Self-averaging in the random two-dimensional Ising ferromagnet

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We study sample-to-sample fluctuations in a critical two-dimensional Ising model with quenched random ferromagnetic couplings. Using replica calculations in the renormalization group framework we derive explicit expressions for the probability distribution function of the critical internal energy and for the specific heat fluctuations. It is shown that the disorder distribution of internal energies is Gaussian, and the typical sample-to-sample fluctuations as well as the average value scale with the system size L like $\sim L \ln\ln(L)$. In contrast, the specific heat is shown to be self-averaging with a distribution function that tends to a δ peak in the thermodynamic limit $L \to \infty$. While previously a lack of self-averaging was found for the free energy, we here obtain results for quantities that are directly measurable in simulations, and implications for measurements in the actual lattice system are discussed.

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I. INTRODUCTION

A varying degree of impurities is present in every material studied in the laboratory. The consequences of disorder vary strongly from system to system, however. While for strong disorder randomness is accompanied by frustration effects and often leads to the absence of long-range order [1], the case of weak disorder is less spectacular in that it cannot destroy the low-temperature ferromagnetic ground state [2–4]. Still, in many cases one observes a change in the character of the transition to this ferromagnetic phase. For pure systems with continuous phase transitions, as revealed by Harris [5], weak disorder is relevant for the critical behavior only if the specific heat is divergent, i.e., the corresponding critical exponent $\alpha > 0$, as in these cases the random fluctuations grow faster with system size than the energy fluctuations. The critical behavior is then governed by a new, random renormalization-group (RG) fixed point, and the pure fixed point becomes unstable. On the other hand, first-order phase transitions in pure systems are softened by the addition of weak disorder and, in some cases, they are turned into continuous transitions [6,7]. These effects of weak disorder have been thoroughly studied both analytically [8–11] as well as numerically [12–14], see Ref. [15] for a review.

An intriguing aspect of systems with quenched disorder is related to the possibility of exceedingly strong disorder induced fluctuations. In some cases, these might lead to a loss of self-averaging [16–19], i.e., the behavior of a large sample with a specific realization of impurities such as an actual material sample in the laboratory will not be well described by the ensemble average normally calculated in an analytical or numerical approach. This clearly has profound consequences for the physical interpretation of the outcomes and the possibilities for comparing theoretical and experimental results. The presence or absence of self-averaging is connected to the question of the relevance of disorder for the system studied [17,18], and it affects static as well as dynamic properties [20,21]. Recently, an explicit expression for the probability distribution function of the critical free-energy fluctuations for a weakly disordered Ising ferromagnet was derived for $d < 4$ and its universal shape was obtained at $d = 3$ [22]. As free energies are not directly accessible in experimental or numerical studies, however, it is desirable to study the self-averaging properties of directly measurable quantities.

A system of particular interest is the Ising model in two dimensions (2D), where a wealth of exact results are available for the pure case [23]. When weak disorder in the form of random but nonfrustrating bonds is added, the Harris criterion is unable to decide its significance as $\alpha = 0$ and the system hence provides a marginal case. Still, it is now well established that such weak disorder "marginally" modifies the critical behavior of this system so the logarithmic singularity of the specific heat is changed into a double logarithmic one [24–27]. While a number of further aspects of this problem have been studied, such as the effect of correlated disorder in the form of extended impurities [28], the question of the disorder distribution of measurable quantities and their (lack of) self-averaging behavior was less studied. Following the seminal works by Wiseman and Domany and Aharony and Harris [16,17], the relative variance of thermodynamic observables was usually studied as a measure to gauge the presence or absence of self-averaging. It was shown that for irrelevant disorder the relative variance weakly decreases as a power of L, indicating the presence of "weak self-averaging," while for relevant disorder this ratio approaches a nonzero constant as $L \to \infty$, indicating a lack of self-averaging [17]. Results of numerical studies of this quantity for the disordered 2D Ising model [16,20,26,29], where the disorder is marginally relevant, were not completely conclusive. Here we derive the form of the distribution functions of sample-to-sample fluctuations and discuss their asymptotics as $L \to \infty$.

The rest of the paper is organized as follows. In Sec. II we recall the description of the critical 2D Ising model in terms of free Majorana fermions and show how this description can be extended to the disordered system. In Sec. III we introduce the replica formalism for the energy distribution of the model. Section IV contains the RG calculations for the disorder distribution of the internal energy, where we show that it lacks self-averaging at criticality. The typical value of its sample-to-sample fluctuations scales with the system size L in the same way as its average $\sim L \ln\ln(L)$. In Sec. V we extend this calculation to the specific heat and see that in contrast to the energy it is self-averaging, and its distribution turns into a δ function in the limit $L \to \infty$. Finally, Sec. VI contains our conclusions.

II. THE MODEL

It is well known that the critical behavior of the 2D ferromagnetic Ising model can be described in terms of free two-component Grassmann-Majorana spinor fields $\psi(\mathbf{r}) = (\psi_1(\mathbf{r}), \psi_2(\mathbf{r}))$ with the following Hamiltonian (see, e.g., Ref. [30]):

$$H_0[\psi;\tau] = \frac{1}{2}\int d^2r\,[\overline{\psi}(\mathbf{r})\hat{\partial}\psi(\mathbf{r}) + \tau\,\overline{\psi}(\mathbf{r})\psi(\mathbf{r})], \tag{1}$$

where $\tau \propto (T - T_c)/T_c \ll 1$ and T_c denotes the critical temperature [in what follows, to simplify formulas we define $\tau \equiv (T - T_c)/T_c$)]. Further,

$$\hat{\partial} = \hat{\sigma}_1\frac{\partial}{\partial x} + \hat{\sigma}_2\frac{\partial}{\partial y}, \tag{2}$$

where

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \hat{\sigma}_1\,\hat{\sigma}_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{3}$$

are the Pauli matrices and $\overline{\psi} \equiv \psi\,\hat{\sigma}_3$. At a given value of the temperature parameter τ the partition function $Z(\tau)$ of the system (1) is

$$Z(\tau) = \int \mathcal{D}\psi\,\exp\{-H_0[\psi;\tau]\}, \tag{4}$$

where the integration measure is defined as

$$\int \mathcal{D}\psi = \prod_{\mathbf{r}}\left[-\int d\psi_1(\mathbf{r})\,d\psi_2(\mathbf{r})\right], \tag{5}$$

and the integration and commutation rules are

$$\int d\psi_\alpha(\mathbf{r}) = 0,$$
$$\int d\psi_\alpha(\mathbf{r})\,\psi_\alpha(\mathbf{r}) = -\int \psi_\alpha(\mathbf{r})\,d\psi_\alpha(\mathbf{r}) = 1, \tag{6}$$

$$\psi_\alpha(\mathbf{r})\psi_\beta(\mathbf{r}') = -\psi_\beta(\mathbf{r}')\psi_\alpha(\mathbf{r})\,, \quad [\psi_\alpha(\mathbf{r})]^2 = 0. \tag{7}$$

Hence the free energy is

$$F(\tau) = -\ln[Z(\tau)]. \tag{8}$$

Note that we did not include the usual temperature prefactor in the definition of the free energy (8). Our analysis is performed close to T_c, which for simplicity is taken to be 1, and we are looking *only* for the leading terms (singularities) in the parameter $\tau = (T - T_c)/T_c = T - 1$. Therefore, in the limit $\tau \to 0$,

$$F(\tau) = -T\,\ln[Z(\tau)] = -\ln[Z(\tau)] - \tau\,\ln[Z(\tau)]$$
$$= -\ln[Z(\tau)] + O(\tau). \tag{9}$$

Simple integration of Eq. (4) yields

$$Z(\tau) = [\det(\hat{\partial} + \tau\hat{\sigma}_0)]^{1/2}, \tag{10}$$

where $\hat{\sigma}_0$ is the unit matrix, and the term on the right-hand side is a symbolic notation for the determinant of the $L^2 \times L^2$ matrix defining the Hamiltonian (1) written in a discrete way on an $L \times L$ lattice. The free energy reads

$$F(\tau) = -\frac{1}{2}\ln[\det(\hat{\partial} + \tau\hat{\sigma}_0)]$$
$$\sim -L^2\int_{|p|<1} d^2p\,\ln(p^2 + \tau^2). \tag{11}$$

Note that the celebrated logarithmic divergence of the specific heat in the limit $\tau \to 0$ follows immediately from Eq. (11):

$$C(\tau) = -\frac{\partial^2}{\partial\tau^2}F(\tau) \sim L^2\int_{|p|<1}\frac{d^2p}{p^2 + \tau^2}$$
$$\sim L^2\int_{|\tau|}^{1}\frac{dp}{p} \sim L^2\ln\frac{1}{|\tau|}. \tag{12}$$

The presence of weak quenched disorder in the considered system can be described by allowing for a spatially varying local transition temperature T_c which, in turn, can be represented by quenched spatial fluctuations of the temperature parameter τ in the Hamiltonian (1) (see, e.g., Ref. [24]). In other words, the critical behavior of the weakly disordered 2D Ising model can be described by the spinor Hamiltonian

$$H[\psi;\tau,\delta\tau] = \frac{1}{2}\int d^2r\,\{\overline{\psi}(\mathbf{r})\hat{\partial}\psi(\mathbf{r})$$
$$+ [\tau + \delta\tau(\mathbf{r})]\overline{\psi}(\mathbf{r})\psi(\mathbf{r})\}, \tag{13}$$

where the random function $\delta\tau(\mathbf{r})$ is characterized as a spatially uncorrelated Gaussian distribution with zero mean, $\overline{\delta\tau(\mathbf{r})} = 0$, and variance

$$\overline{\delta\tau(\mathbf{r})\delta\tau(\mathbf{r}')} = 2g_0\,\delta(\mathbf{r} - \mathbf{r}'), \tag{14}$$

where the parameter $g_0 \ll 1$ defines the disorder strength. For a given realization of the quenched function $\delta\tau(\mathbf{r})$ the partition function of the considered system is

$$Z[\tau;\delta\tau] = \int \mathcal{D}\psi\,\exp\{-H[\psi;\tau,\delta\tau]\} = \exp\{-F[\tau;\delta\tau]\}, \tag{15}$$

where $F[\tau;\delta\tau]$ is a random free-energy function. The internal energy of a given realization is the first derivative of this free energy with respect to the temperature parameter:

$$E[\tau;\delta\tau] = \frac{\partial}{\partial\tau}F[\tau;\delta\tau]. \tag{16}$$

It is clear that $E[\tau;\delta\tau]$ must be a singular function of τ in the limit $\tau \to 0$ (in the pure system $E_0(\tau) \sim \tau\,\ln(1/|\tau|)$). Additionally, $E[\tau;\delta\tau]$ also must be a *random* function exhibiting sample-to-sample fluctuations. The distribution function of these fluctuations is the main target of the present study.

III. REPLICA FORMALISM

From the definition (16) we have

$$E[\tau;\delta\tau] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(F[\tau+\epsilon;\delta\tau] - F[\tau;\delta\tau]). \tag{17}$$

Thus for a given finite value of ϵ (which has to be sent to zero at the end) we have

$$\epsilon\, E[\tau;\delta\tau] = F[\tau+\epsilon;\delta\tau] - F[\tau;\delta\tau]. \tag{18}$$

According to the definition of the free energy, Eq. (15), the above relation can be represented in terms of the ratio of two partition functions,

$$\exp\{-\epsilon\, E[\tau;\delta\tau]\} = Z[\tau+\epsilon;\delta\tau]\, Z^{-1}[\tau;\delta\tau]. \tag{19}$$

Taking the Nth power of both sides of the above equation and performing the disorder average we find

$$\int dE\; P_\tau(E)\; \exp(-\epsilon N E) = \overline{Z^N[\tau+\epsilon;\delta\tau]\, Z^{-N}[\tau;\delta\tau]}. \tag{20}$$

Here $P_\tau(E)$ is the probability distribution over disorder of the internal energy of the system at a given value of the temperature parameter τ and $\overline{(...)}$ denotes the average over the random functions $\delta\tau(\mathbf{r})$. Following the standard tricks of the replica formalism the above relation can be represented in the following way:

$$\begin{aligned}\int dE\; P_\tau(E)\; \exp(-\epsilon N E) &= \lim_{M\to 0} \overline{Z^N[\tau+\epsilon;\delta\tau]\, Z^{M-N}[\tau;\delta\tau]} \\ &\equiv \lim_{M\to 0}\; \mathcal{Z}(M,N;\tau,\epsilon).\end{aligned} \tag{21}$$

In terms of this formalism, first it is assumed that both M and N are integers such that $M > N$. Then, after deriving $\mathcal{Z}(M,N;\tau,\epsilon)$ as an analytic function of M and N, these parameters are analytically continued to arbitrary real values and the limit $M \to 0$ is taken. Finally, we introduce a new analytic parameter $s = \epsilon N$ and, provided that it exists, take the limit $\epsilon \to 0$, such that the relation (21) becomes the Laplace transform of the probability distribution function $P_\tau(E)$,

$$\begin{aligned}\int dE\; P_\tau(E)\; \exp(-s\, E) &= \lim_{\epsilon\to 0}\lim_{M\to 0}\; \mathcal{Z}(M,s/\epsilon;\tau,\epsilon) \equiv \tilde{\mathcal{Z}}(s,\tau).\end{aligned} \tag{22}$$

Thus, the above procedure, although it is not well founded from a mathematical point of view, at least formally allows to reconstruct the function $P_\tau(E)$ by the inverse Laplace transform:

$$P_\tau(E) = \int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\; \tilde{\mathcal{Z}}(s,\tau)\exp(sE). \tag{23}$$

To proceed, consider the structure of the replica partition function $\mathcal{Z}(M,N;\tau,\epsilon)$. According to the definitions (15) and (21),

$$\mathcal{Z}(M,N;\tau,\epsilon) = \int \mathcal{D}\psi\; \overline{\left(\exp\left\{-\sum_{a=1}^{N} H[\psi_a;\tau+\epsilon,\delta\tau] \;-\; \sum_{a=N+1}^{M} H[\psi_a;\tau,\delta\tau]\right\}\right)}. \tag{24}$$

Substituting here the Hamiltonian (13) and performing Gaussian averaging over $\delta\tau(\mathbf{r})$ using Eq. (14) we find:

$$\mathcal{Z}(M,N;\tau,\epsilon) = \int \mathcal{D}\psi\;\; \exp\{-\mathcal{H}_{M,N}[\psi;\tau,\epsilon]\} \equiv \exp\{-\mathcal{F}(M,N;\tau,\epsilon)\}, \tag{25}$$

where $\mathcal{F}(M,N;\tau,\epsilon)$ can be called the "replica free energy" and

$$\mathcal{H}_{M,N}[\psi;\tau,\epsilon] = \int d^2r\left[\frac{1}{2}\sum_{a=1}^{M}\overline{\psi}_a(\mathbf{r})\hat{\partial}\psi_a(\mathbf{r}) + \frac{1}{2}\sum_{a=1}^{M} m_a(\overline{\psi}_a(\mathbf{r})\psi_a(\mathbf{r})) - \frac{1}{4}g_0\sum_{a,b=1}^{M}(\overline{\psi}_a(\mathbf{r})\psi_a(\mathbf{r}))(\overline{\psi}_b(\mathbf{r})\psi_b(\mathbf{r}))\right], \tag{26}$$

where

$$m_a = \begin{cases}(\tau+\epsilon) & \text{for } a = 1,\ldots,N,\\ \tau & \text{for } a = N+1,\ldots,M.\end{cases} \tag{27}$$

The expression obtained, Eq. (26), has the form of an effective Hamiltonian of the random Ising model but with replica-dependent masses. As we will see below, this difference will further influence properties of the internal energy distribution. In the next section we will derive the function $\mathcal{F}(M,N;\tau,\epsilon)$ of Eq. (25) using standard procedures of the RG approach.

IV. RENORMALIZATION GROUP CALCULATIONS

It is well known that the spinor-field theory with four-fermion interactions is renormalizable in 2D, and the renormalization equations lead to "zero-charge" asymptotics for the charge g and mass m (see, e.g., Ref. [24]). Renormalization of the replica Hamiltonian (26) can be achieved in a standard way by integrating out short-wavelength degrees of freedom in the band $\tilde{\Lambda} < p < \Lambda$, where Λ and $\tilde{\Lambda}$ are the old and new ultraviolet momentum cutoffs, respectively. One can easily show that the renormalization of the charge g and the mass m_a in the Hamiltonian (26) is given by the following equations (cf. Eqs. (4.28) in

Ref. [24]):

$$\frac{d}{d\xi}g(\xi) = -\frac{1}{\pi}(2-M)\,g^2(\xi), \tag{28}$$

$$\frac{d}{d\xi}m_a(\xi) = -\frac{1}{\pi}\left[m_a(\xi) - \sum_{b=1}^{M} m_b(\xi)\right]g(\xi), \tag{29}$$

where $\xi = \ln(\Lambda/\tilde{\Lambda})$ and

$$m_a(\xi) = \begin{cases} \tilde{m}(\xi) & \text{for } a = 1,\dots,N, \\ m(\xi) & \text{for } a = N+1,\dots,M, \end{cases} \tag{30}$$

with the initial conditions $g(0) = g_0$, $\tilde{m}(0) = (\tau + \epsilon)$ and $m(0) = \tau$. Substituting Eq. (30) into Eq. (29) we get:

$$\frac{d}{d\xi}\tilde{m}(\xi) = -\frac{1}{\pi}[\tilde{m}(\xi) - N\tilde{m}(\xi) - (M-N)m(\xi)]\,g(\xi), \tag{31}$$

$$\frac{d}{d\xi}m(\xi) = -\frac{1}{\pi}[m(\xi) - N\tilde{m}(\xi) - (M-N)m(\xi)]\,g(\xi). \tag{32}$$

The solution of Eq. (28) is

$$g(\xi) = \frac{g_0}{1 + \frac{1}{\pi}(2-M)\,g_0\,\xi}\,. \tag{33}$$

Equations (28) and (29) have been obtained in the one-loop approximation. The two-loop approximation has been studied in Ref. [31], where it was shown that it gives only a next-order logarithmic correction to the one-loop result. Therefore, being interested in the leading asymptotics, we proceed further within the one-loop approximation. Substituting the above solution (33) into Eqs. (31) and (32) in the limit $M \to 0$, one easily finds [32]:

$$m(\xi) = \left[\tau + \frac{1}{2}(\epsilon N)\ln\left(1 + \frac{2}{\pi}g_0\,\xi\right)\right]\Delta(\xi), \tag{34}$$

$$\tilde{m}(\xi) = m(\xi) + \epsilon\,\Delta(\xi), \tag{35}$$

$$\Delta(\xi) = \frac{1}{\sqrt{1 + \frac{2}{\pi}g_0\,\xi}}. \tag{36}$$

The critical properties of a model with "zero-charge" renormalization [according to Eq. (33), $g(\xi \to \infty) \sim 1/\xi \to 0$] can be studied exactly by RG methods [33,34] (see also Ref. [24]). According to the standard procedure of RG calculations, the singular contribution of thermodynamic quantities in the vicinity of the critical point is obtained by using only the noninteracting part of the renormalized Hamiltonian [the first two terms of the Hamiltonian (26)], in which the mass terms $m_a, a = 1,\dots,M$, become scale-dependent parameters, Eqs. (30) and (34)–(36). In other words, in the process of the RG procedure the contributions originating in the interaction terms of the Hamiltonian (25) are effectively "absorbed" into the mass terms. In this case, similarly to the pure system [see Eqs. (10) and (11)], we get:

$$\begin{aligned}
\mathcal{F}(0,N;\tau,\epsilon) &= -\lim_{M\to 0}\ln[\mathcal{Z}(M,N;\tau,\epsilon)] = -L^2\lim_{M\to 0}\int_{|p|<1}\frac{d^2p}{(2\pi)^2}\,\ln\left\{\prod_{a=1}^{M}\det[i\hat{p} + m_a(p)\hat{\sigma}_0]^{1/2}\right\} \\
&= -L^2\lim_{M\to 0}\int_{|p|<1}\frac{d^2p}{(2\pi)^2}\,\ln\{\det[i\hat{p} + \tilde{m}(p)\hat{\sigma}_0]^{\frac{N}{2}} \times \det[i\hat{p} + m(p)\hat{\sigma}_0]^{\frac{M-N}{2}}\}, \\
&= -L^2\int_{|p|<1}\frac{d^2p}{(2\pi)^2}\,\ln\{\det[i\hat{p} + \tilde{m}(p)\hat{\sigma}_0]^{N/2} \times \det[i\hat{p} + m(p)\hat{\sigma}_0]^{-N/2}\},
\end{aligned} \tag{37}$$

where [cf. Eqs. (2) and (3)]

$$\hat{p} = \hat{\sigma}_1 p_x + \hat{\sigma}_2 p_y. \tag{38}$$

Here the mass parameters $m(p)$ and $\tilde{m}(p)$ are taken to be dependent on the scale according to Eqs. (34)–(36) with $\xi = \ln(1/p)$. Simple calculations yield [cf. Eq. (11)]

$$\begin{aligned}
\mathcal{F}(0,N;\tau,\epsilon) &= -L^2\int_{|p|<1}\frac{d^2p}{(2\pi)^2}\left\{\frac{1}{2}N\ln[p^2 + \tilde{m}^2(p)] - \frac{1}{2}N\ln[p^2 + m^2(p)]\right\} \\
&= -\frac{1}{4\pi}L^2 N\int_0^1 dp\,p\,\ln\left\{\frac{p^2 + [m(p) + \epsilon\Delta(p)]^2}{p^2 + m^2(p)}\right\}.
\end{aligned} \tag{39}$$

Substituting here the solutions (34)–(36) in the leading order in $\epsilon \to 0$ we get (see the Appendix for details):

$$\begin{aligned}
\mathcal{F}(0,N;\tau,\epsilon) &\simeq -\frac{1}{4\pi}L^2 N\int_0^1 dp\,p\,\ln\left[1 + \epsilon\,\frac{2m(p)\Delta(p)}{p^2 + m^2(p)}\right] \\
&\simeq -\frac{1}{2\pi}L^2(\epsilon N)\int_0^1 dp\,p\,\frac{\tau + \frac{1}{2}(\epsilon N)\ln\left[1 + \frac{2}{\pi}g_0\ln(1/p)\right]}{[p^2 + m^2(p)]\left[1 + \frac{2}{\pi}g_0\ln(1/p)\right]} \\
&\simeq -\frac{1}{2\pi}L^2(\epsilon N)\int_{|\tau|}^1\frac{dp}{p}\,\frac{\tau + \frac{1}{2}(\epsilon N)\ln\left[1 + \frac{2}{\pi}g_0\ln(1/p)\right]}{1 + \frac{2}{\pi}g_0\ln(1/p)} = E(\tau)(\epsilon N) - \frac{1}{2}E_*^2(\tau)(\epsilon N)^2,
\end{aligned} \tag{40}$$

where

$$E(\tau) = -\frac{1}{4g_0}L^2\,\tau\,\ln\Big[1+\frac{2}{\pi}g_0\ln(1/|\tau|)\Big], \tag{41}$$

$$E_*(\tau) = \frac{1}{2\sqrt{2g_0}}L\ln\Big[1+\frac{2}{\pi}g_0\ln(1/|\tau|)\Big]. \tag{42}$$

It should be stressed that the expression for the free energy $\mathcal{F}(0,N;\tau,\epsilon)$ obtained in Eq. (40) contains no higher-order terms in powers of ϵN that were neglected. On the other hand, the result is not exact and both $E(\tau)$ and $E_*(\tau)$ contain higher-order logarithmic corrections of the form $\ln\ln(1/\tau)$ which were neglected in the limit $\tau \ll 1$ considered here [see the calculations given in the Appendix as well as the remark below Eq. (33)]. Thus, for the replica partition function on the right-hand side of relation (22) for the Laplace transform, we obtain according to definition (25):

$$\tilde{\mathcal{Z}}(s,\tau) = \lim_{\epsilon\to 0}\exp\{-\mathcal{F}(0,s/\epsilon;\tau,\epsilon)\}$$
$$= \exp\left\{-E(\tau)\,s+\frac{1}{2}E_*^2(\tau)\,s^2\right\}. \tag{43}$$

Substituting this into the inverse Laplace transform relation (23) we get:

$$P_\tau(E) = \int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\,\exp\left\{-E(\tau)\,s+\frac{1}{2}E_*^2(\tau)\,s^2+s\,E\right\} \tag{44}$$

or

$$P_\tau(E) = \frac{1}{\sqrt{2\pi}\,E_*(\tau)}\,\exp\left\{-\frac{[E-E(\tau)]^2}{2E_*^2(\tau)}\right\}. \tag{45}$$

Thus, the sample-to-sample fluctuations of the critical internal energy of the weakly disordered 2D Ising model are described by a Gaussian distribution characterized by the mean value $E(\tau)$ given in Eq. (41) and typical deviations $E_*(\tau)$ as given in Eq. (42). Let us check the behavior of $P_\tau(E)$ as $L\to\infty$ and for $\tau\to 0$. If in this limit the distribution (45) tends towards a δ function, then E is self-averaging; otherwise it is not. For a fixed value of τ, Eqs. (41), (42), and (45) reveal that the distribution function of the energy density $e\equiv E/L^2$ in the thermodynamic limit turns into a δ function: $P(e)=\delta[e-e_0(\tau)]$ with $e_0(\tau) = -(\tau/4g_0)\,\ln[1+\frac{2}{\pi}g_0\ln(1/|\tau|)]$. In other words, in this case the energy density is self-averaging. On the other hand, for finite L the limit $\tau\to 0$ cannot be used directly in formulas (41), (42), and (45), since in this case the correlation length ($R_c(\tau)\sim 1/\tau$) exceeds the system size, which makes no physical sense. The point is that the RG procedure must be stopped at scales of the order of the system size L (provided we take $\tau \ll L$ in the starting Hamiltonian). Therefore, one will get the result given in Eqs. (41), (42), and (45) again, but there the parameter τ has to be replaced by $1/L$ which makes it temperature independent, as it should be.

In close vicinity of the critical point where the disorder induced critical behavior sets in, i.e., at $\tau \ll \exp(-\pi/2g_0)$, one finds:

$$E(\tau) \simeq -\frac{1}{4g_0}L^2\,\tau\,\ln\ln(1/|\tau|), \tag{46}$$

$$E_*(\tau) \simeq \frac{1}{2\sqrt{2g_0}}L\ln\ln(1/|\tau|). \tag{47}$$

At large but finite value of the system size L, we expect the pseudocritical temperature to scale as $\tau_c \sim L^{-\nu} = 1/L$, and hence

$$E_c(L) \equiv E(\tau=1/L) \sim -\frac{1}{g_0}L\,\ln\ln(L), \tag{48}$$

$$E_c^*(L) \equiv E_*(\tau=1/L) \sim \frac{1}{\sqrt{g_0}}L\,\ln\ln(L). \tag{49}$$

Comparing Eqs. (46), (48), (49), and (45) one concludes that at sufficiently large system size, $L \gg \exp(\pi/2g_0)$, the critical internal energy E can be written as a sum of its mean value and a fluctuating part:

$$E \sim -\frac{1}{g_0}L\,\ln\ln(L)+\frac{1}{\sqrt{g_0}}L\,\ln\ln(L)\,f, \tag{50}$$

where the random quantity f does not scale with L, $f\sim 1$, and is described by a standard normal distribution

$$P_c(f) = \frac{1}{\sqrt{2\pi}}\,\exp\left(-\frac{1}{2}f^2\right). \tag{51}$$

Equations (50) and (51) demonstrate that at criticality the internal energy of the 2D random-bond Ising ferromagnet is not self-averaging as the typical value of the sample-to-sample fluctuations, $E_c^*(L)\sim g_0^{-1/2}L\ln\ln(L)$, scales with the system size in the same way as its average value, $E_c(L)\sim g_0^{-1}L\ln\ln(L)$. We note that the RG framework we use in our analysis gives access to the singular part of thermodynamic functions only and is not able to say anything about the behavior of nonsingular background terms that are present in a specific lattice realization. Therefore, it is this singular part of the internal energy that is governed by the distribution (45).

V. SPECIFIC HEAT

We now turn to an investigation of the behavior of the specific heat. To this end, we repeat the steps performed in Secs. II and III for the *second* derivative of the free energy,

$$C[\tau;\delta\tau] = -\frac{\partial^2}{\partial\tau^2}F[\tau;\delta\tau]. \tag{52}$$

In terms of the replica formalism, instead of Eqs. (17)–(21) we get

$$C[\tau;\delta\tau] = -\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}(F[\tau+\epsilon;\delta\tau]+F[\tau-\epsilon;\delta\tau]$$
$$-2F[\tau;\delta\tau]), \tag{53}$$

so

$$\exp\{\epsilon^2\,C[\tau;\delta\tau]\} = Z[\tau+\epsilon;\delta\tau]\,Z[\tau-\epsilon;\delta\tau]\,Z^{-2}[\tau;\delta\tau] \tag{54}$$

and

$$\int dC\, \mathcal{P}_\tau(C)\, \exp(\epsilon^2 N\, C)$$
$$= \lim_{M\to 0} \overline{Z^N[\tau+\epsilon;\delta\tau]\, Z^N[\tau-\epsilon;\delta\tau]\, Z^{M-2N}[\tau;\delta\tau]}$$
$$\equiv \lim_{M\to 0} \mathcal{Z}_c(M,N;\tau,\epsilon), \tag{55}$$

where $\mathcal{P}_\tau(C)$ is the probability distribution function of the specific heat. Correspondingly, instead of Eqs. (22) and (23) we have

$$\int dC\, \mathcal{P}_\tau(C)\, \exp(s\, C)$$
$$= \lim_{\epsilon\to 0}\lim_{M\to 0} \mathcal{Z}_c(M,s/\epsilon^2;\tau,\epsilon) \equiv \tilde{\mathcal{Z}}_c(s,\tau), \tag{56}$$

with $s = \epsilon^2 N$ in this case and

$$\mathcal{P}_\tau(C) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i}\, \tilde{\mathcal{Z}}_c(s,\tau)\exp(-s\, C), \tag{57}$$

where

$$\mathcal{Z}_c(M,N;\tau,\epsilon) = \int \mathcal{D}\psi\, \exp\{-\mathcal{H}^{(c)}_{M,N}[\psi;\tau,\epsilon]\}$$
$$\equiv \exp\{-\mathcal{F}_c(M,N;\tau,\epsilon)\}, \tag{58}$$

and the replica Hamiltonian $\mathcal{H}^{(c)}_{M,N}[\psi;\tau,\epsilon]$ is defined by the right-hand side of Eq. (26) with

$$m_a = \begin{cases} (\tau+\epsilon) & \text{for } a = 1,\ldots,N,\\ (\tau-\epsilon) & \text{for } a = N+1,\ldots,2N,\\ \tau & \text{for } a = 2N+1,\ldots,M. \end{cases} \tag{59}$$

The renormalization of the charge g and the mass m_a of this Hamiltonian is given in Eqs. (28) and (29) where

$$m_a = \begin{cases} m_1(\xi) & \text{for } a = 1,\ldots,N,\\ m_2(\xi) & \text{for } a = N+1,\ldots,2N,\\ m(\xi) & \text{for } a = 2N+1,\ldots,M, \end{cases} \tag{60}$$

with the initial conditions $m_1(0) = (\tau+\epsilon)$, $m_2(0) = (\tau-\epsilon)$, and $m(0) = \tau$. One can easily show that in the limit $M \to 0$, the sum

$$\lim_{M\to 0}\left(\sum_{a=1}^{M} m_a(\xi)\right) = N[m_1(\xi) + m_2(\xi) - 2m(\xi)] \equiv 0, \tag{61}$$

so the solutions of the RG equations (28) and (29) for the masses $m_1(\xi), m_2(\xi)$, and $m(\xi)$ turn out to be effectively decoupled [unlike the situation for the internal energy, Eqs. (34)–(36)]:

$$m_1(\xi) = \frac{\tau+\epsilon}{\sqrt{1+\frac{2}{\pi}\, g_0\, \xi}}, \tag{62}$$
$$m_2(\xi) = \frac{\tau-\epsilon}{\sqrt{1+\frac{2}{\pi}\, g_0\, \xi}}, \tag{63}$$
$$m(\xi) = \frac{\tau}{\sqrt{1+\frac{2}{\pi}\, g_0\, \xi}}. \tag{64}$$

Correspondingly, instead of Eq. (39) we obtain

$$\mathcal{F}_c(0,N;\tau,\epsilon) = -L^2 \int_{|p|<1} \frac{d^2p}{(2\pi)^2}\left\{\frac{1}{2}N \ln\big[p^2 + m_1^2(p)\big] + \frac{1}{2}N\ln\big[p^2+m_2^2(p)\big] - N\ln[p^2+m^2(p)]\right\}$$
$$= -\frac{1}{4\pi}L^2 N \int_0^1 dp\; p\; \ln\left\{\frac{\big[p^2+m_1^2(p)\big]\big[p^2+m_2^2(p)\big]}{[p^2+m^2(p)]^2}\right\}. \tag{65}$$

Substituting here the solutions (62)–(64) in the leading order in $\epsilon \to 0$ we get [cf. Eq. (39)]

$$\mathcal{F}_c(0,N;\tau,\epsilon)$$
$$\simeq -\frac{1}{\pi}L^2\epsilon^2 N \int_0^1 \frac{dp\; p}{[p^2+m^2(p)]\big[1+\frac{2}{\pi}\, g_0\, \ln(1/p)\big]}$$
$$\simeq -\frac{1}{\pi}L^2\epsilon^2 N \int_{|\tau|}^1 \frac{dp}{p}\, \frac{1}{\big[1+\frac{2}{\pi}\, g_0\, \ln(1/p)\big]}$$
$$= -\epsilon^2 N\, C(\tau), \tag{66}$$

where

$$C(\tau) = \frac{1}{2g_0}L^2 \ln\Big[1 + \frac{2}{\pi}g_0 \ln(1/|\tau|)\Big]. \tag{67}$$

Substituting this into the inverse Laplace transform relation (57) we get:

$$\mathcal{P}_\tau(C) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i}\, \exp\{C(\tau)\, s - C\, s\} = \delta[C - C(\tau)]. \tag{68}$$

This result shows that unlike the singular part of the internal energy the specific heat in the vicinity of the critical point is a self-averaging quantity. In particular, at large but finite value of the system size $L \gg L_* \sim \exp(2/\pi g_0)$ at the critical point at $\tau_c \sim 1/L$, according to Eq. (67) the critical specific heat $C(L)$ scales with the system size as

$$C(L) \sim \frac{1}{2g_0}L^2\, \ln\ln(L). \tag{69}$$

Note that the distribution (68) describes only the singular part of the specific heat, similarly to the distributions (45) and (51) which describe the singular part of the internal energy. As a matter of fact, the singular part of the “replica free energy” represented in Eq. (66) is linear in the replica parameter $s = \epsilon^2 N$. Formally, by the inverse Laplace transform, this results in the δ function (68), which may be misleading as the specific heat of the system contains also a regular part that is nonsingular in the limit $\tau \to 0$. As we have already mentioned, this last part is out of control for the present RG approach; however, it is a random quantity, too. According to the central limit theorem, this regular part is normally distributed with its mean value proportional to the volume of the system ($\sim L^2$, in the present case) and with a variance proportional to the square root of the volume of the system ($\sim\sqrt{L^2} = L$, in the present case). In other words, the regular part of the specific heat can be represented as $C_0 L^2 + C_* L \zeta$ where the random variable ζ is normally distributed with zero mean and unit variance, and the values of C_0 and C_* do not scale with L as $L \to \infty$. Correspondingly, in the replica representation this must give two additional contributions to the replica free energy: In addition to the expression presented in Eq. (66) one has two more terms $C_0 L^2 s + (C_* L)^2 s^2$ (where $s = \epsilon^2 N$). Thus, after the inverse Laplace transform the δ function (68) is replaced by a Gaussian distribution with the mean value $C(\tau) + C_0 L^2$, where $C(\tau)$ is given in Eq. (67), and variance $C_* L$. In the limit $L \to \infty$ this results in the following behavior of the specific heat:

$$C(\tau = 1/L) \sim L^2 \ln\ln(L) \gg C_0 L^2 \gg C_* L. \tag{70}$$

Consequently, in this limit the distribution of the specific heat (which includes both the regular and the singular part) turns into a δ function centered at $C(L) \sim L^2 \ln\ln(L)$, corresponding to self-averaging.

VI. CONCLUSIONS

We have derived an explicit expression for the probability distribution function of the sample-to-sample fluctuations of the internal energy of the weakly disordered critical 2D Ising ferromagnet. The result obtained, Eqs. (50) and (51), shows that the internal energy of this system is not self-averaging. Instead, the typical value of its sample-to-sample fluctuations scales in the same way as its average, proportional to $\sim L \ln\ln(L)$. On the other hand, the specific heat was shown here to exhibit self-averaging, with a distribution function that converges to a δ function in the limit of infinite system size. In contrast to the free energy of the system, which was discussed before in Ref. [22], the quantities discussed here are directly observable in numerical simulations. It is not completely obvious at this point how far the singular behavior is masked in a lattice realization by the presence of regular background terms and how clearly the lack or presence of self-averaging could be seen experimentally. A numerical investigation of this system geared towards resolving this intriguing question is the subject of a forthcoming study.

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APPENDIX: DERIVATION OF EQ. (40)

In this Appendix we explain in more detail the derivation of Eq. (40). Let us consider the quantity

$$I(\tau) = \int_{|p|<1} \frac{d^2 p}{p^2 + \tau^2}\, f[\ln(1/p)], \tag{A1}$$

where $f(\xi)$ is a "sufficiently good" (not too divergent) function in the limit $\xi \to \infty$, i.e., $\lim_{\xi\to\infty} f(\xi)\exp\{-\xi\} \to 0$. The leading singularity of $I(\tau)$ in the limit $\tau \to 0$ can be estimated in the standard way:

$$\begin{aligned} I(\tau) &\sim \int_{|\tau|<|p|<1} \frac{d^2 p}{p^2}\, f[\ln(1/p)] \\ &\sim \int_{|\tau|}^{1} \frac{dp}{p}\, f[\ln(1/p)] \sim \int_0^{\ln(1/|\tau|)} d\xi\ f(\xi), \end{aligned} \tag{A2}$$

where $\xi = \ln(1/p)$. Now let us consider the slightly more complicated object

$$\tilde{I}(\tau) \equiv \int_{|p|<1} \frac{d^2 p}{p^2 + m^2(p)}\, f[\ln(1/p)], \tag{A3}$$

where instead of τ^2 in the denominator we have a p-dependent mass term $m^2(p)$,

$$m^2(p) = \frac{\tau^2}{1 + \frac{2}{\pi} g_0 \ln(1/p)}, \tag{A4}$$

which is the case when computing the specific heat singularity of the weakly disordered 2D Ising model. One can consider two limiting cases:

(a) $\frac{2}{\pi} g_0 \ln(1/|\tau|) \ll 1$ or $|\tau| \gg \exp(-\pi/2g_0) \equiv \tau_*$. In this case, while integrating over p one can just drop the presence of the nontrivial denominator in (A4) and we get

$$\tilde{I}(\tau) \sim \int_{|\tau|}^{1} \frac{dp}{p}\, f[\ln(1/p)] \sim \int_0^{\ln(1/|\tau|)} d\xi\ f(\xi), \tag{A5}$$

which coincides with the "pure" case (A2).

(b) $\frac{2}{\pi} g_0 \ln(1/|\tau|) \gg 1$ or $|\tau| \ll \tau_*$. In this case, we have

$$\begin{aligned} \tilde{I}(\tau) &\equiv \int_{|p|<1} \frac{d^2 p}{p^2 + m^2(p)}\, f[\ln(1/p)] \\ &\sim \int_{p_*(\tau)}^{1} \frac{dp}{p}\, f[\ln(1/p)], \end{aligned} \tag{A6}$$

where $p_*(\tau)$ is defined by the condition:

$$p_* \sim \frac{|\tau|}{\sqrt{g_0 \ln(1/p_*)}}, \tag{A7}$$

which yields

$$p_*(\tau) \sim \frac{|\tau|}{\sqrt{g_0 \ln(1/|\tau|)}}. \tag{A8}$$

Substituting this into (A6), we get

$$\tilde{I}(\tau) \sim \int_0^{\xi_*(\tau)} d\xi\ f(\xi), \tag{A9}$$

where in the limit $|\tau| \to 0$,

$$\xi_*(\tau) \sim \ln\left[\frac{\sqrt{g_0 \ln(1/|\tau|)}}{|\tau|}\right] = \ln(1/|\tau|) + \frac{1}{2}\ln\ln(1/|\tau|) + \frac{1}{2}\ln(g_0) \sim \ln(1/|\tau|) + O(\ln\ln(1/|\tau|)). \tag{A10}$$

Thus, in this case we get

$$\tilde{I}(\tau) \sim \int_0^{\ln(1/|\tau|)} d\xi\ f(\xi), \tag{A11}$$

which means that in the limit $\tau \to 0$ in both cases (a) and (b) we can cut the integration over p at $p_* \sim \tau$. Note that in the considered model $|\tau|^{-1} \sim R_c$ is the correlation of the pure system, and the presence of disorder produces not more than a logarithmic correction to the correlation length.

In the case considered in this paper the situation is somewhat more tricky due to the presence of the second term in the brackets of Eq. (34). According to Eqs. (34)–(36) in the limit $\tau \to 0$ (at $|\tau| \ll \tau_*$ where $g_0\xi \gg 1$), we have

$$m(p) \sim \frac{\tau + \frac{1}{2}(\epsilon N)\ln(g_0 \ln(1/p))}{\sqrt{g_0 \ln(1/p)}}. \tag{A12}$$

According to the standard logic of the replica technique, first we have to assume that for a given value of the replica parameter N the value of the parameter ϵ is considered to be less than everything else, such that $(\epsilon N)\ln\ln(1/|\tau|) \ll 1$, so the integration over p is cut at $p_* \sim \tau$ as in the above examples. On the other hand, in the further inverse Laplace transform integration over analytically continued complex parameter N, its relevant value turns out to be of order $1/\epsilon$, which means that the relevant value of the (complex) product (ϵN) turns out to be finite.

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