell+\theta+\phi}. \quad \text{(I1)}$$

Let us define the new sequence $m(k,\theta,\ell,\phi) = \frac{\Gamma(3+\frac{\mu}{\beta}+k+\ell+\theta+\phi)}{(k-1+\theta)!(\ell-1+\phi)!}n(k,\theta,\ell,\phi)$. Using this substitution and applying the properties of the Γ function as well as the δ function, we can rewrite (I1) equivalently as

$$m(k,\theta,\ell,\phi) = m(k,\theta,\ell-1,\phi) + m(k-1,\theta,\ell,\phi) + \frac{\Gamma\left(2+\frac{\mu}{\beta}+k+\beta+\theta+\phi\right)}{(k-1+\theta)!(\beta-1+\phi)!}\rho(\phi)\delta_{\ell,\beta}(k-1+\theta)P(k-1,\theta). \quad \text{(I2)}$$

Using the expression in (16) to rewrite the last term on the right-hand side of this equation, we can express it equivalently as follows:

$$m(k,\theta,\ell,\phi) = m(k,\theta,\ell-1,\phi) + m(k-1,\theta,\ell,\phi) + \rho(\phi)\rho(\theta)\delta_{\ell,\beta}\left(2+\frac{\mu}{\beta}\right)\frac{\Gamma\left(2+\frac{\mu}{\beta}+k+\beta+\theta+\phi\right)}{(\beta-1+\theta)!(\beta-1+\phi)!}\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma\left(k+2+\theta+\frac{\mu}{\beta}\right)}. \quad \text{(I3)}$$

Now define the generating function $\psi(z,\theta,y,\phi) = \sum_k m(k,\theta,\ell,\phi)z^{-k}y^{-\ell}$. Multiplying both sides of (I2) by $z^{-k}y^{-\ell}$, summing over all values of k,ℓ and rearranging the terms, we arrive at

$$\psi(z,\theta,y,\phi) = \frac{\rho(\phi)\rho(\theta)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{(\beta-1+\theta)!(\beta-1+\phi)!}\sum_{j=\beta+1}^{\infty}\frac{\Gamma\left(2+\frac{\mu}{\beta}+j+\beta+\theta+\phi\right)}{\Gamma\left(j+2+\theta+\frac{\mu}{\beta}\right)}\frac{z^{-j}y^{-\beta}}{1-z^{-1}-y^{-1}}. \quad \text{(I4)}$$

[The lower bound of the sum is $\beta+1$ because $P(k-1,\theta)$ is zero for $k<\beta+1$.] The inverse transform of the factor $\frac{z^{-j}y^{-\beta}}{1-z^{-1}-y^{-1}}$ in the summand can be taken through the following steps:

$$\begin{aligned}\frac{z^{-j}y^{-\beta}}{1-z^{-1}-y^{-1}} &\xrightarrow{z^{-1}} \frac{1}{(2\pi i)^2}\oint\oint\frac{z^{k-j-1}y^{\ell-\beta-1}}{1-z^{-1}-y^{-1}}dzdy = \frac{1}{(2\pi i)^2}\oint\oint\frac{z^{k-j}y^{\ell-\beta}}{z-\frac{y}{y-1}}\frac{1}{y-1}dzdy\\ &= \frac{1}{(2\pi i)}\oint\oint\frac{y^{\ell-\beta}}{y-1}\left(\frac{y}{y-1}\right)^{k-j}dzdy = \frac{1}{(k-j)!}\frac{d^{k-j}}{dy^{k-j}}y^{k+\ell-\beta-j}\bigg|_{y=1} = \binom{k-j+\ell-\beta}{\ell-\beta}.\end{aligned} \quad \text{(I5)}$$

So we can invert (I4) term by term. We get

$$m(k,\theta,\ell,\phi) = \frac{\rho(\phi)\rho(\theta)\left(2+\frac{\mu}{\beta}\right)\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{(\beta-1+\theta)!(\beta-1+\phi)!}\sum_{j=\beta}^{\infty}\frac{\Gamma\left(2+\frac{\mu}{\beta}+k+\beta+\theta+\phi\right)}{\Gamma\left(k+2+\theta+\frac{\mu}{\beta}\right)}\binom{k-j+\ell-\beta}{\ell-\beta}. \quad \text{(I6)}$$

From this, we readily obtain

$$\begin{aligned}n(k,\theta,\ell,\phi) = \rho(\phi)\rho(\theta)\left(2+\frac{\mu}{\beta}\right)&\frac{\Gamma\left(\beta+2+\theta+\frac{\mu}{\beta}\right)}{\Gamma\left(3+\frac{\mu}{\beta}+k+\ell+\theta+\phi\right)}\frac{(k-1+\theta)!(\ell-1+\phi)!}{(\beta-1+\theta)!(\beta-1+\phi)!}\\ &\times\sum_{j=\beta}^{k}\frac{\Gamma\left(2+\frac{\mu}{\beta}+j+\beta+\theta+\phi\right)}{\Gamma\left(j+2+\theta+\frac{\mu}{\beta}\right)}\binom{k-j+\ell-\beta}{\ell-\beta}.\end{aligned} \quad \text{(I7)}$$

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