Dynamical stationarity as a result of sustained random growth

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(Received 21 November 2016; revised manuscript received 13 February 2017; published 17 March 2017)

In sustained growth with random dynamics stationary distributions can exist without detailed balance. This suggests thermodynamical behavior in fast-growing complex systems. In order to model such phenomena we apply both a discrete and a continuous master equation. The derivation of elementary rates from known stationary distributions is a generalization of the fluctuation-dissipation theorem. Entropic distance evolution is given for such systems. We reconstruct distributions obtained for growing networks, particle production, scientific citations, and income distribution.

DOI: 10.1103/PhysRevE.95.032130

I. INTRODUCTION

Statistical physics methods are applied to problems related to complex system evolution in an increasing manner. While these are powerful enough to describe essential properties of statistical data and their distributions, the meaning of parameters behind such distributions can be understood more deeply if derived from dynamical models. Following the principle of Occam's razor (among competing hypotheses, the one with the fewest assumptions should be selected), simple rules for the dynamics are welcome.

The dynamics of many complex systems can be studied by using a simple master-equation approach [1–4]. Besides physics, such studies are also popular in network science [5–9], biology [10], economics [11], chemistry [12], epidemics [13,14], scientometrics [15,16], and sociology [17]. Generally, such dynamical processes tend to a stationary state with an invariant limiting distribution [18].

In the master equation approach to the evolution of general probability distributions, we know several statements for systems satisfying the detailed balance condition in their stationary state [19], but much less is known for fast-growing complex systems without detailed balance. In particular, if the microprocesses are not reversible, the entropy growth and the global stability of stationary solutions are not guaranteed even for generalized entropies. Such cases occur in open systems.

In this paper we investigate a promising subset of unbalanced master equations leading to stationary distributions. Such an approach can be applied to understand several complex phenomena. We refer to application examples for emerging particle distributions in high-energy accelerator experiments to income distributions following from redistribution and taxation strategies to scientific citation dynamics and to evolution of growing complex networks.

We focus on dynamics where in an elementary step only growth transitions are allowed from a state with n quanta to a state with $(n+1)$ quanta. A first master equation for such a process writes as

$$\frac{dN_n(t)}{dt} = \mu_{n-1}N_{n-1}(t) - \mu_n N_n(t), \tag{1}$$

where $N_n(t)$ denotes the number of elements with n quanta at time moment t and μ_n is the time-independent transition rate from a state with n quanta to a state with $n+1$ quanta. Such a process alone will not lead to any nontrivial stationary distribution. One can easily realize this by adding up the equations (1) from $n=1$ to ∞:

$$\sum_{n=1}^{\infty}\frac{dN_n}{dt} = \mu_0 N_0. \tag{2}$$

The total number of elements in the system is

$$N(t) = \sum_{n=0}^{\infty} N_n(t) = N_0(t) + \sum_{n=1}^{\infty} N_n(t). \tag{3}$$

For a constant number of elements, $dN/dt = 0$ and Eq. (2) leads to an exponential relaxations of N_0 to 0:

$$\frac{dN_0}{dt} = \frac{dN}{dt} - \mu_0 N_0 = -\mu_0 N_0. \tag{4}$$

This recursively will lead to an exponential relaxation to 0 for all N_n numbers.

In order to get a final stationary distribution, one must complement this process with a finite chance for "resetting" the state with n quanta to the state 0 or to consider a constant dilution of the system by increasing the total number of elements, N. We illustrate such processes by the dynamics sketched in Fig. 1(a).

The master equation is now written as

$$\frac{dN_n(t)}{dt} = \mu_{n-1}N_{n-1}(t) - \mu_n N_n(t) - \beta_n N_n(t), \tag{5}$$

where β_n is the rate of resetting a state with n quanta (naturally, $\beta_0 = 0$). Repeating our previous arguments we get:

$$\frac{dN_0}{dt} = \frac{dN}{dt} + \sum_{n=1}^{\infty}\beta_n N_n - \mu_0 N_0. \tag{6}$$

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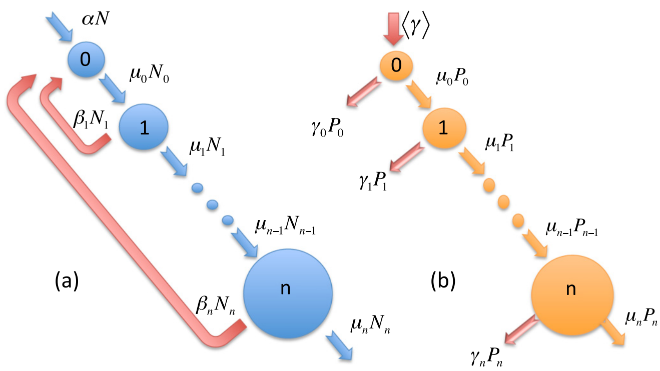

FIG. 1. Schematic illustration for the growth process in the N_n number dynamics (a) and in the corresponding probability dynamics (b).

The total number of elements in the system, N, may increase or not, and in general a nontrivial stationary distribution might emerge.

Turning now, instead of $N_n(t)$, to the probability that a state has n quanta, $P_n(t) = N_n(t)/N(t)$, we replace $N_n(t) = N(t)P_n(t)$ into Eq. (5) to obtain:

$$N\frac{dP_n}{dt} + P_n\frac{dN}{dt} = \mu_{n-1}NP_{n-1} - (\mu_n + \beta_n)NP_n. \tag{7}$$

Dividing this by N and rearranging, one obtains the master equation:

$$dP_n/dt = \mu_{n-1}P_{n-1} - \left(\mu_n + \beta_n + \frac{1}{N}\frac{dN}{dt}\right)P_n. \tag{8}$$

Assuming now a Hubble-like exponential expansion of the system (true for many complex system) $dN/dt = \alpha N$, the master equation of the process writes with time-independent rates

$$\frac{dP_n}{dt} = \mu_{n-1}P_{n-1} - (\mu_n + \gamma_n)P_n, \tag{9}$$

where $\gamma_n = \beta_n + \alpha$. The dynamics in $P_n(t)$ is illustrated in Fig. 1(b).

Possible physical interpretations of the rate γ_n are various. For citation of publications (and for other similar popularity measures) $N_n(t)$ is the number of publications cited n times in a given period of time, t, while $N(t)$ is the total number of publications. Observations indicate an exponential growth in the total number of papers [20]. $P_n(t)$ in this case stands for the fraction of publications among all which were cited exactly n times.

In gambling games and financial market models the dependence of the growth rate μ_n on the already-achieved state can expresses a "rich gets richer" preference. Concurrently, β_n describes the probability rate to lose everything in a single step (characteristic to Black Fridays and similarly rare total resetting events). On the evolution time scale of life on Earth, massive extinctions can also lead to a constant rate β.

In high-energy physics a quark-gluon plasma blob can transform from containing the energy for n hadrons to $n+1$ with the rate μ_n, while it remelts all hadrons with the rate β_n.

Finally, in complex networks n is the number of connections of a node and $P_n(t)$ is the degree distribution. The rate of

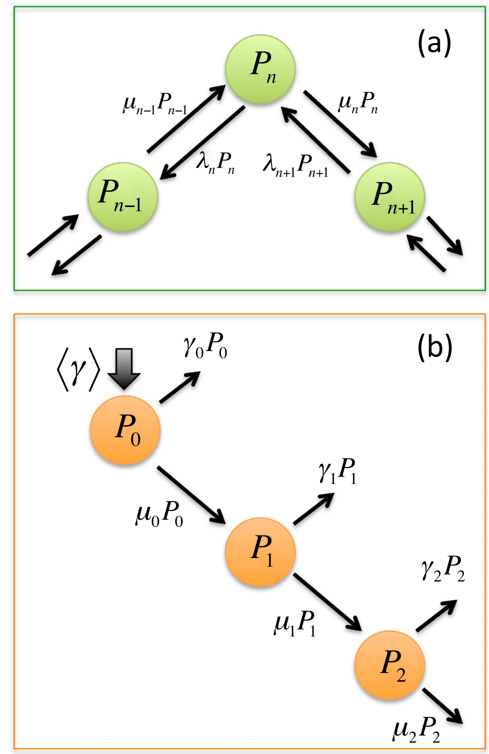

FIG. 2. Schematic view of master equations for balanced (a) and sustained random growth (b) processes. In the balanced growth we denoted by λ_n the reverse rates.

acquiring a new connection while already having n is μ_n, while the probability to become isolated by some technical misfortune or directed attack on the network is β_n. In such applications the number of nodes can also increase in an exponential manner with a rate α.

In our framework, the stationary distribution, Q_n, is determined by two microscopic rates: μ_n describes the transition rate from a state with n quanta to $n+1$ inside a chain of states, while γ_n describes a loss rate for the state n towards zero via an unspecified accident and/or an exponential dilution. We assume that there is no n to $n-1$ process, so the transition dynamics is unidirectional. Already for constant and linearly n-dependent rates a rich structure of possible solutions emerges.

Since there is no reverse process inside the chain of states, a detailed balance condition cannot be fulfilled. We illustrate the difference between the classical scheme allowing detailed balance and the presently discussed one-sided growth picture with the flow diagrams on Fig. 2.

State-dependent loss rates, $\gamma_n \neq 0$, open the door to nontrivial stationary distributions. Usually models are constructed with assumed transition rates μ_n and γ_n and the stationary (limiting) distribution, Q_n, is derived. However, the reverse problem is also interesting: By observing a distribution, Q_n, and knowing the interaction rate with the environment, γ_n, one wishes to reconstruct the internal dynamics of the system governed by the rates, μ_n.

We present both a master equation approach over discrete states labeled by n and its continuous limit. Finally, the stability of stationary distributions obtained from given transition rates is investigated in terms of a generalized entropic distance.

II. MASTER AND FLOW EQUATION FRAMEWORK

Now we turn to the definition of the underlying mathematical formalism. We consider linear and first-order time evolution equations for the distribution, $P_n(t)$, and its continous version, $P(x,t)$. The corresponding stationary distributions, Q_n and $Q(x)$, respectively, shall be determined by the same equations with vanishing time derivative. Beyond finding out what stationary distributions, i.e., results of the long-term evolution, belong to given rates μ_n,γ_n [or $\mu(x)$, $\gamma(x)$], one is interested in the whole process starting from arbitrary initial distributions as well as in the stability and basin of attraction for the final distribution.

A. Discrete state space master equation

The sustained growth master equation hereafter is given as depicted in the lower part of Fig. 2, repeating Eq. (9):

$$\dot{P}_n = \mu_{n-1}P_{n-1} - (\mu_n+\gamma_n)P_n \tag{10}$$

for $n \geqslant 1$. The corresponding equation for the $n=0$ term can be obtained from the normalization condition $\sum_{n=0}^{\infty} P_n(t) = 1$:

$$\dot{P}_0 = \langle\gamma\rangle_P - (\mu_0+\gamma_0)P_0. \tag{11}$$

Here we used the abbreviation $\langle\gamma\rangle_P = \sum_{n=0}^{\infty}\gamma_n P_n$. This system allows for stationary solutions satisfying:

$$\mu_{n-1}Q_{n-1} = (\mu_n+\gamma_n)Q_n \tag{12}$$

for $n \geqslant 1$ and $Q_0 = \langle\gamma\rangle_Q/(\mu_0+\gamma_0)$. Equations (10) and (11) constitute a specific realization of a general, continuous-time Markov process:

$$\dot{P}_n = \sum_m (w_{n\leftarrow m}P_m - w_{m\leftarrow n}P_n) \tag{13}$$

with

$$w_{n\leftarrow m} = \mu_m\delta_{m,n-1} + \gamma_m\delta_{n,0}. \tag{14}$$

The inflow and outflow in each patch in Fig. 2 balance each other in the stationary state. This also offers a strategy to reconstruct the link connection probability rate to a link with already m connections or to increase a conveniently discretized income from m to $m+1$, μ_m, by observing the stationary distribution, Q_n, and the loss rate, γ_n. We simply sum up Eq. (12) from $n=m+1$ to infinity and obtain

$$\mu_m = \frac{1}{Q_m}\sum_{n=m+1}^{\infty}\gamma_n\, Q_n. \tag{15}$$

This relation is like the fluctuation-dissipation theorem, in particular when the stationary distribution is exponential, $Q_n = e^{-\beta n}/Z = (1-q)q^n$, and the loss rate due to environmental effects is constant $\gamma_n = \gamma$. In this case Eq. (15) delivers a constant inner-chain rate that is similar to the quantum Kubo formula:

$$\mu_m^{\exp} = \gamma\,\frac{q}{1-q} = \gamma\,\frac{1}{e^{\beta}-1}. \tag{16}$$

The general solution of the recursion represented by Eq. (12) is given as a ratio of n-fold products,

$$Q_n = Q_0\,\frac{\prod_{i=0}^{n-1}\mu_i}{\prod_{j=1}^{n}(\mu_j+\gamma_j)}. \tag{17}$$

Q_0 can either be obtained from the normalization condition $\sum_{n=0}^{\infty}Q_n = 1$ or by applying Eq. (11) with $\dot{Q}_0 = 0$. It is not trivial that these are equivalent procedures: The product form (17) and the definition of the expectation value delivers

$$Q_0 = \frac{\langle\gamma\rangle_Q}{\mu_0+\gamma_0} = Q_0\sum_{n=0}^{\infty}\frac{\gamma_n}{\mu_n}\prod_{i=0}^{n}\frac{\mu_i}{\mu_i+\gamma_i}. \tag{18}$$

Consistency can easily be reformulated in terms of the basic ratios, $r_i = \gamma_i/\mu_i$, after dividing both sides with $Q_0 \neq 0$:

$$1 = \sum_{n=0}^{\infty} r_n\prod_{i=0}^{n}\frac{1}{1+r_i}. \tag{19}$$

It is at the first glance surprising, but true, that this identity is fulfilled for any infinite series of $r_i \neq -1$ ratios. A short mathematical proof is given in the Appendix.

B. Continuum approach

It is instructive to obtain the above equations in a continuous version. We set up the following Markovian framework:

$$\frac{\partial}{\partial t}P(x,t) = \int [w(x,y)P(y,t) - w(y,x)P(x,t)]\,dy \tag{20}$$

with

$$w(x,y) = \frac{1}{\Delta x}\mu(y)\,\delta(y-x+\Delta x) + \gamma(y)\,\delta(x). \tag{21}$$

Next we take the $\Delta x \to 0$ limit leading to

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}(\mu(x)P(x,t)) - \gamma(x)P(x,t) + \langle\gamma\rangle_P\delta(x). \tag{22}$$

Please note that this is an integrodifferential equation containing

$$\langle\gamma\rangle_P = \int\gamma(y)P(y,t)\,dy - \mu(0)P(0,t). \tag{23}$$

Equation (22) describes a flow with general velocity field, $\mu(x)$, a loss rate, $\gamma(x)$, and a feeding at $x=0$. From now on we restrict the discussion to $x>0$, and all effects stemming from the singular term $\langle\gamma\rangle_P\delta(x)$ are treated by enforcing the normalization condition.

The large-N limit in the familiar approach suppresses the diffusion term by $1/N$ and leaves one with a conserved flow equation [10]. It is equivalent with Eq. (22) for $\gamma(x)\equiv 0$.

The stationary distribution in the continuous model satisfies

$$\frac{d}{dx}(\mu(x)Q(x)) = -\gamma(x)\,Q(x), \tag{24}$$

revealing the solution

$$Q(x) = \frac{K}{\mu(x)}\,e^{-\int\frac{\gamma(x)}{\mu(x)}dx}. \tag{25}$$

The constant, K, is specified by the normalization $\int_0^\infty Q(x)dx = 1$.

We note that this form can also directly be obtained from the discrete solution, Eq. (17), when it is written in the alternative form

$$Q_n = Q_0 \frac{\mu_0}{\mu_n}\, e^{-\sum_{j=1}^{n} \ln(1+\frac{\gamma_j}{\mu_j})}. \tag{26}$$

The approximation, $\gamma_j/\mu_j = \gamma(x)\Delta x/\mu(x) \ll 1$, defines the continuous limit and one arrives at

$$Q_n \approx Q_0 \frac{\mu_0}{\mu_n}\, e^{-\sum_{j=1}^{n} \frac{\gamma(j\Delta x)}{\mu(j\Delta x)} \Delta x}, \tag{27}$$

to an obvious analog of Eq. (25) with $K = \mu(0)Q(0)\Delta x$. Here it is obvious that γ_n and μ_n must scale differently in the continuum limit.

The inner-chain growth rate, $\mu(x)$, can be reconstructed from the known stationary distribution, $Q(x)$, and loss rate, $\gamma(x)$:

$$\mu(x) = \frac{1}{Q(x)} \int_x^\infty \gamma(u) Q(u)\, du. \tag{28}$$

The validity of this formula is tested by applying a derivation with respect to x and Eq. (24). For the exponential distribution, $Q(x) \sim e^{-x/T}$, and constant $\gamma(x) = \gamma$ we obtain the analog of the classical Kubo formula cf. Eq. (16),

$$\mu^{\exp}(x) = \gamma\, T. \tag{29}$$

The temperature-like parameter in the exponential distribution, T, becomes a factor between two elementary rates γ and μ. In a physical picture γ describes dissipation due to drastic resettings from x to zero, $\mu(x)$ random advances towards larger x values.

III. PARTICULAR RATES AND DISTRIBUTIONS

In the followings we discuss the simplest choices for the involved rates. First we keep the loss rate a positive constant, $\gamma_n = \gamma > 0$, and vary the growth rate, μ_n. This is relevant for a wide class of distributions considered in statistics. For a constant $\mu_j = \sigma$ we obtain the *geometrical distribution*,

$$Q_n = \frac{\gamma}{\sigma}(1 + \gamma/\sigma)^{-1-n}, \tag{30}$$

or $Q_n = (1-q)q^n$, with $q = \sigma/(\sigma + \gamma)$. This is also called the *exponential*, or *Boltzmann-Gibbs, distribution* in the form $Q_n = e^{-\beta n}/Z$ with $Z = 1 + \sigma/\gamma$ and $\beta = \ln(1 + \gamma/\sigma) > 0$.

For fast-growing systems, like networks, citations, or energetic hadronization, the most prevalent is the next simplest case, $\mu_j = \sigma(j + b)$, describing a growth rate with thresholded linear preference. Often $b = 1$ is taken when investigating the evolution of aggregates [21]. Equation (17) delivers

$$Q_n = \frac{\gamma}{\sigma b + \gamma} \frac{(b)_n}{(b + 1 + \gamma/\sigma)_n} \tag{31}$$

with the Pochhammer symbol:

$$(b)_n = b(b+1)\cdots(b+n-1) = \frac{\Gamma(b+n)}{\Gamma(b)}. \tag{32}$$

The *Waring distribution* [22–24] in Eq. (31) has a *power-law tail* for large n,

$$\lim_{n\to\infty} Q_n \propto n^{-1-\gamma/\sigma}. \tag{33}$$

This limit is based on the leading-order behavior of Gamma functions for large arguments:

$$\lim_{n\to\infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1. \tag{34}$$

Our result Eq. (31) coincides with Eq. (7) in Ref. [21] at $b = 1$. The asymptotic power is steeper than -1 for positive rate factors γ and σ, but it can be anything, depending on the ratio of the universal driving rate γ and the preference scale of the individual growth rates, $\sigma = \mu_n - \mu_{n-1}$. For $\gamma \to 0^+$ the stationary distribution tail (33) leads to *Zipf's law* [25].

Now we analyze these particular growth rates by a constant loss rate, $\gamma(x) = \gamma$, in the continuous model. We expect the same asymptotic behavior for the tail of the distribution. For a constant growth rate, $\mu(x) = \sigma$, the stationary probability density function (PDF) becomes again the *exponential distribution*,

$$Q(x) = \frac{\gamma}{\sigma}\, e^{-(\gamma/\sigma)x}. \tag{35}$$

This is the $\gamma \ll \sigma$ limit of the result in the discrete case, Eq. (30). For a linearly preferential rate, $\mu(x) = \sigma(x + b)$, we obtain a *cut power law* in the form of the *Tsallis-Pareto distribution* [25–29],

$$Q(x) = \frac{\gamma}{b\sigma}\left(1 + \frac{x}{b}\right)^{-1-\gamma/\sigma}. \tag{36}$$

Beyond these reassuring results a further question arises: What is the stationary distribution for weaker or stronger than linear preferences in the attachment probability rate [30,31]? By assuming a general power, $\mu(x) = \sigma(x + b)^a$, one obtains the *stretched exponential distribution*,

$$Q(x) = \frac{\gamma}{\sigma(x+b)^a}\, e^{-\alpha(x+b)^{1-a}} \tag{37}$$

for $a < 1$ with $\alpha = \gamma/[\sigma(1-a)]$. For $a > 1$ it delivers a $Q(x) \sim \gamma/\mu(x)$ tail. Equation (37) represents also a three-parameter *Weibull distribution*, with $a = 1 - k$, $b = -\theta$, and $\gamma/\sigma = k\lambda^{-k}$ [32]. On the other hand, for an exponential preference rate, $\mu(x) = \sigma e^{\alpha x}$, one obtains the *Gompertz distribution* [33]:

$$Q(x) = \frac{\gamma/\sigma}{1 - e^{-\gamma/\sigma\alpha}}\, e^{-\alpha x}\, e^{-\frac{\gamma}{\alpha\sigma}(1-e^{-\alpha x})}. \tag{38}$$

We note by passing that the particular form for the stationary distribution, $Q(x)$ in Eq. (25) with constant γ, is term by term compatible with the usual notions used in survival analysis in demography, finance and insurance statistics [34]: The PDF has in this case the form

$$Q(x) = h(x)\, e^{-H(x)} \tag{39}$$

with $h(x)$ being the *hazard rate* and $H(x) = \int_0^x h(t)dt$ the *cumulative hazard*. The factor $R(x) = e^{-H(x)}$ is called *survival rate*. The growth rate inside the chain is simply related to the hazard rate: $\mu(x) = \gamma/h(x)$. This is again a special case for the fluctuation-dissipation correspondence summarized in

Eq. (28). The same relation has been called the truncated expectation value theorem in Refs. [35,36]. A similar result has been derived by generalizing the thermodynamical fluctuation-dissipation relation between the diffusion and damping coefficients for a general Fokker-Planck equation stemming from a particularly colored, i.e., energy-dependent, multiplicative noise Langevin equation [37,38] and Eq. (5.46) in Ref. [39].

Finally, we mention two important examples, frequently encountered in complex system applications, which do not fit into the above scheme. We consider loss rates, γ_n, which can be *negative* for some low n. Such a mechanism has been suggested in Ref. [40] for describing the multiplicity distribution of hadrons in high-energy collision events. The linear rates

$$\gamma_n = \sigma(n - kf), \qquad \mu_n = \sigma f(n+k) \tag{40}$$

will lead to a *negative binomial* stationary distribution:

$$\begin{aligned} Q_n &= Q_0 \frac{(f\sigma)^n \prod_{j=0}^{n-1}(j+k)}{[\sigma(1+f)]^n \prod_{i=1}^{n} i} \\ &= \binom{n+k-1}{n} f^n (1+f)^{-n-k}. \end{aligned} \tag{41}$$

We note that in this case $\langle\gamma\rangle_Q = 0$. In order to achieve a normalized stationary distribution, obviously $\gamma_n + \mu_n > 0$ for all n.

A similar arrangement of the rates in the continuous model,

$$\gamma(x) = \sigma(ax - c), \qquad \mu(x) = \sigma x, \tag{42}$$

leads to the *two-parameter gamma distribution*,

$$Q(x) = \frac{K}{\sigma x} e^{-\int (a-c/x)\,dx} = \frac{a^c}{\Gamma(c)} x^{c-1} e^{-ax}. \tag{43}$$

This stationary distribution emerges as a result of a pure (unthresholded) linear preference in the growth rate and a linear, but not overall positive, loss rate to the environment. The negative values of $\gamma(x)$ actually mean a feeding from the environment (Fig. 3).

Such a gamma distribution fits income data very well [41]. We risk the conclusion that in the background of such processes, beyond the linear prefrence rate, $\mu(x) = \sigma x$ (often cited as the Matthias principle in market economies), a taxation and a social welfare redistribution system acts.

Finally, the question about the rapidness of establishing a stationary distribution in various systems arises naturally. The characteristic approach from an arbitrary $P_n(0)$ distribution towards Q_n in the simplest cases contains an exponential factor $e^{-\gamma t}$ for a state-independent reset rate $\gamma_n = \gamma$. In general, however, it is a lengthy calculation to solve for the time-dependent $P_n(t)$. The introduction of $\xi(x,t) = P(x,t)/Q(x)$ in the continuous model approach is somewhat simpler, delivering a flow picture on this ratio:

$$\frac{\partial \xi}{\partial t} + \mu(x)\frac{\partial \xi}{\partial x} = 0. \tag{44}$$

Its solution is soliton-like, diminishing the deviation from $Q(x)$ in a manner depending on $\mu(x)$. For constant $\mu(x) = \sigma$, one obtains $\xi(x,t) = \xi(x - \mu t, 0)$, and for a linear thresholded preference rate, $\mu(x) = \sigma(x+b)$, the solution becomes $\xi(x,t) = \xi(z,0)$ with $z = xe^{-\sigma t} - b(1 - e^{-\sigma t})$. We save further details on this for a future work.

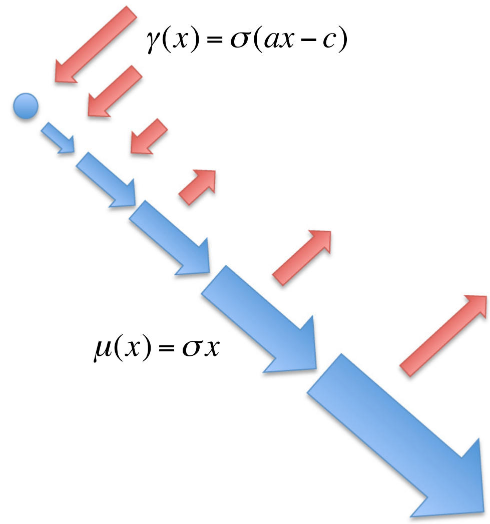

FIG. 3. Schematic illustration of the linear rates leading to the gamma distribution: At x below the average the environment feeds the chain, above it detracts from the system. Since $\langle\gamma\rangle_Q = 0$, there is no extra feed at the beginning of the chain.

IV. EVOLUTION OF ENTROPIC DISTANCE

The entropy-probability connection is also interpreted as a measure of a distance to the minimal information state. The well-known Boltzmann-Gibbs-Shannon formula is a special instance of the more general entropic distance, $\rho(P,\Pi)$, between two distributions. For such generalized entropic distances, the following requirement should hold: $\rho(P,\Pi) \geqslant 0$ and reaches zero only for identical distributions [$\rho(P,P) = 0$ and $\rho(P,\Pi) > 0$ for $P \neq \Pi$]. We consider in this paper univariate distributions, P_n, Π_n indexed with a natural number, n, and normalized as $\sum_{n=0}^{\infty} P_n = 1$ and $\sum_{n=0}^{\infty} \Pi_n = 1$, respectively.

By considering random dynamics in fast-growing complex systems, dominantly unidirectional changes in the quantity n are considered. The general question arises whether there exists a quantity, possibly expressed as an expectation value of a function of the respective probability values at the same state indexed by n, which changes only in one direction during the dynamical evolution. In particular, the entropic distance to a stationary distribution, Q_n, from any starting distribution, $P_n = P_n(0)$, should decrease during such an evolution:

$$\frac{d}{dt}\rho(P,Q) \leqslant 0. \tag{45}$$

A trace form for the entropic distance from a nonconstant stationary Q_n is given by

$$\rho(P,Q) = \sum_{n=0}^{\infty} \mathfrak{s}(P_n, Q_n)\, Q_n. \tag{46}$$

It is very common to deal with entropic distances defined via a function of the ratio of the respective probabilities only, $\mathfrak{s}(\xi_n)$ with $\xi_n = P_n/Q_n$. Then, from the property of zero distance from itself one concludes that $\mathfrak{s}(1) = 0$, and differs from zero if there is an index n such that $\xi_n \neq 1$.

The change of the entropic distance is governed by its definition and the evolution equation for the distribution. The entropic distance of an actual, time-dependent distribution, $P_n(t)$, to the stationary distribution, Q_n, has the trace form [42]:

$$\rho = \sum_n \mathfrak{s}(\xi_n) Q_n. \tag{47}$$

For all concave $\mathfrak{s}(\xi)$ functions, the following Jensen inequality applies:

$$\rho \geqslant \mathfrak{s}\left(\sum_n \xi_n Q_n\right) = \mathfrak{s}\left(\sum_n P_n\right) = \mathfrak{s}(1) = 0. \tag{48}$$

The time derivative of this entropic distance is given by

$$\dot{\rho} = \sum_n \mathfrak{s}'(\xi_n)\dot{\xi}_n Q_n = \sum_n \mathfrak{s}'(\xi_n)\dot{P}_n. \tag{49}$$

Now we utilize Eq. (13) and obtain

$$\dot{\rho} = \sum_{n,m} \mathfrak{s}'(\xi_n)\,[w_{n\leftarrow m} Q_m \xi_m - w_{n\leftarrow m} Q_n \xi_n]. \tag{50}$$

As a first step, we write $\xi_m = \xi_n + (\xi_m - \xi_n)$ and use the property

$$0 = \sum_m [w_{n\leftarrow m} Q_m \; - \; w_{m\leftarrow n} Q_n] \tag{51}$$

for the stationary state. The above formula transforms to

$$\dot{\rho} = \sum_{n,m} \mathfrak{s}'(\xi_n)\,(\xi_m - \xi_n) w_{n\leftarrow m} Q_m. \tag{52}$$

In the second step, we use the remainder theorem for Taylor series in its Lagrange form:

$$\mathfrak{s}(\xi_m) = \mathfrak{s}(\xi_n) + (\xi_m - \xi_n)\mathfrak{s}'(\xi_n) + \tfrac{1}{2}(\xi_m - \xi_n)^2 \mathfrak{s}''(c_{nm}) \tag{53}$$

with the internal point c_{nm} lying between ξ_n and ξ_m. Expressing the first-order term in Eq. (53), Eq. (52) becomes

$$\dot{\rho} = \sum_{m,n} [\mathfrak{s}(\xi_m) - \mathfrak{s}(\xi_n)] w_{n\leftarrow m} Q_m - \frac{1}{2}\sum_{n,m} (\xi_m - \xi_n)^2 \mathfrak{s}''(c_{nm})\, w_{n\leftarrow m} Q_m. \tag{54}$$

Here the first sum on the right-hand side vanishes due to the stationarity (51). This can be seen by exchanging the summation indices m and n in the first part, leading to

$$\sum_n \mathfrak{s}(\xi_n) \sum_m [w_{m\leftarrow n} Q_n - w_{n\leftarrow m} Q_m] = 0. \tag{55}$$

For positive transition rates, $w_{n\leftarrow m} > 0$, the remainder term is always negative for concave, $\mathfrak{s}''(\xi) > 0$ functions.

In the special case of the avalanche dynamics with loss, $w_{n\leftarrow m} = \mu_m \delta_{m,n-1} + \gamma_m \delta_{n,0}$, we obtain

$$\dot{\rho} = -\frac{1}{2}\sum_n (\xi_n - \xi_{n-1})^2 \mathfrak{s}''(c_{n,n-1})\mu_{n-1} Q_{n-1} - \frac{1}{2}\sum_n (\xi_n - \xi_0)^2 \mathfrak{s}''(c_{n,0})\gamma_n Q_n. \tag{56}$$

For positive rates γ_n and μ_n therefore $\dot{\rho} < 0$ unless the stationary state is achieved where $\xi_n = 1$ for all n.

Finally, let us briefly discuss cases when some γ_n can be negative. We encountered this for processes leading to negative binomial or gamma distributions. The remainder result (56) in such a case does not guarantee a steady approach towards the stationary distribution in terms of a general entropic distance. Henceforth further investigations are necessary also with respect to nonequilibrium thermodynamics [43].

V. CONCLUSION

In the present work we have proposed a unified mathematical framework based on a master equation approach to complex systems governed by random dynamics. In particular, we have focused on transition rates which do not lead to a detailed balance. A wide variety of stationary distributions known from complex network research, particle physics, scientometrics, econophysics, biology, and demography are successfully reproduced.

This view is able to clarify why only the linear preference rate leads to a power-law tailed degree distribution in random networks as well as to a Pareto-type distribution of wealth when the preference expressed by "the rich gets richer" principle is linear. Similarly, the distribution of scientific citations, known to be power-law tailed, has been explained earlier on the basis of such an evolution equation [15]. The exponential (geometrical) distribution is obtained for constant rates and the power-law tailed Waring (in the continuum limit Tsallis-Pareto) distribution for a linear pereference growth rate. The method outlined in this paper is able to deliver further well-known and frequently used distributions, such as the Weibull or the Gompertz distribution or the stretched exponential.

Beyond the above-mentioned practical application possibilities, we have established connections to the fundamental fluctuation-dissipation relation central in statistical physics. In the simplified version with a constant loss rate, $\gamma_n = \gamma$, the stationary PDF, $Q(x)$, are proved to be related to quantities familiar from general statistics: The necessary growth rate is reciprocal to the hazard rate, $\mu(x) = \gamma/h(x)$. The correspondence between this hazard rate and the cumulated hazard was generalized to a "fluctuation-dissipation"-type relation between the growth rate, $\mu(x)$, and the loss rate, $\gamma(x)$, in Eq. (28). A similar general relation was derived for the discrete version in Eq. (15). The specific case $m = 0$ gives the key to reconstruct the first attachment rate μ_0 from observing Q_n and measuring $\langle\gamma\rangle_Q$: $\mu_0 = \langle\gamma\rangle_Q/Q_0 - \gamma_0$.

Finally, while seeking answer to the question regarding which entropy formula could be the optimal one for such unbalanced growth processes in random systems, we proved that any entropic distance based on a general concave function of the probability ratio, $\mathfrak{s}(\xi)$, will decrease to zero for $\gamma(x) > 0$.

Generalizing further the dynamics for γ_n containing both positive and negative elements, we have discussed two models. First, with $\gamma_n = \sigma(n - fk)$ and $\mu_n = \sigma f(n + k)$, a model for high-energy hadron production, leading to a negative binomial stationary distribution was evoked. Second, with $\gamma(x) = \sigma(ax - b)$ and $\mu(x) = \sigma x$, a continuous model for the income distribution was recited. This model assumes a constant percentage taxation and social welfare amendments, leading to a gamma distribution.

The unified mathematical treatment outlined in this paper should be a primary tool in understanding intriguing universality classes reported in complex systems. Important questions are left open for further research, including what the precise conditions are for entropy growth in cases involving partially negative $\gamma(x)$ rates [while $\gamma(x) + \mu(x) > 0$ is still satisfied], what the minimal conditions are for gaining a stationary distribution in unbalanced random processes, or how the transient dynamics towards the stationary state is displayed with time.

ACKNOWLEDGMENTS

This work has been supported by NKFIH (Project No. 104260), and by a University Babeş-Bolyai STAR fellowship. Discussions with A. Telcs and Zs. Lázár are gratefully acknowledged. Z.N. acknowledges support by UEFISCDI (Grant No. PN-II-IDPCE-2011-3-0348).

APPENDIX

Here we prove Eq. (19). We define the summed expression of product chain as

$$S_0 = \sum_{n=0}^{\infty} r_n \prod_{i=0}^{n} \frac{1}{1+r_i}. \tag{A1}$$

The first few terms are

$$S_0 = \frac{r_0}{1+r_0} + \frac{1}{1+r_0}\frac{r_1}{1+r_1} + \frac{1}{1+r_0}\frac{1}{1+r_1}\frac{r_2}{1+r_2} + \dots. \tag{A2}$$

By rearranging the sum starting at the second term,

$$S_0 = \frac{r_0}{1+r_0} + \frac{1}{1+r_0}\left(\frac{r_1}{1+r_1} + \frac{1}{1+r_1}\frac{r_2}{1+r_2} + \dots\right), \tag{A3}$$

we realize that

$$S_0 = \frac{r_0}{1+r_0} + \frac{1}{1+r_0} S_1 \tag{A4}$$

with an obvious notation, S_1, for the same infinite sum starting with terms containing r_1. After a linear rearrangement it is convincing that this relation,

$$(S_0 - S_1) = r_0(1 - S_0), \tag{A5}$$

holds for an arbitrary r_0 if and only if $S_1 = S_0 = 1$. The same proof is valid for starting at any mth element. $S_0 = 1$ proves the original statement.

[1] H. P. Breuer and F. Petruccione, Continuum Mech. Thermodyn. **7**, 439 (1995).
[2] N. Boccara, *Modeling Complex Systems* (Springer, New York, 2004).
[3] R. Cohen and S. Havlin, *Complex Networks: Structure, Robustness, and Function* (Cambridge University Press, Cambridge, 2010).
[4] M. Mitchell, *Complex Systems: A Gudied Tour* (Oxford University Press, Oxford, 2009).
[5] J. Schnakenberg, Rev. Mod. Phys. **48**, 571 (1976).
[6] A. L. Barabási, R. Albert, and H. Jeong, Physica A **272**, 173 (1999).
[7] R. Albert and A. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[8] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. Lett. **85**, 4633 (2000).
[9] S. N. Dorogovtsev and J. F. F. Mendes, Phys. Rev. E **62**, 1842 (2000).
[10] L. R. de Oliveira, G. Castellani, and G. Turchetti, Commun. Nonlinear Sci. Numer. Simul. **20**, 461 (2015).
[11] R. N. Mantegna and H. E. Stanley, *Introduction to Econophysics* (Cambridge University Press, Cambridge, 1999).
[12] N. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (Elsevier, Oxford, 2007).
[13] M. E. J. Newman, Phys. Rev. E **66**, 016128 (2002).
[14] M. E. J. Newman, *Networks: An Introduction* (Oxford University Press, New York, 2010).
[15] A. Schubert and W. Glänzel, Scientometrics **6**, 149 (1984).
[16] A. L. Barabási, H. Jeong, Z. Néda, E. Ravasz, A. Schubert, and T. Vicsek, Physica A **311**, 590 (2002).
[17] P. Sen and B. K. Chakrabarti, *Sociophysics: An Introduction* (Oxford University Press, Oxford, 2013).
[18] R. G. Gallager, *Stochastic Processes: Theory for Applications* (Cambridge University Press, New York, 2014).
[19] T. S. Biró and Z. Néda, Physica A **474**, 355 (2017).
[20] Z. Néda, L. Varga, and T. S. Biró, *Science and Facebook: The same popularity law!* (2017), arXiv:1701.05347.
[21] L. Kullmann and J. Kertész, Phys. Rev. E **63**, 051112 (2001).
[22] J. O. Irwin, J. Roy. Stat. Soc. A **138**, 18 (1975); **138**, 374 (1975).
[23] J. Rodríguez-Avi, A. Conde-Sánchez, A. J. Sáez-Castillo, and M. J. Olmo-Jiménez, Comput. Stat. Data Anal. **51**, 6138 (2007).
[24] H. Jeong, Z. Néda, and A. L. Barabási, Europhys. Lett. **61**, 567 (2003).
[25] G. K. Zipf, *Human Behavior and Principle of Least Effort* (Addison-Wesley, Cambridge, MA, 1949).
[26] V. Pareto, Giornale degli Economisti **10**, 59 (1895).
[27] S. Thurner, F. Kyriakopoulos, and C. Tsallis, Phys. Rev. E **76**, 036111 (2007).
[28] C. Tsallis and M. P. de Albequerque, Eur. J. Phys. B **13**, 777 (2000).
[29] S. N. Dorogovtsev, J. F. F. Mendes, Nat. Phys. **11**, 882 (2015).
[30] P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).
[31] P. L. Krapivsky, G. J. Rodgers, and S. Redner, Phys. Rev. Lett. **86**, 5401 (2001).

[32] W. Weibull, J. Appl. Mech.-Trans. ASME **18**, 293 (1951).
[33] B. Gompertz, Philos. Trans. R. Soc. London **115**, 513 (1825).
[34] M. Rausand and A. Høyland, *System Reliability Theory: Models, Statistical Methods and Applications*, 2nd ed. (John Wiley & Sons, Hoboken, NJ, 2004).
[35] A. Telcs, W. Glänzel, and A. Schubert, Math. Soc. Sci. **10**, 169 (1985).
[36] W. Glänzel, A. Telcs, and A. Schubert, Z. Wahrscheinlichkeitstheorie verw, Gebiete **66**, 173 (1984).
[37] T. S. Biró and A. Jakovác, Phys. Rev. Lett. **94**, 132302 (2005).
[38] T. S. Biró, G. Purcsel, G. Györgyi, A. Jakovác, and Z. Schram, Nucl. Phys. A **774**, 845 (2006).
[39] T. S. Biró, Is there a temperature? In *Energy Dependent Noise*, Fundamental Theories of Physics 171, Chap. 5.3 (Springer, New York, 2011).
[40] T. Osada, N. Nakajima, M. Biyajima, and N. Suzuki, Prog. Theor. Phys. **98**, 1289 (1997).
[41] V. M. Jakovenko and J. B. Rosser Jr, Rev. Mod. Phys. **81**, 1703 (2009).
[42] T. S. Biró and Z. Schram, Entropy **18**, 42 (2016).
[43] T. Tomé and M. J. de Oliveira, Phys. Rev. Lett. **108**, 020601 (2012).