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Mutually avoiding paths in random media and largest eigenvalues of random matrices

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Recently, it was shown that the probability distribution function (PDF) of the free energy of a single continuum directed polymer (DP) in a random potential, equivalently to the height of a growing interface described by the Kardar-Parisi-Zhang (KPZ) equation, converges at large scale to the Tracy-Widom distribution. The latter describes the fluctuations of the largest eigenvalue of a random matrix, drawn from the Gaussian unitary ensemble (GUE), and the result holds for a DP with fixed end points, i.e., for the KPZ equation with droplet initial conditions. A more general conjecture can be put forward, relating the free energies of $N > 1$ noncrossing continuum DP in a random potential, to the sum of the Nth largest eigenvalues of the GUE. Here, using replica methods, we provide an important test of this conjecture by calculating exactly the right tails of both PDFs and showing that they coincide for arbitrary N.

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Introduction. Remarkable connections have emerged in the last decade between random matrix theory, growth models, and glassy systems. The celebrated Kardar-Parisi-Zhang (KPZ) equation [1] provides the simplest description for the growth of an interface in the presence of noise. This equation sits at the center of a wide universality class [2], encompassing several models and physical systems, such as the polynuclear growth model (PNG) [3], the asymmetric exclusion processes (ASEPs) [4–6], and Burgers turbulence [7].

Additionally, the height $h(x,t)$ of the KPZ interface in d dimensions can be exactly mapped into (minus) the free energy of a directed polymer (DP) of length t in a quenched random potential in $1+d$ dimension [8,9]. The DP is one of the most straightforward realizations of a glass, with applications including domain walls in magnets [10], vortex lines in superconductors [11], localization paths in Anderson insulators [12], and even to problems in biophysics [13] and economics [14].

The link between KPZ and DP has been particularly fruitful in $d = 1$, where a hidden integrable structure comes to light. In this case, several exact solutions, first for zero temperature [15], and later for finite temperature discrete [16–18] and continuum DP models [19–24], unveiled an astounding connection: The probability distribution of the (scaled) KPZ height field $h(x,t)$ coincides with the (scaled) distribution of the largest eigenvalue of a random matrix drawn from the famous Gaussian ensembles. This is the so-called Tracy-Widom (TW) distribution [25], recurring in a broad variety of contexts [26]. In particular, if we define as $\hat{\mathcal{Z}}_1(t)$ the partition function in the continuum of a DP which starts and ends at the same point $x(t) = x(0)$ (see below for an explicit definition), it was found in Refs. [19–21,27] that at large time one can write $\ln \hat{\mathcal{Z}}_1(t) \simeq -t/12 + \hat{\gamma}_1 t^{-1/3}$, where $\hat{\gamma}_1$ follows the $\beta = 2$ TW distribution associated with the Gaussian unitary ensemble (GUE). Here, we use a caret to denote a random variable. Furthermore, rigorous mathematical treatments have confirmed these results [28,29], suggesting the existence of a refined mathematical connection between the KPZ equation and Whittaker-Macdonald processes, which degenerate in certain limits to the GUE ensemble. Nevertheless, a more direct interpretation of the deep reason behind this correspondence is still lacking. A natural direction to shed some light on this problem is to extend this connection beyond the maximal eigenvalue $\hat{\gamma}_1$, to the full portion of the GUE spectrum around the edge.

In the present Rapid Communication, we consider an ensemble of N mutually avoiding polymers, i.e., several directed paths constrained not to intersect with one another, and competing to optimize their total energy in the same random media. We extend the study of the single polymer partition function $\hat{\mathcal{Z}}_1$ to the one of N noncrossing paths $\hat{\mathcal{Z}}_N$. We build on a general method which we recently developed to treat any number N of DPs, but until now, only applied in the specific case $N = 2$ to analyze the noncrossing probability [30–32] (see Fig. 1).

Here, we corroborate the conjecture that the N-path free energy takes the form at large time [33],

$$\ln \hat{\mathcal{Z}}_N(t) \simeq -Nt/12 + t^{1/3}\hat{\zeta}^{(N)}, \tag{1}$$

where the random variable $\hat{\zeta}^{(N)}$ coincides in law with the *partial sum* of the N-largest eigenvalues $\hat{\gamma}_1, \ldots, \hat{\gamma}_N$ of a GUE random matrix,

$$\hat{\zeta}^{(N)} \overset{\text{in law}}{=} \sum_{i=1}^{N} \hat{\gamma}_i =: \hat{\gamma}. \tag{2}$$

The validity of this conjecture for the continuum, finite temperature model, is suggested by an argument of universality [30,31,34,35], together with exact results on discrete DP models *at zero temperature*, specifically, the last passage percolation model [36,37] and the semidiscrete directed polymer [16,38–40]. Furthermore, a more abstract conjecture was recently proposed relating the KPZ equation and the Airy line ensemble [41]: If proved, it would imply Eq. (2) as a particular case.

Obviously, showing the equality of the probability distribution functions (PDF) $P_N^{\mathrm{DP}}(\zeta)$ and $P_N^{\mathrm{GUE}}(\gamma)$ is a major challenge. Here, we will provide a test, by showing that

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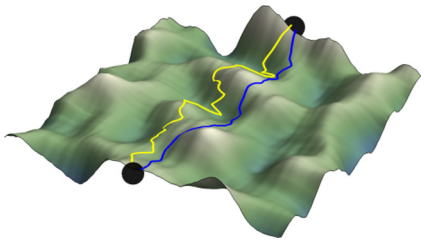

FIG. 1. Sketch of the $N = 2$ case: Two polymers start and end at (almost) coinciding points. The blue and yellow lines represent the point with maximal probability at any time slice t and the disorder realization is chosen to have well-separated paths.

their leading (stretched exponential order) *tail* approximant functions are identical. More precisely, we will show that at large arguments

$$P_N^{\rm DP}(\zeta) = \rho_N^{\rm DP}(\zeta)[1 + O(e^{-a_N \zeta^{3/2}})], \tag{3}$$

$$P_N^{\rm GUE}(\gamma) = \rho_N^{\rm GUE}(\gamma)[1 + O(e^{-a'_N \gamma^{3/2}})], \tag{4}$$

with $a_N, a'_N > 0$ and exactly the *same* function $\rho_N^{\rm DP}(\gamma) = \rho_N^{\rm GUE}(\gamma) = O(e^{-\frac{4\gamma^{3/2}}{3\sqrt{N}}})$. Here and below, $O(e^{-a\gamma^{3/2}})$ means at leading exponential accuracy. Note that the function $\rho_N^{\rm GUE}(\gamma)$ is nontrivial, hence the coincidence is a strong hint for the conjecture to hold. For instance, in the simpler case of $N = 1$, where the conjecture is known to hold, one has $\rho_{N=1}^{\rm GUE}(\gamma) = {\rm Ai}'(\gamma)^2 - \gamma\,{\rm Ai}(\gamma)^2$. Likewise, we will provide a (more complicated) formula for $N > 1$.

Note that nonintersecting Brownian motions (sometimes dubbed "watermelon configurations") have already been put in relation with Airy processes and Tracy-Widom distributions [42]. These studies hold, however, in a very different context, in particular, in the absence of any quenched disorder.

The GUE ensemble. To fix the notation we take the GUE specified by the measure $\propto \exp(-{\rm Tr}\, H^2)(dH)$, where H is a complex $\mathcal{N} \times \mathcal{N}$ Hermitian matrix. For large $\mathcal{N}$, the support of the spectrum concentrates in $(-\sqrt{2\mathcal{N}}, \sqrt{2\mathcal{N}})$. Nevertheless, there is a finite probability for the eigenvalues $\hat{\lambda}_1 > \cdots > \hat{\lambda}_{\mathcal{N}}$ to fall outside this interval. In particular, introducing the rescaled eigenvalues $\hat{\gamma}_l = (\hat{\lambda}_l - \sqrt{2\mathcal{N}})\sqrt{2}\mathcal{N}^{1/6}$, the mean spacing for the variables close to $\hat{\gamma}_1$ becomes of order unity. In the limit $\mathcal{N} \to \infty$, this results in a well-defined determinantal point process, characterized by the correlation functions [43,44]

$$r_N(x_1, \ldots, x_N) = \det[K_{\rm Ai}(x_i, x_j)]_{i,j=1}^N, \tag{5}$$

for the density probability that there is a scaled eigenvalue in each interval $[x_i, x_i + dx_i]$, $i = 1, \ldots, N$. Here, the Airy kernel has been introduced as

$$K_{\rm Ai}(x, y) = \int_0^\infty dw\, {\rm Ai}(x + w)\, {\rm Ai}(y + w). \tag{6}$$

Correlation functions in Eq. (5) can then be used to recover the distribution of the N-largest (scaled) eigenvalues $\gamma_1 > \cdots > \gamma_N$: One has to remove all the configurations where an arbitrary number of other eigenvalues $x_1, \ldots, x_k$ fall in between. In order to avoid overcounting, the inclusion-exclusion principle can be employed, and we refer to Ref. [45] for the details. This leads to

$$p_N(\boldsymbol{\gamma}_N) = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \prod_{j=1}^k \int_{\min_{i=1}^N \gamma_i}^\infty dx_j\, r_{N+k}(\boldsymbol{\gamma}_N, \mathbf{x}_k), \tag{7}$$

where the bold symbol $\boldsymbol{\gamma}_N$ stands for $\gamma_1, \ldots, \gamma_N$ (and similarly for $\mathbf{x}_p$). Here, we conventionally normalize to 1 the integral of such probability in the domain $\gamma_1 > \cdots > \gamma_N$. In the particular case $N = 1$, this expression can be recast as the derivative of a Fredholm determinant: $p_1(\gamma) \equiv f_2(\gamma)$ is the GUE Tracy-Widom function. Setting $f_2(\gamma) = dF_2(\gamma)/d\gamma$, the cumulative distribution function $F_2(\gamma)$ is expressed as

$$F_2(\gamma) = {\rm Det}(1 - \Pi_\gamma K_{\rm Ai} \Pi_\gamma), \tag{8}$$

with Π_γ the projector onto $[\gamma, +\infty)$.

Sums of largest eigenvalues. We now introduce the partial sum of the N-largest eigenvalues, defined as

$$\hat{\gamma}^{(N)} = \hat{\gamma}_1 + \cdots + \hat{\gamma}_N, \tag{9}$$

and in the following we will omit the superscript N when not explicitly necessary. The probability distribution $P_N^{\rm GUE}(\gamma)$ for this quantity can be inferred from Eq. (7),

$$P_N^{\rm GUE}(\gamma) = \frac{1}{N!} \int d\gamma_1 \cdots d\gamma_N\, \delta\left(\gamma - \sum_{k=1}^N \gamma_k\right) p_N(\boldsymbol{\gamma}_N). \tag{10}$$

It is useful to introduce the double-sided Laplace transform (LT) of $P_N(\gamma)$ as

$$\tilde{P}_N^{\rm GUE}(u) := \overline{\exp(u\hat{\gamma})} = \int_{-\infty}^\infty d\gamma\, P_N^{\rm GUE}(\gamma) e^{u\gamma}. \tag{11}$$

We are interested in the right tail $\gamma \gg 1$, which governs the integral when u is large. Because of the behavior of the tail $K_{\rm Ai}(\gamma_i, \gamma_j) \sim e^{-\frac{2}{3}(\gamma_i^{3/2} + \gamma_j^{3/2})}$, this regime is dominated by the configuration minimizing the sum $\sum_i \gamma_i^{3/2}$ at fixed $\gamma = \sum_i \gamma_i$: This suggests that large values of the sum γ require all the N-largest eigenvalues to be of the same order of magnitude, i.e., $\gamma_k \simeq \gamma/N$. Then, in order to estimate the tail $\rho_N^{\rm GUE}(\gamma)$ defined by (3) of the distribution of the sum in Eq. (10), we can limit the expansion in Eq. (7) to the first term $p_N(\boldsymbol{\gamma}_N) \simeq r_N(\boldsymbol{\gamma}_N)$. Rearranging the determinant in $r_N(\boldsymbol{\gamma}_N)$, we obtain

$$\tilde{\rho}_N^{\rm GUE}(u) = \frac{1}{N!} \prod_{i=1}^N \int_0^\infty dv_i \times \det\left[\int_{-\infty}^\infty d\gamma e^{\gamma u}\, {\rm Ai}(\gamma + v_j)\, {\rm Ai}(\gamma + v_k)\right]_{j,k=1}^N, \tag{12}$$

which, after some simple manipulations [46], leads to our main result,

$$\tilde{\rho}_N^{\rm GUE}(u) = \frac{e^{\frac{{\rm Nu}^3}{12}} u^{-\frac{3N}{2}}}{\pi^{N/2} N!} \prod_{i=1}^N \int_{v_i > 0} e^{-2v_i} \det\left[e^{-\frac{(v_j - v_k)^2}{u^3}}\right]_{j,k=1}^N, \tag{13}$$

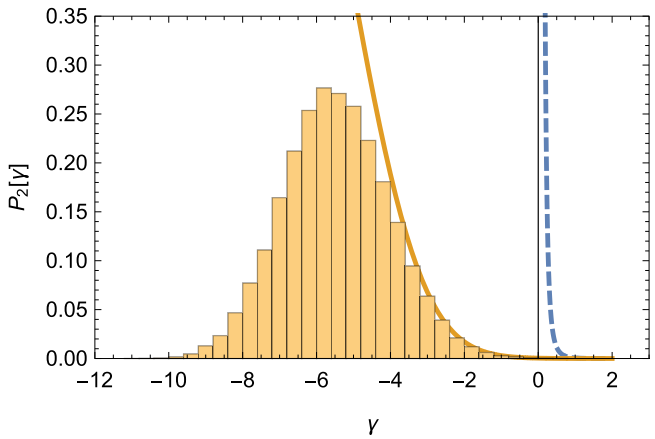

FIG. 2. The empirical distribution $P_2(\gamma)$ for the rescaled sum $\gamma = (\lambda_1 + \lambda_2 - \sqrt{8\mathcal{N}})\sqrt{2}\mathcal{N}^{1/6}$ of the first $N = 2$ eigenvalues in 10^5 realizations of GUE matrices of size $\mathcal{N} = 250$. The continuous lines are two different approximations for the tail obtained: from the inverse Laplace transformation of Eq. (12) (orange and continuous line); from the simple approximation $R_N(\gamma) = 1$ in Eq. (16) (blue and dashed line).

which generalizes the remarkably simple $N = 1$ result for the LT of the tail of the Tracy-Widom distribution,

$$\tilde{\rho}^{\mathrm{GUE}}_{N=1}(u) = \frac{e^{\frac{u^3}{12}}}{2\pi^{1/2}u^{\frac{3}{2}}}. \tag{14}$$

For $N > 1$ it takes the general form

$$\tilde{\rho}^{\mathrm{GUE}}_{N}(u) = \frac{e^{\frac{Nu^3}{12}}G(N+1)}{2^{N(N+1)/2}\pi^{N/2}u^{3N^2/2}}Q_N\left(\frac{1}{u^3}\right), \tag{15}$$

where $G(x)$ is the Barnes function, $Q_N(0) = 1$, and $Q_N(z)$ admits a series expansion around $z = 0$, since at large large u, this last determinant can be computed explicitly [46]. The Laplace inversion of (15) gives the general form of the tail function $\rho^{\mathrm{GUE}}_N(\gamma)$ where the leading behavior at large γ is apparent [with $R_N(+\infty) = 1$],

$$\rho^{\mathrm{GUE}}_{N}(\gamma) = \frac{N^{\frac{3N^2-1}{4}}G(N+1)e^{-\frac{4\gamma^{3/2}}{3\sqrt{N}}}}{2^{2N^2}(2\pi)^{\frac{N+1}{2}}\gamma^{\frac{3N^2+1}{4}}}R_N(\gamma). \tag{16}$$

The function $R_N(\gamma)$ can be obtained from subdominant orders in a saddle-point expansion and has the form of a double series in $1/\gamma$ and $1/\gamma^{3/2}$. In Fig. 2, we compare these predictions with the empirical distribution for $N = 2$. Note that the exact form of $\rho^{\mathrm{GUE}}_N(\gamma)$ is a major improvement compared to the naive approximation for the tail obtained by setting $R_N(\gamma) = 1$ in Eq. (16). From considerations of Airy function asymptotics it is easy to see that the corrections in Eq. (3) to $\rho^{\mathrm{GUE}}_N(\gamma)$ itself, calculated as above, are indeed subdominant by $O(e^{-a'_N\gamma^{3/2}})$ with $a'_N = \frac{2}{3}N^{-3/2}$.

Mutually avoiding directed polymers. We introduce the partition function of a directed polymer with fixed end points x, y,

$$\hat{Z}_\eta(x;y|t) \equiv \int_{x(0)=x}^{x(t)=y} Dx e^{-\int_0^t d\tau[\frac{1}{4}(\frac{dx}{d\tau})^2 - \sqrt{2\bar{c}}\hat{\eta}(x(\tau),\tau)]}, \tag{17}$$

in a given realization of a random potential with white-noise correlations $\overline{\hat{\eta}(x,t)\hat{\eta}(x',t')} = \delta(x-x')\delta(t-t')$. In the following, to simplify the notation, we rescale time and space and set $\bar{c} = 1$. Considering N polymers starting respectively at $\mathbf{x} = x_1, \ldots, x_N$ and arriving at $\mathbf{y} = y_1, \ldots, y_N$, the partition function constrained to nonintersecting paths can be expressed, using Ref. [47], as a single determinant,

$$\hat{Z}^{(N)}_\eta(\mathbf{x};\mathbf{y}|t) = \det[\hat{Z}_\eta(x_i;y_j|t)]^N_{i,j=1}. \tag{18}$$

This expression involves arbitrary space dependence; in order to simplify it, we consider therefore the limit where all the initial and final points coincide: $x_i = y_i = \epsilon u_i$. In the limit $\epsilon \to 0$, $\hat{Z}^{(N)}_\eta(\mathbf{x};\mathbf{y}|t) \simeq \frac{\epsilon^{N(N-1)}}{G(N+1)^2}\prod_{i<j}(u_i - u_j)^2\hat{\mathcal{Z}}_N(t)$, where [30,31,34,48]

$$\hat{\mathcal{Z}}_N(t) = \det\left[\partial_x^{i-1}\partial_y^{j-1}\hat{Z}_\eta(x;y|t)|_{x=y=0}\right]^N_{i,j=1}. \tag{19}$$

This random variable will be our quantity of interest. Its integer moments can be treated in the framework of the nested Bethe ansatz (NBA) [30] and of Macdonald processes [28]. As showed in Ref. [30], both methods lead to an expansion in terms of a sum over eigenstates of the (integrable) quantum Hamiltonian associated with the attractive δ-Bose gas, i.e., the Lieb-Liniger model. In particular, using a residue expansion of the contour-integral formula of [49], one obtain a series over integer partitions,

$$\overline{\hat{\mathcal{Z}}_N(t)^m} = \sum_{n_s=1}^{n}\frac{n!}{n_s!(2\pi)^{n_s}}\sum_{(m_1,\ldots,m_{n_s})_n}\prod_{j=1}^{n_s}\int_{-\infty}^{+\infty} \times \frac{dk_j}{m_j}e^{-tE[\mathbf{k},\mathbf{m}]}\Phi[\mathbf{k},\mathbf{m}]\,\mathcal{B}_{N,m}[\mathbf{k},\mathbf{m}], \tag{20}$$

where $(m_1, \ldots, m_{n_s})_n$ indicates the sum over all integers $m_j \geqslant 1$ whose sum equals $\sum_{j=1}^{n_s} m_j = n = mN$ and the energy of the string configuration has the form $E[\mathbf{k},\mathbf{m}] = \sum_{j=1}^{n_s} m_j k_j^2 + \frac{1}{12}(m_j - m_j^3)$. Equation (20) can be interpreted as an expansion over the Lieb-Liniger eigenstates composed by n_s strings of sizes $m_1, \ldots, m_{n_s}$. Then, the factor $\Phi[\mathbf{k},\mathbf{m}]$ can be obtained from the normalization of the string eigenstates and has the form [19,50]

$$\Phi[\mathbf{k},\mathbf{m}] = \prod_{1\leqslant i<j\leqslant n_s}\frac{(k_i-k_j)^2 + (m_i-m_j)^2/4}{(k_i-k_j)^2 + (m_i+m_j)^2/4}. \tag{21}$$

The factor $\mathcal{B}_{N,m}[\mathbf{k},\mathbf{m}]$ encodes the noncrossing constraint and contains all the dependence on N and m. It is expressed by introducing ($\mu_{jk} = \mu_j - \mu_k$)

$$\mathcal{B}_{N,m}[\boldsymbol{\mu}] = \frac{1}{N!^m}\sigma_{\boldsymbol{\mu}}\left[\frac{\prod_{i=1}^m\prod_{(i-1)N<j<k\leqslant iN}h(\mu_{jk})}{\prod_{1\leqslant j<k\leqslant n}f(\mu_{kj})}\right], \tag{22}$$

where the functions $h(u) = u(u+i)$ and $f(u) = u/(u+i)$ and $\sigma_{\boldsymbol{\lambda}}[W(\boldsymbol{\lambda})] = \sum_R W(R\boldsymbol{\lambda})/n!$ is the symmetrization of $W(\boldsymbol{\lambda})$ over the variables $\boldsymbol{\lambda}$. Then, to obtain $\mathcal{B}_{N,m}[\mathbf{k},\mathbf{m}]$, one needs to specialize the n variables $\boldsymbol{\mu} = \{\mu_1, \ldots, \mu_n\}$ with $\{\tilde{k}_1, \tilde{k}_1 + i, \ldots, \tilde{k}_1 + i(m_1 - 1), \tilde{k}_2, \tilde{k}_2 + i, \ldots\}$ and $\tilde{k}_j = k_j - i(m_j - 1)/2$.

The standard way to extract the PDF of the random variable $\hat{\zeta}$ in Eq. (1) from the knowledge of the moments in Eq. (20) is

to introduce a generating function by

$$g_N(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m \overline{\hat{\mathcal{Z}}_N(t)^m} = \overline{\exp(-e^{-\lambda s + t^{1/3}\hat{\zeta}})}, \quad (23)$$

where x is related to s by $xe^{-\frac{N_t}{12}} = e^{-\lambda s}$ and we introduce the rescaled time $\lambda = (Nt/4)^{1/3}$. Equation (23) has two advantages: (i) It lifts the constraint over the sum of m_i in Eq. (20); (ii) in the limit $t \to \infty$, $g_N(s) \to \text{Prob}[\hat{\zeta} < (N/4)^{1/3}s]$, i.e., the cumulative distribution function (CDF) of the random variable $\hat{\zeta}$. Unfortunately, even without the constraint, it is difficult to perform the sum (23) exactly for $N > 1$: Indeed, already obtaining a closed expression for $\mathcal{B}_{N,m}[\boldsymbol{\mu}]$ is a nontrivial task, which, apart from $\mathcal{B}_{1,m}[\boldsymbol{\mu}] = 1$, has been overcome only for $N = 2$ [32], where, however, the sum in Eq. (20) remains an open challenge.

Fortunately, however, we can still deal with (23) by replacing each moment with its asymptotics at large time. Although this *does not* give the exact large time behavior of $g_N(s)$, it is sufficient, as discussed below, to obtain the *exact tail behavior* of the PDF of $\hat{\zeta}$. Indeed, such properties already appear in the studies of the case $N = 1$,

At large times t and for fixed N,m, the sum in Eq. (20) is dominated by the configurations $\mathbf{m}$ with smallest energy $E[\mathbf{k},\mathbf{m}]$. In general, the energy $E[\mathbf{k},\mathbf{m}]$ will be minimized by the configurations with the largest possible m_i. For a single polymer $N = 1$, this simply translates into $n_s = 1$ and $m_1 = n$. However, for $N > 1$, this configuration gives a vanishing contribution: A general property of $\mathcal{B}_{N,m}[\mathbf{k},\mathbf{m}]$ is that it vanishes on any configuration with at least one $m_j > m$ [32]. This condition has the simple physical interpretation: A bound state (i.e., a string) cannot be formed joining particles which have been constrained to avoid each other. Surprisingly, this property is sufficient to completely determine the value of $\mathcal{B}_{N,m}[\mathbf{k},\mathbf{m}]$ on the lowest energy configuration with a nonvanishing contribution, which is the one consisting of a set of N m-strings, i.e., $n_s = N$ and every $m_j = m$ [46]. Combining Eq. (21) with Eq. (22) on this configuration, we have (omitting now the trivial dependence on $m_j = m$, and noting $k_{ij} = k_i - k_j$)

$$\Phi[\mathbf{k}]\mathcal{B}_{N,m}[\mathbf{k}] = \frac{m!^N}{(mN)!} \prod_{1<i<j<N} (-\imath k_{ij})_m (\imath k_{ij})_m, \quad (24)$$

where $(x)_m$ indicates the Pochhammer symbol. Inserting in the formula for the mth moment (20) and keeping only the configuration $m_j = m$, $j = 1, \ldots, n_s$ with $n_s = N$, one finds $\mathcal{Z}^{(0)}_{N,m}(t)$, defined as the leading contribution at large t and fixed N,m, to $\overline{\hat{\mathcal{Z}}_N(t)^m}$ in Eq. (20) (see Ref. [46]).

We now calculate $g^0_N(s) = \sum_{m=0}^{\infty}(-1)^m x^m \mathcal{Z}^{(0)}_{N,m}(t)$. In order to deal with the summation over m we follow two steps: (i) We use the Airy trick [19,20] to get rid of the factor m^3 in the exponent:

$$\int_{-\infty}^{\infty} dy\, \text{Ai}(y) e^{yw} = e^{w^3/3}; \quad (25)$$

(ii) we rewrite the sum over m using the Mellin-Barnes representation

$$\sum_{m\geqslant 1}(-1)^m \mathsf{f}(m) = -\frac{1}{2i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dz}{\sin(\pi z)} \mathsf{f}(z), \quad (26)$$

where $\epsilon \in]0,1[$ has to be chosen such that the function $\mathsf{f}(z)$ does not have singularities for $\text{Re}(z) > \epsilon$. After some manipulations (see Ref. [46]), one arrives at

$$g^{(0)}_N(s) \overset{t\to\infty}{=} 1 - \frac{1}{N!}\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\frac{dk_i}{2\pi}\int_0^{\infty} dy \times \text{Ai}\left(y + \sum_i k_i^2 + s\right)\int_{\epsilon-\imath\infty}^{\epsilon+\imath\infty}\frac{dz}{2\pi\imath z} e^{\sqrt{N}zy} \times \det\left[\frac{1}{2z + \imath k_{jk}}\right]_{j,k=1}^{N}. \quad (27)$$

We observe how a nice determinantal structure emerges at this level, reminiscent of the $N \times N$ determinant appearing in Eq. (12). To compare further, we obtain the PDF by differentiating with respect to s and we take again the Laplace transform,

$$\tilde{\rho}^{\text{DP}}_N(u) = \overline{\exp(u\hat{\zeta})} = \int_{-\infty}^{\infty} ds\, \partial_s g^{(0)}_N(s) e^{(\frac{N}{4})^{1/3}us}. \quad (28)$$

The integral over s in Eq. (28) can now be computed by a simple variation of Eq. (25),

$$\int_{-\infty}^{\infty} ds\, \text{Ai}'(s + x) e^{ws} = -we^{-xw}e^{w^3/3}, \quad (29)$$

where in order to simplify the notation we set $\tilde{u} \equiv (N/4)^{1/3}u$. When inserting this equality back in Eq. (28), the integral over y can be easily performed as $\epsilon > 0$ and leads to a simple pole at $z = \tilde{u}/\sqrt{N}$. This allows us to perform the integral over z, by closing the contour in the positive $\text{Re}[z]$ half plane and arriving at

$$\tilde{\rho}^{\text{DP}}_N(u) = \frac{e^{\frac{\tilde{u}^3}{3}}}{N!}\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\frac{dk_i}{2\pi} e^{-\frac{\tilde{u}k_i^2}{N}} \det\left[\frac{1}{2\tilde{u} + \imath k_{jk}}\right]_{j,k=1}^{N}. \quad (30)$$

We now check that this expression is equivalent to Eq. (13). Indeed, employing the standard expansion of a determinant as a sum over the permutation group $\mathcal{S}_N$ of N and introducing auxiliary variables $v_1, \ldots, v_N$, we have

$$\det[\frac{1}{2\tilde{u} + \imath k_{jk}}] = \sum_{P\in\mathcal{S}_N}(-1)^{\sigma_P}\prod_{j=1}^{N}\int_0^{\infty} e^{-2\tilde{u}v_j - \imath(k_j - k_{P_j})v_j}, \quad (31)$$

where σ_P is the signature of P. We can now easily perform the Gaussian integrals over the $k_1, \ldots, k_N$ variables and, relabeling $P \to P^{-1}$ in the sum, one obtains exactly the expansion of the determinant in Eq. (13) (see Ref. [46] for more details), i.e., the two Laplace transforms coincide, $\tilde{\rho}^{\text{GUE}}_N(u) = \tilde{\rho}^{\text{DP}}_N(u)$. Via a Laplace inversion, this shows our main statement, below Eq. (2), namely, that the two PDF exactly coincide in the tails, i.e., $\rho^{\text{GUE}}_N(\gamma) = \rho^{\text{DP}}_N(\gamma)$.

Note that we have assumed that the restriction to the N m-string states gives the exact tail of the PDF of $\hat{\zeta}$ at large time, in other words, that $\lim_{t\to+\infty} g_N^{(0)}(s)|_{s=(4/N)^{1/3}\zeta} = 1-\int_\zeta^{+\infty} d\zeta' \rho_N^{\rm DP}(\zeta')$, and that the neglected terms give a contribution subdominant by $O(e^{-a_N\zeta^{3/2}})$ as in Eq. (4). This, however, can be justified by examining the contributions of the remaining states, which necessarily contain a larger number of strings. As in the case of $N=1$, these lead to a larger number of Airy functions, hence to subdominant asymptotics.

Conclusion. We analyzed a general correspondence between random variables arising in very different contexts of statistical mechanics: on the one hand, the sum of the N-largest eigenvalues in the GUE, and on the other, the free energy of N noncrossing directed polymers in a $d=1+1$ random media. We provided a striking indication that these two quantities have the same distributions for any N, by comparing the tails of their PDFs at large positive values. Indeed, the perfect agreement found between the Laplace transforms associated with the leading stretched exponential decays implies the nontrivial matching of an infinite series of coefficients. This naturally extends the well-known $N=1$ case, where the single-polymer free energy, in turn, the KPZ height, maps to the largest eigenvalue of a GUE random matrix. In view of existing results for DP discrete zero temperature models, it also nicely suggests universality for $N>1$. Along the same line of ideas, one can put forward a more general conjecture, where the joint distributions of the ensemble of noncrossing free energies in the same random medium are mapped into the joint distribution of the N-largest eigenvalues, i.e.,

$$\lim_{t\to\infty}\left\{\mathfrak{F}_t\left(\frac{\mathcal{Z}_1}{\mathcal{Z}_0}\right),\ldots,\mathfrak{F}_t\left(\frac{\mathcal{Z}_N}{\mathcal{Z}_{N-1}}\right)\right\} \overset{\text{in law}}{\equiv} \{\gamma_1,\ldots,\gamma_N\}, \quad (32)$$

with $\mathfrak{F}_t(z)=t^{-1/3}(\ln z + t/12)$. It seems natural [34,41] as both ensembles of random variables involve strong correlations which reflect in the distributions of the marginals (i.e., the partial sums) studied in this Rapid Communication. A proof of this conjecture would be beneficial for a full understanding of the ubiquitous appearance of random matrix extreme statistics.

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