Spatiotemporal velocity-velocity correlation function in fully developed turbulence

Léonie Canet,[1] Vincent Rossetto,[1] Nicolás Wschebor,[2] and Guillaume Balarac[3]

[1] *Université Grenoble Alpes and Centre National de la Recherche Scientifique, LPMMC, UMR 5493, 38042 Grenoble, France*

[2] *Instituto de Física, Facultad de Ingeniería, Universidad de la República, J.H.y Reissig 565, 11000 Montevideo, Uruguay*

[3] *Université Grenoble Alpes and Centre National de la Recherche Scientifique, LEGI, UMR 5519, 38042 Grenoble, France*

(Received 24 August 2016; published 16 February 2017)

Turbulence is a ubiquitous phenomenon in natural and industrial flows. Since the celebrated work of Kolmogorov in 1941, understanding the statistical properties of fully developed turbulence has remained a major quest. In particular, deriving the properties of turbulent flows from a mesoscopic description, that is, from the Navier-Stokes equation, has eluded most theoretical attempts. Here, we provide a theoretical prediction for the functional *space and time* dependence of the velocity-velocity correlation function of homogeneous and isotropic turbulence from the field theory associated to the Navier-Stokes equation with stochastic forcing. This prediction, which goes beyond Kolmogorov theory, is the analytical fixed point solution of nonperturbative renormalization group flow equations, which are exact in the limit of large wave numbers. This solution is compared to two-point two-times correlation functions computed in direct numerical simulations. We obtain a remarkable agreement both in the inertial and in the dissipative ranges.

DOI: 10.1103/PhysRevE.95.023107

I. INTRODUCTION

Kolmogorov made a fundamental step in the understanding of the statistical properties of homogeneous and isotropic three-dimensional turbulence in his seminal K41 theory [1,2]. This theory shows that energy, which is injected at a typical large (integral) length scale L by the external stirring, is conserved across an inertial range of scales, through a constant-flux transfer mechanism (the energy cascade), until it is dissipated by molecular viscosity at a small (Kolmogorov) length scale η. Assuming universality and scale invariance in the inertial range, one then deduces from dimensional considerations scaling predictions such as the power-law decay of the kinetic energy spectrum with the well-known $-5/3$ exponent. These predictions quite reliably describe most experimental and numerical observations, at least for the energy spectrum and low-order structure functions (moments of equal-time velocity differences) [3]. However, despite many theoretical efforts, the derivation of these scaling predictions from fundamental principles, that is, from the Navier-Stokes (NS) equation for the fluid dynamics, is still unsatisfactory [3]. Moreover, deviations from K41 scalings are observed in experiments and numerical simulations and are large for high-order structure functions. These deviations are related to what is named “intermittency,” which refers to the full-fledged complexity of turbulence beyond K41 theory [3]. Calculating intermittency effects from NS equations is a longstanding unsolved issue.

In this paper, we derive analytical solutions of fixed point nonperturbative renormalization group (NPRG) (also called functional) equations associated with the NS equation. These equations are the exact leading behavior at large wave numbers. We obtain the full space and time dependence of correlation and response functions in the turbulent steady state at distances smaller than the integral scale L. Its spatial Fourier transform is found to take a form $\propto \exp(-\alpha k^2 t^2)$, where k is the wave number, t the time interval, and α a (nonuniversal) constant which could be calculated from the (numerical) solution of the full flow equations. Let us emphasize that the dependence in tk is a noticeable result. Indeed, scaling theory would imply that the correlation function depends on the scaling variable tk^z, where z is the dynamical exponent, which is $z = 2/3$ for Navier-Stokes turbulence in $d = 3$. This result shows that an effective exponent $z = 1$ arises, which is a signature of violations of scale invariance, that is, intermittency. It follows in particular that the presented solution correctly accounts for the “sweeping effect,” which is imposed by the large-scale motion on the Eulerian velocities at small scales [4]. Indeed, the energy spectrum calculated as a function of the frequency displays a $\omega^{-5/3}$ decay, as observed in experiments or numerical simulations for Eulerian velocity correlations [3,5]. This is a nontrivial result from a theoretical point of view [6–9] which reflects deviations from K41 scalings.

On the other hand, at *coinciding times*, the violations to scale invariance are found to be subleading at large k, and are hence not captured by the exact equations at leading order studied in this paper. In particular, determining corrections to the $k^{-5/3}$ decay of the energy spectrum in the inertial range would require the study of subleading equations, which goes beyond the present paper. However, the behavior of this spectrum can be determined for scales in the dissipative range, beyond the Kolmogorov regime. Several mostly empirical expressions were proposed to describe this behavior [10]. They all suggest “an approximately exponential decay” [8]. Our analytical solution shows that a crossover occurs at a scale given by Taylor scale λ, from the $k^{-5/3}$ power law in the inertial range to a stretched exponential decay following $\propto \exp[-\hat{\mu}(\lambda k)^{2/3}]$ in the dissipative range, where $\hat{\mu}$ is a (nonuniversal) constant (which could be computed by numerically integrating the full flow equations).

In order to test these predictions, we perform direct numerical simulations of fully developed isotropic turbulence from the NS equation, recording in particular the time dependence of the correlation function. The numerical solution precisely exhibits the Gaussian dependence in kt. Moreover, the behavior of the numerical energy spectrum in the dissipative range is in agreement with the predicted stretched exponential.

In summary, we provide an analytical expression for the spatiotemporal correlation and response function, accounting in particular for intermittency corrections at finite time differences, and which is confirmed by the numerical data. This expression is directly derived from the NS equation without approximation for large wave numbers. This constitutes a major step in the theoretical understanding and modeling of isotropic and homogeneous turbulence, and opens promising perspectives for the calculation of higher-order correlation and structure functions for three-dimensional and also two-dimensional turbulence.

The paper is organized as follows. The principles of the NPRG formalism and main results obtained in [11] are reviewed in Sec. II. Our starting point is the flow equation for the two-point functions in an asymptotic form which is exact at large wave numbers. We focus on the fixed point equations, and derive and analyze their general solution in Sec. III. We study in particular the behavior of the solution in the dissipative range in Sec. III C. We briefly describe in Sec. IV the direct numerical simulations, before stressing concluding remarks and perspectives.

II. NPRG FORMALISM

The results presented in this paper are based on the mesoscopic description of fluid dynamics embodied in the Navier-Stokes equation

$$\partial_t \vec{v} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{v} + \vec{f}\,. \qquad (1)$$

In this equation, the velocity field $\vec{v}$, the pressure field p, and the external stirring force $\vec{f}$ depend on the space-time coordinates $(t,\vec{x})$; ν is the kinematic viscosity; and ρ is the density of the fluid. This continuous hydrodynamical description is typically valid at scales much smaller than the Kolmogorov scale defined by

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \qquad (2)$$

where ϵ is the mean rate of injection of energy per unit mass. The presence of the external forcing $\vec{f}$ in Eq. (1) is necessary to sustain a stationary turbulent state. We consider incompressible flows, satisfying $\vec{\nabla}\cdot\vec{v} = 0$.

At characteristic distances much smaller than the integral scale, the statistical behavior of the velocity field is observed to be independent of the actual details of the forcing. Therefore, one can conveniently perform a statistical average over stochastic forcings peaked at scales of order L. These forcings are chosen Gaussian distributed, with a correlator

$$\langle f_\alpha(t,\vec{x}) f_\beta(t',\vec{x}\,')\rangle = 2\delta_{\alpha\beta}\delta(t-t')N_L(|\vec{x}-\vec{x}\,'|), \qquad (3)$$

where the profile N_L is peaked at the scale L. The NS equation with stochastic forcing can then be cast into a field theory following the standard Martin-Siggia-Rose-Janssen-de Dominicis formalism [12–14].

Since universality and power-law behaviors are expected in the inertial range, the renormalization group (RG) appears as a natural theoretical approach to study the NS field theory, and to calculate its scaling properties [15]. However, applying the perturbative RG to turbulence has a long history, dating back to the 1970s [16–19], and has turned out to be extremely challenging [20,21]. In this context, some of us have developed an alternative RG approach, based on a nonperturbative and functional RG. The NPRG is a modern implementation of Wilson's original idea [22], which is to calculate the physical properties of a system by progressively, scale by scale, averaging over fluctuations. It is an efficient procedure to compute large-scale properties even in the presence of strong correlations and fluctuations at all scales (as in critical phenomena) [23].

The NPRG consists in constructing a series of scale-dependent effective models, each of which describing the physics of the system at a given momentum scale κ. An initial condition can be specified when κ is a large wave-number scale Λ, chosen much larger than the inverse Kolmogorov scale η^{-1}, where the dynamics of the velocity field is given by the NS equation (or equivalently by NS "bare" action) [11,23]. The physical statistical properties of the model are obtained in the "infinite volume" limit $\kappa \to 0$, when all fluctuations have been taken into account.

The NPRG formalism provides exact RG flow equations, governing the evolution of these effective models when the renormalization scale κ runs from Λ to zero [23]. Solving these RG flow equations is thus a way to solve the model. However, these equations are partial-differential and functional equations for the n-point functions of the theory, that is, generalized n-point correlation and response functions. Moreover, the flow equations for the n-point functions involve the $(n+1)$- and $(n+2)$-point functions, such that one should consider in practice an infinite hierarchy of flow equations. As is common in many theoretical approaches, the usual way to deal with such an infinite hierarchy is to devise an approximation scheme to truncate it and obtain a closed equation at a given order n (a closure scheme).

In this paper, we focus on two-point (i.e., two-space point and two-time) functions ($n = 2$) in the stationary turbulent state. More precisely, we consider the scale-dependent correlation function $C_\kappa(t,\vec{x}) = \langle \vec{v}(t,\vec{x})\cdot\vec{v}(0,0)\rangle_\kappa$ and response function $G_\kappa(t,\vec{x})$ (translational invariance in time and space is assumed), and their Fourier transforms, denoted $C_\kappa(t,\vec{k})$ and $G_\kappa(t,\vec{k})$ where $\vec{k}$ is the wave vector, and t is the time difference in the stationary state. The response function is related to the mean value $\langle \vec{v}(t,\vec{x})\cdot\vec{f}(0,0)\rangle_\kappa$ at scale κ through the relation

$$\langle \vec{v}(t,\vec{x})\cdot\vec{f}(0,0)\rangle_\kappa = 2\,(d-1)\int_{\varpi,\vec{q}} N_\kappa(\vec{q})e^{-i\varpi t}G_\kappa(\omega,\vec{q}), \qquad (4)$$

where $N_\kappa(\vec{q})$ is the Fourier transform of the correlator of the external stochastic forcing defined in Eq. (3), peaked at scale κ. In particular, one can show that the mean energy injection rate is given by

$$\epsilon = \langle \vec{v}(0,0)\cdot\vec{f}(0,0)\rangle_\kappa = D_\kappa\kappa^d\gamma^{-1}, \qquad (5)$$

where D_κ is the effective (renormalized) forcing strength, d is the space dimension, and γ is a pure (nonuniversal) number depending on the precise choice of the forcing profile $N(q)$ [24].

A. Closure of the flow equations

As mentioned above, the exact NPRG flow equations for the two-point functions involve three- and four-point functions. Usually, in order to solve these equations, one must truncate them in some way. Such a truncation (usually called closure) was achieved for the NS problem within the NPRG context in several related works [11,25,26]. The approximation implemented in these works, referred to as leading order (LO) approximation, is very much inspired from similar ones developed in the context of the closely related Kardar-Parisi-Zhang (KPZ) equation describing interface growth and kinetic roughening [27]. This approximation scheme has yielded for KPZ very accurate results [28–30], and it can be quite straightforwardly transposed to NS since the KPZ and NS field theories share many common features, in particular (time-gauged) symmetries [31].

For NS, the LO approximation consists in a truncation at quadratic order in the velocity fields (velocity and the associated Martin-Siggia-Rose response velocity), neglecting all higher-order functions but the unrenormalized nonlinearity (three-point vertex). This approximation is well controlled in the small wave-number sector $|\vec{k}| \ll \kappa$ of the theory, and thus provides an accurate description of this regime (see detailed discussion in [11]). It was shown at LO that the NPRG flow reaches a fixed point, which encompasses the universal properties of the turbulent steady state [11,25,26]. The existence of this fixed point was also well known from the perturbative RG [20,21].

Besides these studies, some of us made a decisive step in [11], by showing that in the regime of large wave numbers truncation can be *avoided*. The closure of the flow equations for the two-point functions can be achieved without approximation in the large wave-number regime, by only exploiting the symmetries of the NS action. This result, exceptional in the NPRG framework, relies on the existence of very constraining symmetries (time-gauged – or time-dependent, ones, in particular a time-gauged shift unveiled in [31]) and the extensive use of the related Ward identities, and also on other specificities of the NS equation referred to as "nondecoupling" (see below).

B. Exact flow equations in the limit of large wave numbers

The equations derived in [11] following from a symmetry-based closure are our starting point. They give the flow equations for the two-point (response and correlation) functions in Fourier space, and are exact in the limit of large wave number $|\vec{k}| \gg \kappa$:

$$\kappa\partial_\kappa C_\kappa(\omega,\vec{k}) = -\frac{2}{3}k^2 \int_\varpi \frac{C_\kappa(\omega+\varpi,\vec{k}) - C_\kappa(\omega,\vec{k})}{\varpi^2} J_\kappa(\varpi),$$
$$\kappa\partial_\kappa G_\kappa(\omega,\vec{k}) = -\frac{2}{3}k^2 \int_\varpi \frac{G_\kappa(\omega+\varpi,\vec{k}) - G_\kappa(\omega,\vec{k})}{\varpi^2} J_\kappa(\varpi), \tag{6}$$

in units where $\nu = \eta = 1$. The Fourier conventions used in this paper are

$$f(t,\vec{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega,\vec{k})\, e^{-i\omega t},$$
$$f(\omega,\vec{k}) = \int_{-\infty}^{\infty} dt\, f(t,\vec{k})\, e^{i\omega t}, \tag{7}$$

keeping the same notation for the function and its Fourier transform. In Eq. (6), J_κ is the integral

$$J_\kappa(\varpi) = -\int_{\vec{q}} \{2\partial_s N_s(\vec{q})\, |G_\kappa(\varpi,\vec{q})|^2 - 2\partial_s R_s(\vec{q})\, C_\kappa(\varpi,\vec{q}) \Re[G_\kappa(\varpi,\vec{q})]\}, \tag{8}$$

where $s = \ln(\kappa/\Lambda)$ is the "RG time," with $\partial_s = \kappa\partial_\kappa$. N_s is the forcing profile defined in Eq. (3), and R_s is a momentum profile involved in the NPRG procedure to freeze all fluctuations with wave numbers $|\vec{k}| \lesssim \kappa$, and which vanishes in the limit $\kappa \to 0$ (see [11] for details).

As shown in [11], these flow equations exhibit a very peculiar property compared to ordinary critical phenomena, named the nondecoupling property: the flow does not vanish when the RG scale is much smaller than a given wave number of the correlation function, that is,

$$\lim_{|\vec{k}|\gg\kappa} \frac{\kappa\partial_\kappa X_\kappa(\omega,\vec{k})}{X_\kappa(\omega,\vec{k})} \neq 0, \tag{9}$$

where X_κ stands for C_κ or G_κ. This nondecoupling property opens the door for some intermittency effects. Indeed, the stationary turbulent state corresponds to a fixed point of the flow, which leads to universality and power-law behaviors. However, the nondecoupling property implies that the behavior at large wave numbers can deviate from dimensional scaling (see [11]). This is indeed what we find in the following. On the other hand, the decoupling property is restored at *equal times*. Indeed, integrating Eq. (6) over frequencies leads to

$$\int_\omega \kappa\partial_\kappa X_\kappa(\omega,\vec{k}) = 0, \tag{10}$$

which means that standard K41 scaling results are recovered at equal times. Intermittency corrections are hence absent at leading order in $\vec{k}$ for equal-time correlation and response functions, which implies that these corrections must be small, in agreement with experimental and numerical observations [3]. Possible corrections come from the subleading orders, which are not included in the equations studied in this paper. These corrections would be accessible by integrating the full flow equations, given in [11], using some approximations to close them, for instance along the lines of [29].

The flow equations for the correlation function and response function in real time can be deduced from Eqs. (7) and (6), which yields

$$\kappa\partial_\kappa X_\kappa(t,\vec{k}) = -\frac{2}{3}k^2 X_\kappa(t,\vec{k}) \int_\varpi \frac{\cos(\varpi t) - 1}{\varpi^2} J_\kappa(\varpi). \tag{11}$$

In the following, we consider the regime of small time differences $t \ll \kappa^{-2/3}$, or equivalently large frequencies $\omega \gg \kappa^{2/3}$. In this limit, Eq. (11) takes a simple form

$$\kappa\partial_\kappa X_\kappa(t,\vec{k}) = \tfrac{1}{3} k^2 t^2 I_\kappa X_\kappa(t,\vec{k}), \tag{12}$$

where I_κ is the integral

$$I_\kappa = \int_\varpi J_\kappa(\varpi). \tag{13}$$

Note that this integral is a function of the RG scale κ only. It does not depend on the external wave vector $\vec{k}$ and time difference t of the correlation or response functions X_κ. Moreover, it is determined by the small wave-number sector. Indeed, it is shown in [11] that the internal wave vector $\vec{q}$ and frequency ϖ appearing in the integral (13) are dominated by values of order $|\vec{q}| \lesssim \kappa$ and $\varpi \lesssim \kappa^{2/3}$, respectively, which hence belong to the opposite regime as the one studied here. This integral can be reliably (but approximately) computed within an approximation controlled in the small wave-number sector, such as the LO one.

C. Fixed point equation

As mentioned previously, the RG flow associated to NS turbulence leads to a fixed point. We are interested in the vicinity of this fixed point where universality is expected. As usual in RG studies, the fixed point is most conveniently studied in terms of dimensionless quantities, such that all explicit κ dependence is absorbed. In the NPRG procedure for NS, two renormalized coefficients ν_κ and D_κ are introduced, which correspond, respectively, to the effective viscosity and forcing strength (see [11] for the precise definitions). Close to a fixed point, these running coefficients behave as power laws $\nu_\kappa \sim \kappa^{-4/3}$ and $D_\kappa \sim \kappa^{-3}$ in $d = 3$. The effective viscosity is hence approximately related to the microscopic viscosity as

$$\nu_\kappa \simeq \nu(\kappa\eta)^{-4/3} = \epsilon^{1/3}\kappa^{-4/3}, \tag{14}$$

using Eq. (2) and neglecting the small evolution of ν_κ at the beginning of the flow when it is not a power law yet. The effective forcing strength D_κ is related to the mean injection rate ϵ following Eq. (5).

We introduce dimensionless variables, denoted with a hat symbol. Momenta are measured in units of κ and times are measured in units of $(\nu_\kappa\kappa^2)^{-1} = \epsilon^{-1/3}\kappa^{-2/3}$. The response function $G_\kappa(t,\vec{k})$ is dimensionless, and the dimension of $C_\kappa(t,\vec{k})$ is $D_\kappa/(\nu_\kappa\kappa^2) = \gamma\epsilon^{2/3}\kappa^{-11/3}$. Let us hence define the dimensionless functions

$$G_\kappa(t,\vec{k}) = \hat{G}_s(\hat{y} = \epsilon^{1/3}\, tk^{2/3}, \hat{x} = (k/\kappa)^{2/3}),$$
$$C_\kappa(t,\vec{k}) = \frac{\gamma\epsilon^{2/3}}{k^{11/3}}\hat{C}_s(\hat{y} = \epsilon^{1/3}\, tk^{2/3}, \hat{x} = (k/\kappa)^{2/3}). \tag{15}$$

We also introduce the dimensionless integral $\hat{I}_s$ as

$$I_\kappa = \frac{D_\kappa\kappa^3}{\nu_\kappa\kappa^2}\hat{I}_s = \gamma\epsilon^{2/3}\,\kappa^{-2/3}\hat{I}_s. \tag{16}$$

According to Eq. (12), the dimensionless functions satisfy the flow equation

$$\partial_s\hat{X}_s(\hat{y},\hat{x}) - \tfrac{2}{3}\hat{x}\partial_{\hat{x}}\hat{X}_s(\hat{y},\hat{x}) = \tfrac{1}{3}\,\hat{y}^2\,\hat{x}\,\gamma\,\hat{I}_s\,\hat{X}_s(\hat{y},\hat{x}). \tag{17}$$

The fixed point equation corresponds by definition to $\partial_s\hat{X}_s = 0$, and $\hat{I}_s \to \hat{I}_*, \hat{X}_s(\hat{y},\hat{x}) \to \hat{X}_*(\hat{y},\hat{x})$ tend to fixed quantities independent of s. Hence, the fixed point equation reads as

$$\partial_{\hat{x}}\hat{X}_*(\hat{y},\hat{x}) = -\tfrac{1}{2}\gamma\,\hat{I}_*\hat{y}^2\,\hat{X}_*(\hat{y},\hat{x}) \equiv -\hat{\alpha}\,\hat{y}^2\,\hat{X}_*(\hat{y},\hat{x}), \tag{18}$$

for both functions $\hat{C}_s$ and $\hat{G}_s$, with $\hat{\alpha}$ a nonuniversal constant (depending on the precise forcing profile.) At LO approximation, one finds that $\hat{I}_s$ tends to a positive constant $\hat{I}_*$ of the order 4×10^{-2} at the fixed point, and $\hat{\alpha}$ is hence positive. Let us analyze the solution of this equation.

III. GENERAL SOLUTION

The general solution of Eq. (17) is given by

$$\hat{X}(\hat{y},\hat{x}) = F_X(\hat{y})\,\exp[-\hat{\alpha}\,\hat{y}^2\,\hat{x}], \tag{19}$$

where $F(\hat{y})$ is an arbitrary function that must be determined by boundary conditions. We are interested in the regime of large wave numbers $|\vec{k}| \gg \kappa$ and small time differences $t \ll \kappa^{-2/3}$. In this regime $\hat{y} \ll tk$. Moreover, the RG flow ensures that all functions are analytic smooth functions. Hence, in this regime, $F_X(\hat{y})$ can be replaced by $F_X(\hat{y}) \simeq F_X(0) \equiv c_X$.

A. Time dependence

To analyze this solution, let us first restore the dimensions. The physical functions are obtained when the RG scale κ tends to zero (infinite volume limit). In fact, the RG flow reaches a fixed point, and the relevant scale is the inverse integral scale L^{-1}: the functions are essentially unchanged when κ further decreases. Hence, one has

$$C(t,\vec{k}) = c_C\,\frac{\gamma\epsilon^{2/3}}{k^{11/3}}\,\exp(-\hat{\alpha}\epsilon^{2/3}L^{2/3}t^2k^2)$$
$$= c_C\,\frac{\gamma\epsilon^{2/3}}{k^{11/3}}\,\exp(-\tilde{\alpha}\epsilon^{2/3}\eta^{2/3}t^2k^2), \tag{20}$$

and similarly for G, with $\tilde{\alpha} = a\hat{\alpha}\mathrm{Re}^{1/2}$ (with a a numerical factor of order 1), using that $\eta/L \sim \mathrm{Re}^{-3/4}$ where Re is the Reynolds number.

The expression (20) hence yields that the dominant behavior of the correlation function is a Gaussian dependence in the variable tk, and not in the scaling variable $tk^{2/3}$. This is a nontrivial result, since it implies a violation of standard scale invariance, which is a signature of intermittency. It means that the value of the critical dynamical exponent $z = 2/3$ is effectively changed to the value $z = 1$, which represents a strong correction to Kolmogorov scaling. A physical manifestation of this is what is called the sweeping effect, discussed below.

Whereas equal-time quantities are easily measured in experiments and numerical simulations, recording the time dependence is more difficult. The expression (20) hence provides an interesting prediction, that we tested in numerical simulations, and that could be studied in experimental settings. We performed direct numerical simulations of the NS equation at two different Taylor-scale Reynolds numbers $R_\lambda = 90$ and 160. The detail of the simulation is described in Sec. IV. We computed for both R_λ the function $C(t,k)$. The result for $R_\lambda = 160$ is represented in Fig. 1, where C is plotted as a function of the dimensionless variable $\hat{z}^2 = \epsilon^{2/3}\eta^{2/3}k^2t^2$ for different values of $k\eta$. The upper plot is in log scale. The different curves for each $k\eta$ appear as parallel straight lines as expected from the predicted form of Eq. (20). The

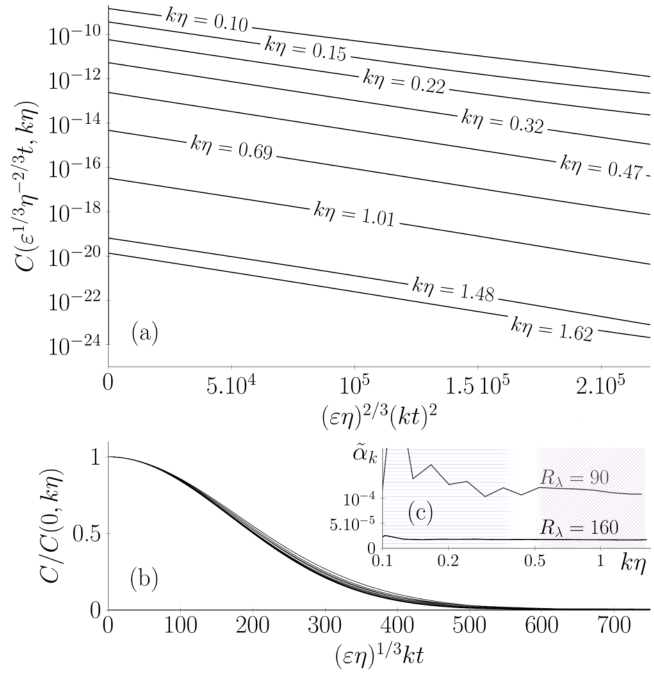

FIG. 1. Time dependence of the correlation function $C(t,k)$ in k space (in dimensionless form) computed from direct numerical simulations at $R_\lambda = 90$ and 160. (a) Correlation function $C(t,k)$ for $R_\lambda = 160$ in log scale as a function of the dimensionless variable $\hat{z}^2 = (\epsilon\eta)^{2/3}(kt)^2$, for different values of $k\eta$. In this representation, the curves for the different $k\eta$ are parallel lines, with the same slope $\tilde{\alpha}$, confirming the NPRG prediction. (b) Illustration of the collapse of the normalized correlation function $C(t,k)/C(0,k)$ for $R_\lambda = 160$ for all values of k, on a single Gaussian curve in the dimensionless variable $\hat{z}$. (c) Value of the parameter $\tilde{\alpha}_k$ estimated from Gaussian fits of the data for all the different values of k, for both $R_\lambda = 90$ and 160. $\tilde{\alpha}_k = \tilde{\alpha}$ is perfectly constant for $R_\lambda = 160$ (approximatively for $R_\lambda = 90$).

lower plot shows the precise collapse of the normalized function $C(t,k)/C(0,k)$ (in dimensionless form) on a unique Gaussian. The numerical data hence accurately confirm the NPRG prediction. The values of the parameter $\tilde{\alpha}$ estimated from the numerical data are given in Table I.

TABLE I. Parameters of the numerical simulations for the different Taylor Reynolds number R_λ: Taylor microscale λ, viscosity ν, Kolmogorov scale η, mean energy injection rate ϵ; and fitting parameters: Kolmogorov constant C_K, dimensionless parameter of the Gaussian time dependence $\tilde{\alpha}$, and dimensionless parameter of the stretched exponential decay $\hat{\mu}$ (see text).

R_λ	90	160	433
λ	0.236(7)	0.161(4)	0.118
ν	10^{-4}	10^{-4}	1.8×10^{-4}
η	0.01264(7)	0.00642(4)	0.00287
ϵ	0.0000392(9)	0.00059(2)	0.0928
C_K	2.26	1.90	2.24
$\tilde{\alpha}$	1.2×10^{-4}	1.2×10^{-5}	–
$\hat{\mu}$	1.00	0.78	0.46

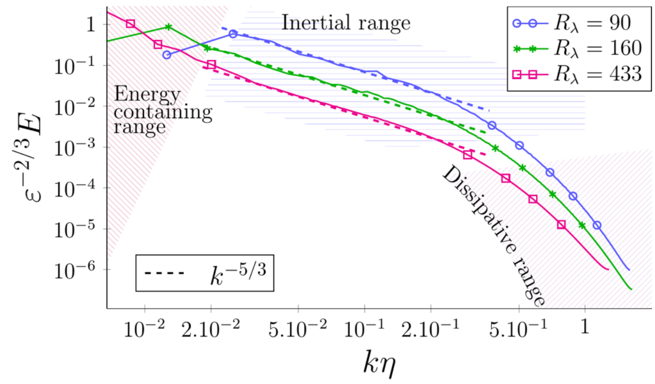

FIG. 2. Kinetic energy spectra in $d = 3$ obtained from direct numerical simulations at different Taylor-scale Reynolds numbers $R_\lambda = 90,160,433$ (from top to bottom), plotted in a dimensionless form: $\epsilon^{-2/3}E$ as a function of $k\eta$. The data for $R_\lambda = 90$ and 160 are from this work; the data for $R_\lambda = 433$ come from the Johns Hopkins Turbulence Database [32]. The numerical spectra display an inertial range, extending with R_λ, with a $k^{-5/3}$ decay (up to possible small corrections).

B. Inertial range of the energy spectrum

Equal-time quantities can also be deduced from the solution (20). In particular, the kinetic energy spectrum is obtained as

$$E(k) = 4\pi k^2 C(t=0,k) = 4\,\pi\,\gamma\,c_C\,\epsilon^{2/3}\,k^{-5/3}. \tag{21}$$

It decays as a power-law with the Kolmogorov exponent $-5/3$. This is in accordance with our statement below Eq. (10) that the decoupling property is restored at equal times. To illustrate this result, the energy spectra obtained from numerical simulations are shown in Fig. 2, in a dimensionless form. The numerical data come from the simulations we ran for $R_\lambda = 90$ and 160, and from the Johns Hopkins Turbulence database [32] for $R_\lambda = 433$. These spectra exhibit a substantial inertial range, extending with R_λ, with a clear $k^{-5/3}$ decay (up to possible small corrections).

More interestingly, let us compute the energy spectrum as a function of the frequency. This yields

$$\begin{aligned} E(\omega) &= 4\pi \int_0^\infty dk\, k^2 C(\omega,k) \\ &= 4\pi \int_0^\infty dk\, k^2 \int_{-\infty}^\infty dt\, C(t,\vec{k})\, e^{i\omega t} \\ &= 2^{8/3} c_C\, \pi^{3/2} \gamma\, \epsilon^{8/9}\, \eta^{2/9}\, \hat{\alpha}^{1/3}\, \Gamma(5/6)\, \omega^{-5/3}, \end{aligned} \tag{22}$$

using Eq. (20). Hence, the decay of the energy spectrum as a function of the frequency is found to also decay with the exponent $-5/3$. This result corresponds to what is observed in experiments and numerical simulations for velocities measured in a fixed reference frame (Eulerian velocities) [3]. The equality of the exponents in Eqs. (21) and (22) is rooted in the fact that the wave number and time interval appear as the combination tk in Eq. (20), that is, with an effective dynamical exponent $z = 1$, and not $z = 2/3$, as emphasized previously. The latter would yield a ω^{-2} decay for the energy spectrum in frequency, which is characteristic of velocities measured along the flow (Lagrangian velocities), but is not observed for

Eulerian ones [5]. Here, the solution (20) correctly predicts the same power in frequency or wave number. Let us underline that we consider a fluid without mean flow. This observation is hence not related to Taylor's frozen turbulence hypothesis. In contrast, in the absence of mean flow, the $\omega^{-5/3}$ decay is usually attributed to what is named the sweeping effect, which is the random advection of small eddies past the observation point by large energy-containing eddies. This can be viewed as a statistical form of Taylor's hypothesis, as introduced by Tennekes [4]. This result is nontrivial from a field-theoretical point of view, since it implies that standard scale invariance is violated, which can be attributed to intermittency. This shows that this effect is properly taken into account in the solution (20).

C. Dissipative range of the energy spectrum

The solution (20) is valid for large wave numbers and small time intervals. Let us study more precisely the limit of vanishing time differences. In the previous section, t was sent strictly to zero. This supposes in turn that one can consider arbitrary small time differences, which is equivalent to sending Kolmogorov scale η to zero. This is of course justified in the inertial range, but not on smaller scales, that is, in the dissipative range. In practice, the smaller time difference is bounded by Kolmogorov time:

$$\tau = \left(\frac{\nu}{\epsilon}\right)^{1/2}. \tag{23}$$

Let us hence consider the limit $t \to \tau$, and $k \gg \kappa \sim L^{-1}$. It is reasonable to assume that the scaling variable $\hat{y}$ saturates in this limit to a constant value given by

$$\hat{y} = \epsilon^{1/3} t k^{2/3} \to \hat{y}_0 = \epsilon^{1/3}\tau L^{-2/3} = \eta^{2/3} L^{-2/3}. \tag{24}$$

The fixed point Eq. (18) then becomes

$$\partial_{\hat{x}} \hat{X}_*(\hat{y} \to \hat{y}_0, \hat{x}) = -\hat{\alpha}\eta^{4/3} L^{-4/3} \hat{X}_*(\hat{y},\hat{x}). \tag{25}$$

The general solution of this equation reads

$$\hat{X}_*(\hat{y} \to \hat{y}_0,\hat{x}) = c_X \exp(-\hat{\alpha}\eta^{4/3} L^{-4/3} \hat{x}) \tag{26}$$

and the energy spectrum is thus given in this limit by a stretched exponential:

$$E(k) = 4\pi\gamma\epsilon^{2/3} k^{-5/3} c_C \exp[-\hat{\alpha}\eta^{4/3} L^{-2/3} k^{2/3}]. \tag{27}$$

Remarkably, a new scale emerges in the exponential, which is Taylor scale. Indeed, the latter is related to the Kolmogorov and integral scale through

$$\frac{\lambda}{\eta} \sim \mathrm{Re}^{1/4}, \quad \text{and } \frac{\lambda}{L} \sim \mathrm{Re}^{-1/2}, \tag{28}$$

from which one deduces that

$$\frac{\eta^{2/3}}{L^{1/3}} \sim \lambda^{1/3}\,\mathrm{Re}^{-3/4}, \tag{29}$$

and thus

$$E(k) = 4\pi\gamma\epsilon^{2/3} k^{-5/3} c_C \exp[-\hat{\mu}(\lambda k)^{2/3}], \tag{30}$$

with $\hat{\mu}$ a nonuniversal constant $\hat{\mu} = b\hat{\alpha}\mathrm{Re}^{-3/2}$, where b is a numerical factor of order 1. Hence, the expression (30) shows that, beyond the $k^{-5/3}$ Kolmogorov decay, a crossover to a

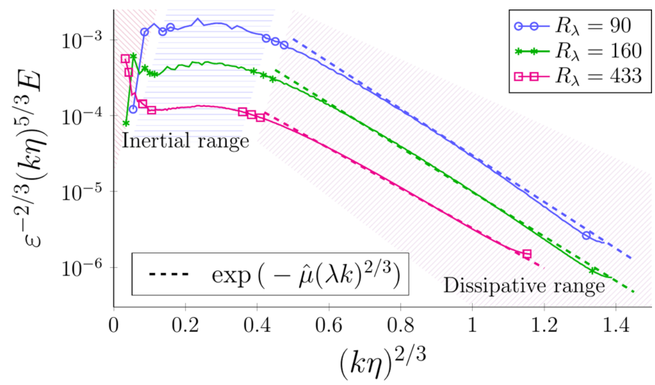

FIG. 3. Same dimensionless kinetic energy spectra as in Fig. 2, multiplied by $(k\eta)^{5/3}$ and represented in log scale as a function of $(k\eta)^{2/3}$. The NPRG predicts a crossover from the $k^{-5/3}$ power law to a stretched exponential decay $\exp[-\hat{\mu}(\lambda k)^{2/3}]$ in the dissipative range (dashed lines), which is observed in the numerical data (plain lines with symbols).

stretched exponential decay with argument $k^{2/3}$ occurs typically below the Taylor scale, that is, for wave numbers in the dissipative range. Several expressions have been proposed for this regime, mainly under the form of a modified exponential $\exp(-ck^y)$, but with different values for y (1/2 [33], 3/2 [34], 4/3 [35], or 2 [36,37]). These expressions are mostly based on approximate fits of the experimental data or (approximate) analytical considerations [10]. The common wisdom is that the spectrum decay is "approximately exponential" in the dissipative range [8]. In order to assess the prediction (30), we analyzed the numerical energy spectra in the dissipative range. The stretched exponential on the scale $k^{2/3}$ is indeed observed, as illustrated in Fig. 3, although the number of decades available in the data is of course limited. The value of the parameter $\hat{\mu}$ estimated from the numerical data is given in Table I.

IV. DIRECT NUMERICAL SIMULATIONS

The numerical simulations performed to obtain the data of incompressible forced homogeneous isotropic turbulence at $R_\lambda = 90$ and 160 are based on a pseudospectral code with second-order explicit Runge-Kutta time advancement [38]. The simulation domain is discretized using 256^3, respectively, 512^3, grid points on a domain of length 2π for $R_\lambda = 90$, respectively, 160. A classic 3/2 rule is used for dealiasing the nonlinear convection term and a projection method in spectral space is used to enforce the divergence-free condition. The forcing is a fully random forcing concentrated at small wave numbers [39]. The simulation parameters are chosen such that $k_{\max}\eta > 1.5$, where $k_{\max}$ is the maximum wave number in the domain. A typical configuration of the modulus of the velocity field obtained in the simulation for $R_\lambda = 160$ is represented at different times in Fig. 4. Additional data from the Johns Hopkins Turbulence Database [32] are also used for the kinetic energy spectra in Figs. 2 and 3. They correspond to simulations on 1024^3 nodes of isotropic turbulence with $R_\lambda \simeq 433$. The relevant parameters for the simulations are gathered in Table I.

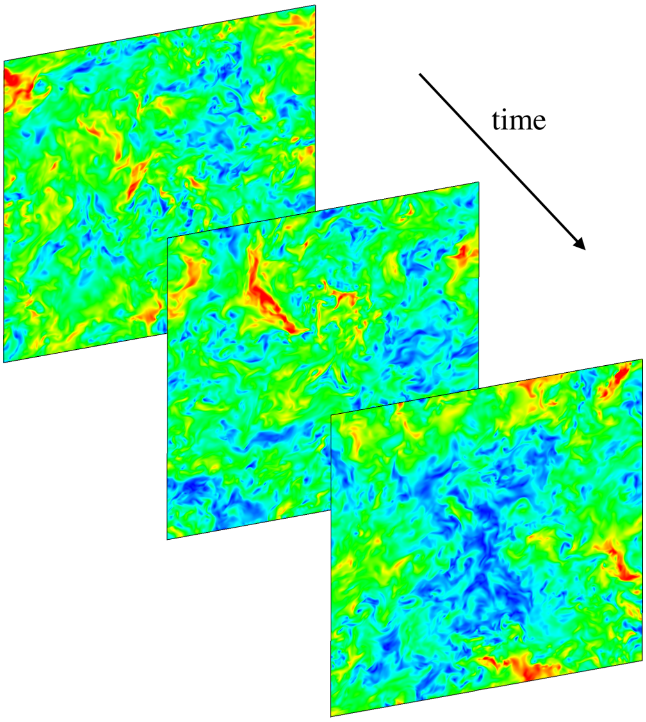

FIG. 4. Typical map of the modulus of the velocity field in simulated turbulence for $R_\lambda = 160$, at different times.

V. CONCLUSION

In this paper, we provide an analytical expression for the space- and time-dependent correlation (and response) functions of a fluid in a forced turbulent state. These expressions are derived from NPRG flow equations which are exact in the limit of wave numbers large compared to the inverse integral scale. We show that these expressions yield predictions beyond the standard observations and Kolmogorov theory. The essential aspects are (i) the time dependence for the correlation function in k space $\propto \exp(-\alpha t^2 k^2)$ with an effective dynamical exponent $z = 1$, which implies a strong correction to standard scaling theory; (ii) a related $\omega^{-5/3}$ decay of the energy spectrum, as observed for Eulerian velocities, and which reflects the sweeping effect; and (iii) a stretched exponential decay as $k^{-5/3}\exp[-\hat{\mu}(\lambda k)^{2/3}]$ of the spectrum in the dissipative range. We believe that deriving such analytical solutions, directly from the NS equation, and not on a phenomenological basis, constitutes a major progress in the theoretical understanding of isotropic and homogeneous turbulence, and its modeling at all scales.

It opens perspectives in many respects. An important issue to be addressed is the numerical integration of the complete flow equations (in both the small and large wave-number sectors), to assess intermittency corrections for equal-time quantities. This analysis would require one to make some approximations in order to truncate the full flow equations. Another important direction is the investigation of two-dimensional turbulence, and the derivation of correlation functions for scales both below the integral scale (direct cascade) and above (inverse cascade). Moreover, a promising perspective is the computation of higher-order structure functions, and the determination of intermittency effects in this case, which are expected to be much more pronounced as the order increases.

ACKNOWLEDGMENTS

The authors thank B. Delamotte, L. Chevillard, and M. Tarpin for fruitful discussions. Simulations were performed using high-performance computing resources from GENCI-IDRIS (Grant No. 020611).

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941) [Proc. Roy. Soc. London A **434**, 9 (1991)].
[2] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **32**, 16 (1941) [Proc. Roy. Soc. London A **434**, 15 (1991)].
[3] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University, Cambridge, England, 1995).
[4] H. Tennekes, J. Fluid Mech. **67**, 561 (1975).
[5] L. Chevillard, S. G. Roux, E. Leveque, N. Mordant, J. F. Pinton, and A. Arneodo, Phys. Rev. Lett. **95**, 064501 (2005).
[6] V. Yakhot, S. A. Orszag, and Z.-S. She, Phys. Fluids A **1**, 184 (1989).
[7] S. Chen and R. H. Kraichnan, Phys. Fluids A **1**, 2019 (1989).
[8] M. Nelkin and M. Tabor, Phys. Fluids A **2**, 81 (1990).
[9] T. Gotoh, R. S. Rogallo, J. R. Herring, and R. H. Kraichnan, Phys. Fluids A **5**, 2846 (1993).
[10] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence*, 2nd ed. (MIT, Cambridge, 1973).
[11] L. Canet, B. Delamotte, and N. Wschebor, Phys. Rev. E **93**, 063101 (2016).
[12] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
[13] H.-K. Janssen, Z. Phys. B **23**, 377 (1976).
[14] C. de Dominicis, J. Phys. (Paris) Colloq. **37**, C1-247 (1976).
[15] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed. (Oxford University, New York, 1989).
[16] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. Lett. **36**, 867 (1976).
[17] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[18] C. DeDominicis and P. C. Martin, Phys. Rev. A **19**, 419 (1979).
[19] J. D. Fournier and U. Frisch, Phys. Rev. A **28**, 1000 (1983).
[20] L. T. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon and Breach, London, 1999).
[21] Y. Zhou, Phys. Rep. **488**, 1 (2010).
[22] K. G. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1974).
[23] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002).
[24] Note that the limit $t \to 0$ in Eq. (4) must be taken with care, as it would naively yield zero in Ito's prescription, but is in fact nonvanishing (see [11]).

[25] P. Tomassini, Phys. Lett. B **411**, 117 (1997).
[26] C. Mejía-Monasterio and P. Muratore-Ginanneschi, Phys. Rev. E **86**, 016315 (2012).
[27] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[28] L. Canet, H. Chaté, B. Delamotte, and N. Wschebor, Phys. Rev. Lett. **104**, 150601 (2010).
[29] L. Canet, H. Chaté, B. Delamotte, and N. Wschebor, Phys. Rev. E **84**, 061128 (2011).
[30] T. Kloss, L. Canet, and N. Wschebor, Phys. Rev. E **86**, 051124 (2012).
[31] L. Canet, B. Delamotte, and N. Wschebor, Phys. Rev. E **91**, 053004 (2015).
[32] Y. Li, E. Perlman, M. Wan, Y. Yang, C. Meneveau, R. Burns, S. Chen, A. Szalay, and G. Eyink, J. Turbulence **9**, 31 (2008).
[33] V. I. Tatarskii, *Line of Sight Propagation Fluctuations* in *Atmospheric Turbulence and Radio Wave Propagation*, edited by A. M. Yaglom and V. I. Tatarskii (Nauka, Moscow, 1967), pp. 314–329.
[34] M. S. Uberoi and P. Freymuth, Phys. Fluids **12**, 1359 (1969).
[35] Y.-H. Pao, Phys. Fluids **8**, 1063 (1965).
[36] E. A. Novikov, Dokl. Akad. Nauk SSSR **139**, 331 (1961).
[37] A. S. Gurvich, B. M. Koprov, L. R. Tsvang, and A. M. Yaglom, *Data on the Small-Scale Structure of Atmospheric Turbulence* in *Atmospheric Turbulence and Radio Wave Propagation*, edited by A. M. Yaglom and V. I. Tatarskii (Nauka, Moscow, 1967), pp. 30–52.
[38] J.-B. Lagaert, G. Balarac, and G.-H. Cottet, J. Comp. Phys. **260**, (2014).
[39] K. Alvelius, Phys. Fluids **11**, 1880 (1999).