Noise, delocalization, and quantum diffusion in one-dimensional tight-binding models

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As an unusual type of anomalous diffusion behavior, namely (transient) superballistic transport, has been experimentally observed recently, but it is not yet well understood. In this paper, we investigate the white noise effect (in the Markov approximation) on quantum diffusion in one-dimensional tight-binding models with a periodic, disordered, and quasiperiodic region of size L attached to two perfect lattices at both ends in which the wave packet is initially located at the center of the sublattice. We find that in a completely localized system, inducing noise could delocalize the system to a desirable diffusion phase. This controllable system may be used to investigate the interplay of disorder and white noise, as well as to explore an exotic quantum phase.

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I. INTRODUCTION

Quantum diffusion in one-dimensional (1D) tight-binding models has a rich background [1–3]. First, numerical evidence was presented by Hufnagel *et al.* [4] to support the position that the variance of a wave packet in 1D tight-binding models can exhibit a superballistic increase ($\sigma^2 = t^\nu$ with $2 < \nu \leqslant 3$) for parametrically large time intervals with the appropriate model. They replaced the disordered part by a point source in which anything emitted from it could move with a constant velocity modeling the dynamics of a perfect lattice [4]. The model explains this phenomenon, and its predictions were verified numerically for various periodic, disordered, and quasiperiodic systems. Then, superballistic diffusion of entanglement was constructed in disordered spin chains [5]. Zhang *et al.* [6] found a superballistic increase in variance $\sigma^2 = t^\nu$ with $3 < \nu \leqslant 4.7$ numerically, and they extended the interpretation given in Ref. [4] to diffusion rates beyond cubic. The first experimental observation of superballistic growth of the variance occurred for optical wave packets in 2013 [7].

Fractal [1,8,9] and multifractal [10] analysis of the width of a spreading wave packet revealed that for systems in which the shape of the wave packet is preserved, the kth moment evolves as $t^{k\beta}$ with $\beta = D_2^\mu / D_2^\Psi$, where in general $t^{k\beta}$ is an optimal lower bound, D_2^μ is the correlation dimension of the spectral measure μ (i.e., the local density of states), and D_2^Ψ is the correlation dimension of the (suitably averaged) eigenfunctions. The (disorder or phase-averaged) diffusion exponent is of particular physical importance because it characterizes the low-temperature behavior of the direct conductivity as given by Kubo's formula in the relaxation-time approximation [11].

In this article we describe the system in universal terms, not specific to matter waves, as manifested by the analogy between the Schrödinger equation and the paraxial wave equation. Hyper-diffusion is in fact a universal concept that should be observable in a variety of systems beyond matter waves, such as optics, sound waves, plasma, as well as conducting electron transport in semiconductors [12]. Furthermore, fundamentally, once such temporal acceleration causes high velocities to be reached, the relativistic effects have to be included. These ideas have opened a range of exciting possibilities. However, in view of the recent experiments on quantum walks of correlated photons [13] and on localization with entangled photons [14], it will be very interesting to determine whether the phenomenon of hyper-transport occurs not only with entangled spin chains, but with entangled photons as well [5].

Taking the decoherence problem into account, the effect of temperature on wave-packet spreading is an essential feature [15]. The theoretical description of relaxation and decoherence processes in open quantum systems often leads to a non-Markovian dynamics, which is determined by pronounced memory effects. The following can lead to long memory times and to a failure of the Markovian approximation [16]: (i) strong system-environment couplings, correlations, and entanglement in the initial state; (ii) interactions with environments at low temperatures and with spin baths; (iii) finite reservoirs; and (iv) transport processes in nanostructures. However, since we intend here to investigate the effect of white noise on super-ballistic diffusion in 1D tight-binding lattices, we do not deal with these restrictions; we can use the Markov approximation and the Lindblad equation. In strong system-environment couplings, we can also use a similar method called Non-Markovian generalization of the Lindblad theory of open quantum systems [16,17].

In this work, we examined the noise effect on the quantum wave-packet dynamics in several nonuniform 1D tight-binding lattices, where a sublattice with on-site potential is embedded in a lattice with uniform potential, irrespective of whether the sublattice on-site potential is periodic, disordered, or quasiperiodic (some cases were studied in the absence of any environment in Refs. [4,6]). We found the threshold values of the white noise strength, beyond which quantum superballistic diffusion does not occur (in the disordered case, the observed disappearance of super-ballistic diffusion is based on a fixed number of realizations of the sublattice). Such threshold values for the disappearance of quantum super-ballistic diffusion should be one of the key elements in real experimental studies, where environment and noise have a significant dephasing effect. Furthermore, based on our numerical studies, we predict that the quantum diffusion exponent can be tuned extensively via the amount of induced white noise. Thus, we can manually induce noise in a system to drive it to a desired diffusion rate. The results must be within reach of present-day cold-atom experiments.

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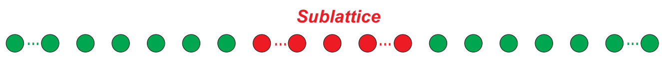

FIG. 1. Schematic of the 1D lattice. The red circles (central part) represent sites with on-site potential, while the green circles (in the left and right parts) represent sites without any on-site potential.

II. THE SYSTEM

A. The lattice

In this work, we examine quantum wave-packet dynamics in several nonuniform 1D tight-binding lattices, where a sublattice with on-site potential is embedded in a uniform lattice without on-site potential (see Fig. 1). Assume the following 1D tight-binding Hamiltonian:

$$H = -\Sigma_{i,j}(t_{ij}c_i^\dagger c_j + t_{ij}^* c_j^\dagger c_i) + \Sigma_i V_i c_i^\dagger c_i, \qquad (1)$$

where t_{ij} is the tunneling rate from site i to site j, and we set $t_{ij} = t_{ij}^* = -1$, where $i = j \pm 1$ and $t_{ij} = t_{ij}^* = 0$ elsewhere. $c_i^\dagger$ and c_j are the usual creation and annihilation operators, V_i represents the dimensionless on-site potential scaled by a tunneling rate, and the sublattice of length $2L + 1$ is located at the center of the perfect lattice.

The on-site potential is as follows:

$$V_i = \begin{cases} 0 & \text{if } r_i \notin [-L,L], \\ W_i, & \end{cases}$$

where W_i can be periodic, semiperiodic, or disordered.

At time zero, a localized wave packet was launched in the sublattice's center, with $\rho_{m,n} = \delta_{c,c}$, where c denotes the central site in the lattice. This initial state was a coherent superposition of many quasimomentum eigenstates. We have used the master equation of Lindblad form, which can be written as

$$\frac{\partial \rho}{\partial t} = L\rho = -\frac{i}{\hbar}[H,\rho] + \sum_i \frac{1}{2}\gamma_i M_i (2A_i\rho A_i^\dagger - \rho A_i^\dagger A_i - A_i^\dagger A_i \rho) + \sum_i \frac{1}{2}\gamma_i N_i (2A_i\rho A_i^\dagger - \rho A_i^\dagger A_i - A_i^\dagger A_i \rho), \qquad (2)$$

where H is a Hermitian, ρ is the density matrix, M_i and N_i are real dimensionless non-negative c numbers (either of which can be larger), and A_i are arbitrary dimensionless operators. The coefficient $\frac{1}{2}\gamma_i$ is a positive damping rate, and it could in principle be absorbed into the other two coefficients, which would then acquire its dimension of inverse time. If a master equation is of Lindblad form, the time evolution is completely consistent with quantum mechanics. In particular, this means that (i) the solution for the density operator is always a positive-definite operator, i.e., no negative probabilities occur, and (ii) the trace of ρ is time-independent, so that probability is conserved.

Therefore, the new Lindblad equation for white noise can be rewritten as

$$\frac{\partial \rho}{\partial t} = L\rho = -\frac{i}{\hbar}[H,\rho] + \Gamma \sum_i (2A_i\rho A_i^\dagger - \rho A_i^\dagger A_i - A_i^\dagger A_i \rho), \qquad (3)$$

where Γ denotes the noise intensity. We assume that the interaction of the system with the environment is dominated by white noise captured within the Haken-Strobl model (pure dephasing) [18]. The dephasing term damps all the off-diagonal entries of the density matrix via the generators $A_i = |i\rangle\langle i|$, suppressing a superposition of localized states at a rate Γ_i, which is called the dephasing rate. Note that the pure-dephasing (Haken-Strobl) model is a simplified but useful model that has been used successfully in numerous studies in quantum optics, quantum-information science, physical chemistry, and condensed-matter physics. Its prediction becomes more realistic when the system is interacting with a thermal bath at high temperatures, where its effects can be modeled by white noise [19].

B. Measuring the spreading of the wave packet

We measure the spreading of the wave packet by its variance. The variance of the wave packet is defined as

$$\text{variance} \equiv \sigma^2 \equiv \sum_n n^2 |\psi_n|^2, \qquad (4)$$

where n is the lattice site index, and ψ_n depicts a normalized time-evolving wave packet.

III. COMPUTATION CASES

A. Periodic case

We choose a periodic on-site potential in the sublattice, i.e., we let $W_i = 0.5(-1)^i$. The time dependence of the variance of the wave packet $\sigma^2(t)$ for a sublattice potential intensity $V = 0.5$ is shown in Fig. 2. The spreading of the wave packet in the absence of any noise is shown in Fig. 2(a). We study the effect of the white noise with small noise intensity $\Gamma = 0.01$ in Fig. 2(b). As can be seen, the white noise reduces the diffusion exponent by 0.05. If we increase the noise intensity to $\Gamma = 0.04$ [see Fig. 2(c)], the white noise reduces the diffusion exponent to 1.90, which means that in this case the diffusion rate is changed from the super-ballistic to the sub-ballistic regime. Increasing the noise intensity further to $\Gamma = 0.1$ [as can be seen in Fig. 2(d)] can decrease the diffusion exponent by 0.37 with respect to the noise-free case. It is seen that the white noise suppresses the diffusion significantly for all the above cases. However, if we increase the sublattice potential intensity to $V = 1.5$, the white noise affects the diffusion differently. In the absence of any noise, when we set the sublattice potential intensity to $V = 1.5$, it appears that the diffusion exponent increases to 2.09 [see Fig. 3(a)]. The effect of white noise with small noise intensity $\Gamma = 0.01$ changes dramatically from what can be seen in Fig. 2(b). We see that the white noise has a counterintuitive effect on the diffusion rate. Here noise is no longer a nuisance to be avoided, rather it improves the diffusion exponent by 0.10 [see Fig. 3(b)]. This bizarre behavior is due to

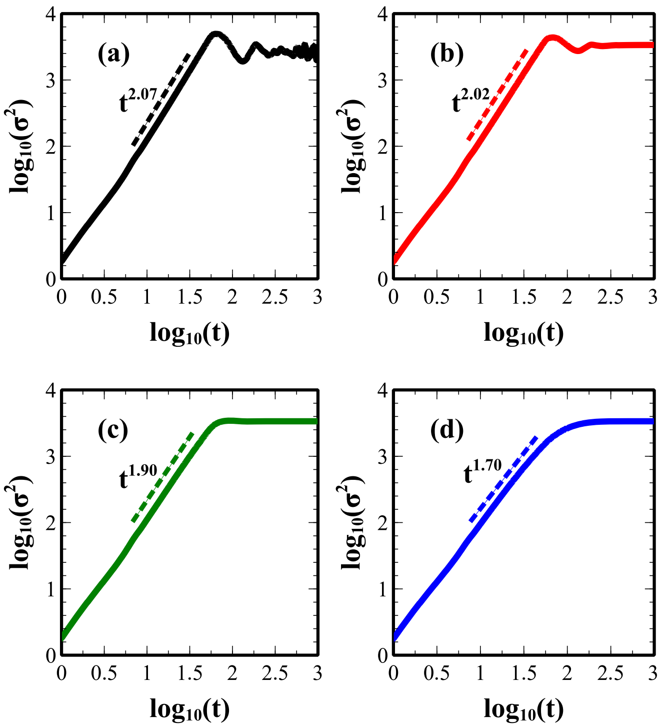

FIG. 2. Time dependence of the variance of the wave packet (σ^2) for a periodic potential with a sublattice potential intensity, $V = 0.5$ (with $L = 10$, where $2L + 1$ is the size of the sublattice). From the top left to the bottom right for noise intensity $\Gamma = 0$, 0.01, 0.04, and 0.1, respectively, here and in all the other figures, the dashed lines represent power-law fitting. All quantities are dimensionless.

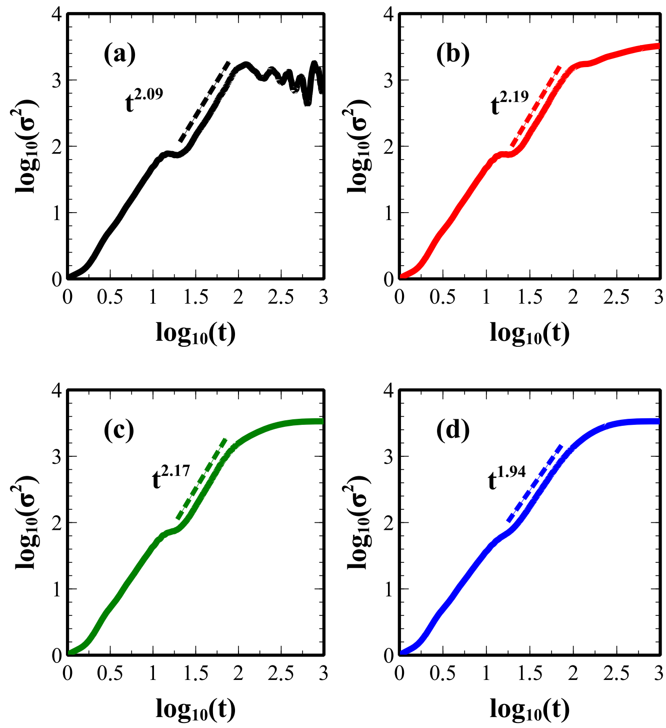

FIG. 3. Same as Fig. 2 for a periodic potential with a sublattice potential intensity $V = 1.5$.

the fact that the white noise has delocalized some of the system states. More importantly, the delocalized states are found to separate energy domains corresponding to two distinct types of localized states: the usual localized states centered at sites in the sublattice, and new states that are localized at the contact with the perfect lattice (these new states were termed antilocalized states [20]). The antilocalized states are expected to play a special role in improving the diffusion rate. When noise intensity Γ increases from 0.01 to 0.04 [see Fig. 3(c)], the diffusion exponent decreases a little (0.02). Here the stochastic resonance between the matter-wave antilocalized states and the white noise becomes less than what we see in Fig. 3(b). So the system's diffusion gets a little slower with respect to the case with smaller noise intensity. Increasing the noise intensity to $\Gamma = 0.1$ makes the stochastic resonance between the system's antilocalized states and the white noise disappear. As can be seen in Fig. 3(d), this decreases the diffusion exponent more and leads to a regime change in the system's diffusion from super-ballistic to sub-ballistic.

B. Disordered case

We choose a disordered on-site potential in the sublattice, i.e., we let the W_i take $+V$ or $-V$ randomly. Therefore, we obtain slightly different results of $\sigma^2(t)$ from different disorder realizations in $[-L,L]$. Finally, we present the result of $\sigma^2(t)$ after first averaging them over 50 different disorder realizations. The time dependence of the averaged variance of the wave packet $\sigma^2(t)$ for sublattice potential intensity $V = 0.5$ is shown in Fig. 4. The spreading of the wave packet in the absence of any noise is shown in Fig. 4(a). We also study the effect of white noise with small noise intensity $\Gamma = 0.01$, and

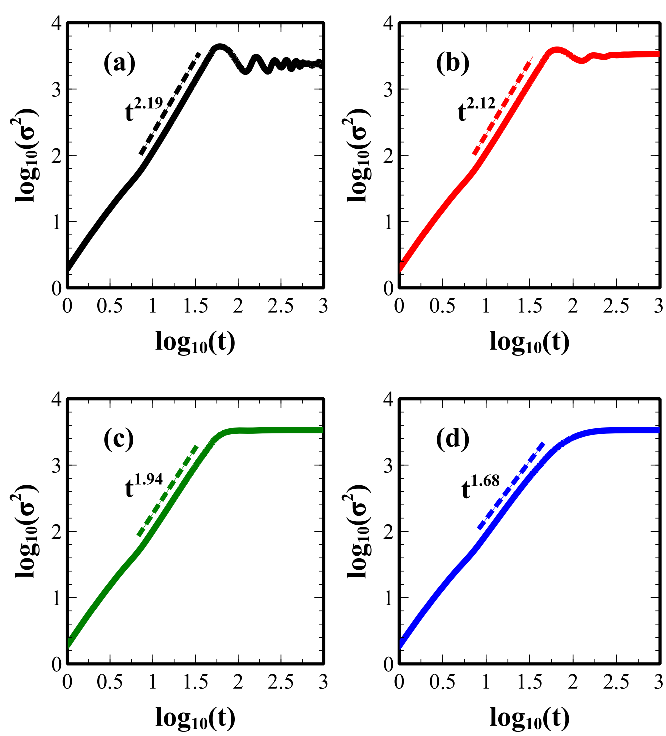

FIG. 4. Same as Fig. 2 for a disordered sublattice with a sublattice potential intensity $V = 0.5$.

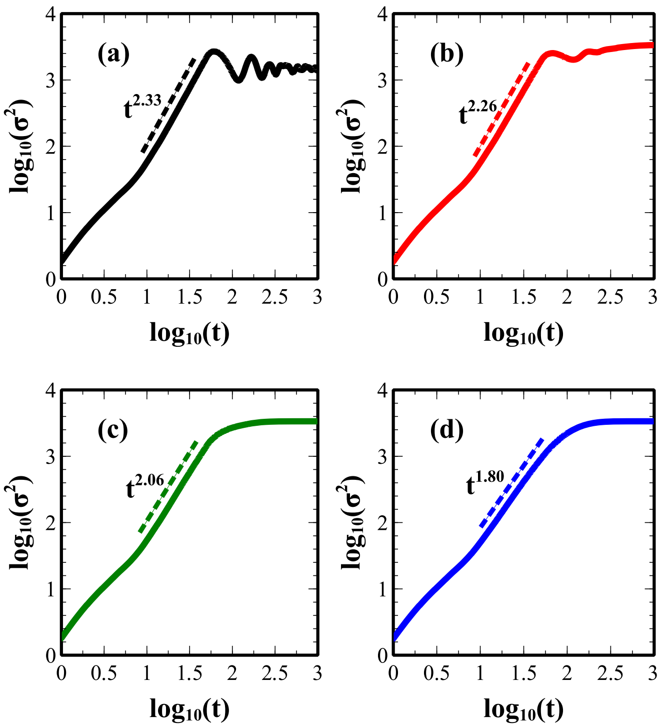

FIG. 5. Same as Fig. 2 for a disordered sublattice with a sublattice potential intensity $V = 0.8$.

it can be seen from Fig. 4(b) that the white noise reduces the diffusion exponent by 0.07. If we increase the noise intensity to $\Gamma = 0.04$ [see Fig. 4(c)], the white noise reduces the diffusion exponent to 1.94. This means that in this case, the diffusion rate is changed from the super-ballistic to the sub-ballistic regime. A further increase in the noise intensity to $\Gamma = 0.1$ [as can be seen in Fig. 4(d)] can decrease the diffusion exponent by 0.51 with respect to the noise-free case.

In the absence of any noise when we set the sublattice potential intensity to $V = 0.8$, it appears that the diffusion exponent increases to 2.33 [see Fig. 5(a)]. If we induce small noise intensity $\Gamma = 0.01$, it decreases the diffusion exponent by 0.07 [see Fig. 5(b)]. When noise intensity Γ increases from 0.01 to 0.04, the diffusion exponent decreases more [see Fig. 5(c)]. Increasing the noise intensity to $\Gamma = 0.1$ reduces the diffusion exponent to 1.80 [see Fig. 5(d)]. This means that in this case, the diffusion changes from super-ballistic (hyper-diffusion) diffusion to sub-ballistic diffusion. Because in a disordered sublattice there is higher intrinsic disorder with respect to the periodic case, there is no stochastic resonance between the system's antilocalized states and the white noise. Therefore, inducing white noise to the system cannot reduce the total noise of the system, so the white noise plays its usual suppression role here.

C. Quasiperiodic case

A quasiperiodic potential is intermediate between a truly random potential, which may cause localization in 1D systems, and periodic potentials, which lead to energy bands and extended states [21]. Quasiperiodic systems possess intriguing energy spectra and eigenstate structure [22–24].

1. Fibonacci case

The other sublattice's potential that we will consider here is the simplest model of a quasicrystal, called the diagonal model. It is obtained when W_i has two possible values, which we denote by W_A and W_B, following the Fibonacci sequence (FS). The FS is built as follows: consider two letters, A and B, and the substitution rules, $A \to B$ and $B \to AB$: if one defines the first generation sequence as $F_1 = A$ and the second one as $F_2 = B$, the subsequent chains are generated using the two previous rules. For instance, $F_3 = AB$: starting with an A, we construct the following sequences: A, B, AB, BAB, $ABBAB$, $BABABBAB$, and so on. Each generation obtained by iteration of the rules is labeled with an index l: The number of letters in each generation l is given by the Fibonacci numbers $F(l)$ of generation l, which satisfy $F(l) = F(l-1) + F(l-2)$ with the initial condition $F(0) = 1$, $F(1) = 1$. It is well known that a Fibonacci lattice yields singular continuous energy spectra and critical eigenstates (that are neither localized nor extended) [22–25].

Here the sublattice has 55 sites, and we choose the sublattice Fibonacci potential with intensity $V = 0.5$. If we set $W_A = +0.5$ and $W_B = -0.5$ in the absence of any noise, it appears that the diffusion exponent increases to a high amount of 2.79 [see Fig. 6(a)]. If we induce a small noise intensity $\Gamma = 0.01$, it decreases the diffusion exponent by 0.24 [see Fig. 6(b)]. When the noise intensity Γ increases from 0.01 to 0.04, the diffusion exponent decreases more [see Fig. 6(c)]. Increasing the noise intensity to $\Gamma = 0.1$ reduces the diffusion exponent to 1.94 [see Fig. 6(d)]. This

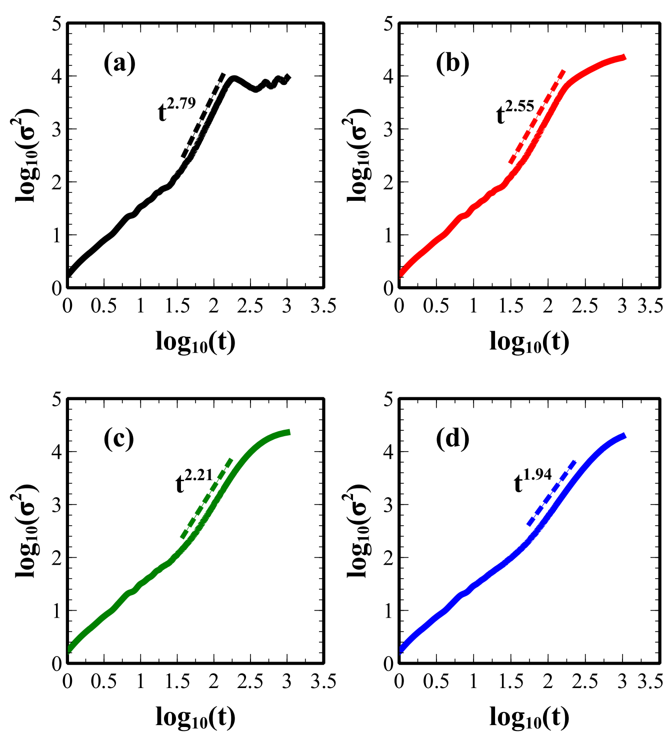

FIG. 6. Same as Fig. 2 for a Fibonacci sublattice with a sublattice potential intensity $V = 0.5$ (with sublattice size $2L + 1 = 55$). It is evident that the saturation point is much higher than cases with $L = 10$).

means that in this case, the diffusion changes from super-ballistic diffusion (hyper-diffusion) to sub-ballistic diffusion.

In the Fibonacci case, the intrinsic disorder is higher than the periodic case but still smaller than the disordered case. Yet there is no stochastic resonance between the system's antilocalized states. Therefore, the white noise suppresses the diffusion here. We see that the diffusion exponent can be tuned by changing Γ.

2. *Harper case*

We now turn to a noninteracting Harper sublattice [26] with an Aubry-Andre Hamiltonian [27],

$$H = J\sum_i(|w_i\rangle\langle w_{i+1}| + |w_{i+1}\rangle\langle w_i|) + \Delta \sum_{i\in[-L,L]} \cos(2\pi\beta + \phi)|w_i\rangle\langle w_i|, \tag{5}$$

where $|w_i\rangle$ is the Wannier state localized at the lattice site i, J is the site-to-site tunneling energy, and Δ is the strength of the potential.

$\beta = k_2/k_1$ is the ratio of the two lattice wave numbers, and ϕ is an arbitrary phase (we set $\phi = 0$). For a rational β, the above equation can be solved by Bloch's theorem, although its value is very limited since the coefficients in Fourier space of the solution form a dense set. For irrational β, the spectrum depends on the value of Δ/J. We have studied it for $\Delta/J = 0.5$ (see Fig. 7), $\Delta/J = 1.5$ (see Fig. 8), and $\Delta/J = 2.5$ (see Fig. 9). In the experiment, the two relevant energies J and Δ can be controlled independently by changing the heights of the primary and secondary sublattice potentials, respectively. For a maximally incommensurate ratio $\beta = (\sqrt{5}-1)/2$, the

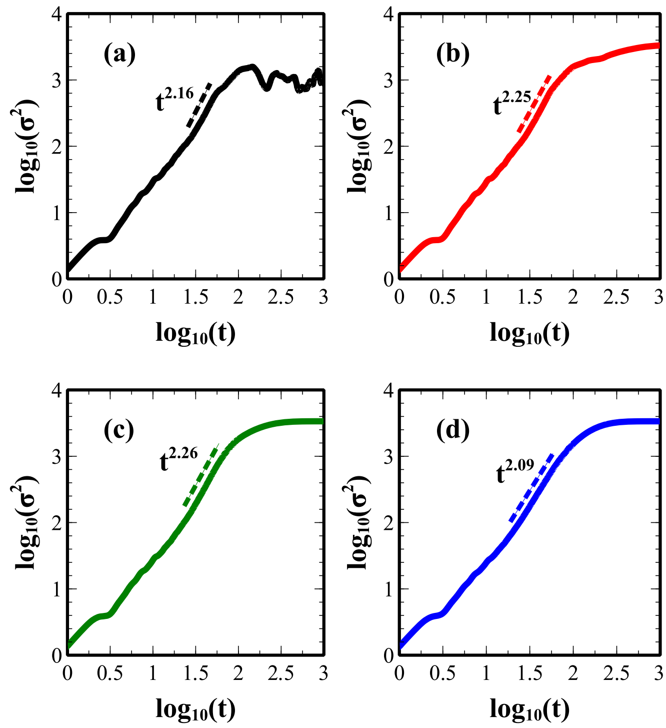

FIG. 8. Same as Fig. 2 for a Harper sublattice with a sublattice potential with $\Delta/J = 1.5$.

model exhibits a sharp transition from extended to localized states at $\Delta/J = 2$ [27]. In Fig. 7, we present the diffusion rate for the Harper potential for $\Delta/J = 0.5$ in the absence of any noise. It turns out that the diffusion rate increases

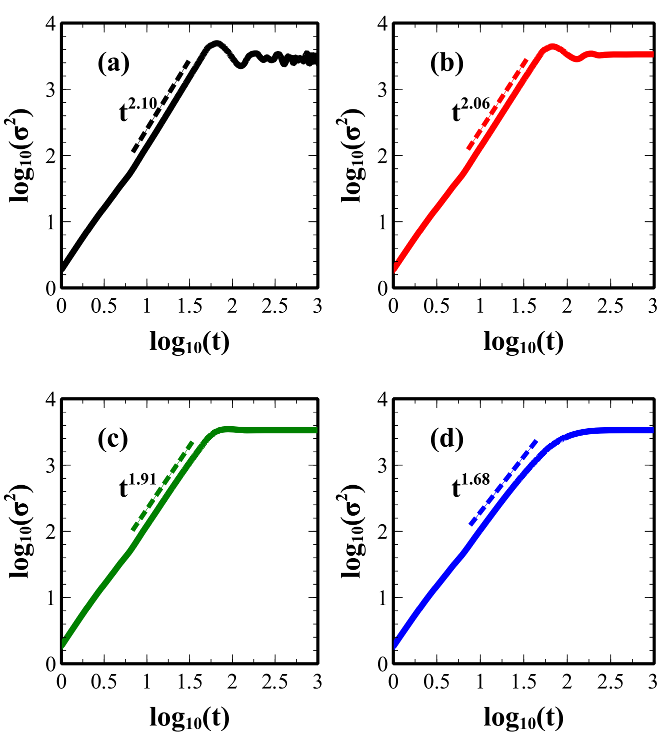

FIG. 7. Same as Fig. 2 for a Harper sublattice with a sublattice potential with $\Delta/J = 0.5$.

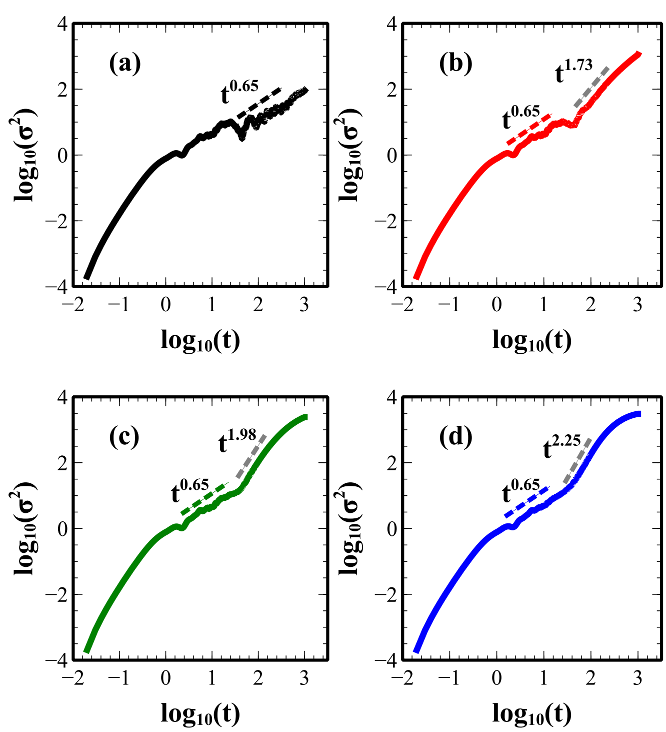

FIG. 9. Same as Fig. 2 for a Harper sublattice with a sublattice potential with $\Delta/J = 2.5$.

to $\sigma^2 = t^{2.1}$ [see Fig. 7(a)]. If we induce small noise intensity $\Gamma = 0.01$, it decreases the diffusion exponent by 0.04 [see Fig. 7(b)]. When noise intensity Γ increases from 0.01 to 0.04, the white noise reduces the diffusion exponent to 1.91 [see Fig. 7(c)]. This means that in this case, the diffusion changes from super-ballistic diffusion (hyper-diffusion) to sub-ballistic diffusion. Increasing the noise intensity to $\Gamma = 0.1$ [as can be seen in Fig. 7(d)] can decrease the diffusion exponent by 0.42 with respect to the noise-free case.

In the Harper case for $\Delta/J = 0.5$, the intrinsic disorder is higher than the periodic case but smaller than the disordered and Fibonacci cases. Hence there is no stochastic resonance between the system's antilocalized states and the white noise. Therefore, the white noise suppresses the diffusion rate.

It is worth mentioning that β controls the transition from periodic to quasiperiodic sequences. Thus, the present approach also leads to the possibility of studying the electronic properties as a function of such a parameter. Furthermore, it can be proven that the parameter β can also be related to a magnetic field, as happens in the Harper potential. In that case, instead of having a constant magnetic field in space, one has a space-modulated magnetic field [28]. In real systems, the changes in β are simple to study using many different devices, since its effect is only a change in the sequence of the binary potential. For example, one can use microwaves in a cavity, a dielectric superlattice, or a space-modulated magnetic field in a semiconductor [29].

If we set $\Delta/J = 1.5$, the white noise effects would be different. In the absence of any noise, it appears that the diffusion exponent increases to 2.16 [see Fig. 8(a)]. The effect of white noise with small noise intensity $\Gamma = 0.01$ changes dramatically from what we have seen in Fig. 7(b). The white noise delocalizes some of the system states and shows its counterintuitive effect again, and it improves the diffusion exponent [see Fig. 8(b)]. When noise intensity Γ increases from 0.01 to 0.04 [see Fig. 8(c)], the diffusion exponent is still higher than the case in the absence of the noise.

As can be seen in Fig. 8(d), increasing the noise intensity to $\Gamma = 0.1$ reduces the damping effect on the diffusion, making the stochastic resonance between the system's antilocalized states and the white noise smaller.

For $\Delta/J = 2.5$ in the absence of any noise, it appears that the diffusion exponent is as low as 0.65 [see Fig. 9(a)]. The white noise with small noise intensity $\Gamma = 0.01$ does not change the first subdiffusion region. But as time goes on, the white noise delocalizes some of the system states and drives the system's subdiffusion ($0 < \nu < 1$) to sub-ballistic diffusion ($1 < \nu < 2$) instantly [see Fig. 9(b)]. When the noise intensity Γ increases from 0.01 to 0.04 [see Fig. 9(c)], first there is no change in the diffusion rate, but then the white noise delocalizes some of the system states and causes the system to diffuse almost ballistically. The stochastic resonance between the matter-wave antilocalized states and the white noise then becomes higher than what can be seen in Fig. 9(b). So the system's diffusion becomes faster compared to the smaller noise intensity. Increasing the noise intensity to $\Gamma = 0.1$ enhances the stochastic resonance between the system's antilocalized states and the white noise, such that it changes the system's diffusion from subdiffusion to the super-ballistic regime instantly [see Fig. 9(d)].

IV. FINITE-SIZE EFFECT

We study the variance of the wave packet for finite-size lattices. As can be seen in all the figures, the variance is saturated at a point that is different for different lattices. It is notable that in the Fibonacci case with $W_A = +0.5$ and $W_B = -0.5$ [see Fig. 6(a)], we find a diffusion exponent 2.79 in the absence of noise, while this diffusion exponent is considerably larger than that found in [6]. This means that sometimes the finite size could improve the diffusion rate significantly. This can be explained via the structure of the underlying eigenstates. There has been a great deal of research on the finite-size effect in different physical systems [30–37]. The finite-size effect in the absence of coupling with the environment is manifested in Fig. 10(a). As we increase the size of the lattice, the final saturation point gets higher. Due to practical limits, we cannot solve the master equation for large lattices. But as we know from the Schrödinger equation's solution for much bigger lattices (in the absence of any noise), this finite-size effect is usually just at the point of saturation and the diffusion exponent. So our results will be different for other lattice and sublattice site numbers. In the laboratory, however, even a lattice with 200 sites is usually big enough for many experiments. In the calculation of the finite-size effect in systems under the influence of white noise [see Figs. 10(b)–10(d)], we can see that as we increase the noise intensity, it dampens the fluctuation more and more until eventually there is none [see Fig. 10(d)]. As the lattice size increases from 200 to 600 sites in a periodic sublattice, the final saturation point becomes higher. We have studied other potentials as well in which the lattice size changes the saturation point in all cases. So, this long-time limit must be characterized by localization

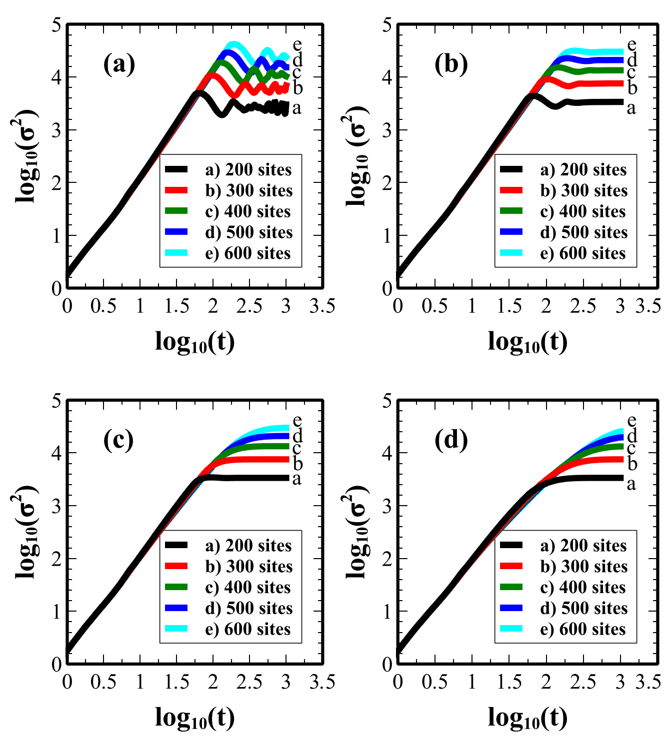

FIG. 10. Same as Fig. 2, showing the finite-size effect on periodic sublattices with different lattice sizes (200, 300, 400, 500, and 600).

due to the finite-size effect. The white noise does not change the total saturation point because it is a finite-size effect.

V. LOCALIZATION TO DELOCALIZATION TRANSITION

The localization to delocalization transition has long-established background [38–40]. However, the subject has attracted more attention recently [41,42]. Yamada studied delocalization in one-dimensional tight-binding models with fractal disorder [43]. Such models with an ergodic and stationary random potential have a positive Lyapunov exponent of a wave function with probability 1 (Goldsheid-Molchanov-Pastur theorem). A survey was performed of some of the mathematical results and techniques for Schrödinger operators with random and quasiperiodic potentials in Refs. [44,45]. The existence of the positive Lyapunov exponent is a necessary and sufficient condition for a pure point set spectrum of the operators, and all the eigenfunctions then exhibit exponential decay in the thermodynamic limit. Kotani's theory states that if the potential sequence is nondeterministic under the following conditions: (i) stationarity, (ii) ergodicity, (iii) integrability, then there is no absolutely continuous (a.c.) spectrum of the operators. These theorems can be proven true for continuous and discrete one-dimensional disordered systems (1DDSs) [46]. Usually in the presence of a background noise, an increased effort put into controlling a system stabilizes its behavior. In rare cases, increased control of the system can lead to a looser response and, therefore, to a poorer performance [47]. For example, here we consider a triangular potential with $W_i \in (W_1 = V_{\min}, \ldots, W_{N/2} = V_{\max}, \ldots, V_1 = W_{\min})$, where N is the total number of sites in the sublattice (an odd number), and $V_{\max}$ ($V_{\min}$) is the biggest (smallest) on-site potential.

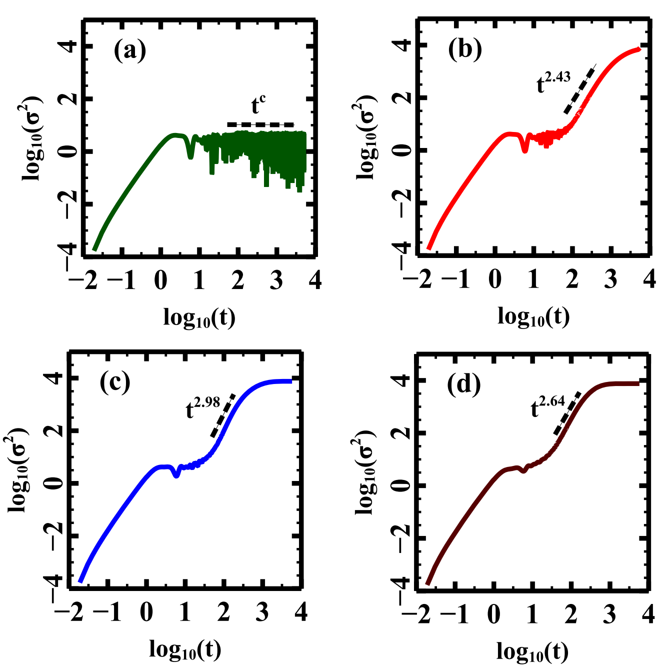

FIG. 11. Same as Fig. 2 for a sublattice with a triangular potential (with $L = 10$ but a larger perfect lattice) for more time steps.

Keeping a quantum system in a given instantaneous eigenstate is a control problem [48]. As can be seen in Fig. 11(a), in the absence of any noise (the increased control of the system) this Hamiltonian generates localization, and this is a manifestation of Anderson localization [see Fig. 11(a)] because, as is shown, the saturation point is far beyond the amount that is dictated by the finite-size effect [see Figs. 11(b)–11(d)]. When we induce small noise intensity $\Gamma = 0.01$ to the system (causing the system to destabilize its behavior), it takes some time for the white noise to delocalize some of the system states, and this drives the system to sub-diffusion very smoothly. Then at some time step suddenly stochastic resonance occurs and the system changes its diffusion regime instantly to the super-ballistic regime [see Fig. 11(b)]. If we increase the noise intensity to $\Gamma = 0.04$ [see Fig. 11(c)], the white noise delocalizing effect is similar to the latter case, but stochastic resonance occurs earlier with respect to the latter case. Increasing the noise intensity further to $\Gamma = 0.1$ [as can be seen in Fig. 11(d)] results in earlier stochastic resonance with respect to the former cases. As can be seen here, stochastic resonance can have an increasing effect on the diffusion rate at certain points. Further assessment of this stochastic resonance is presented in the next section. Here the finite size and the stochastic resonance are correlated.

VI. SIGNS OF STOCHASTIC RESONANCE

The process whereby noise operates on the quantum system enhancing the response to an external noisy signal has been termed stochastic resonance. Upon decreasing the temperature, quantum tunneling becomes increasingly important [49]. Above a crossover noise intensity, noise-activated transitions dominate over quantum tunneling events. The effects of quantum noise then result in a quantum correction factor of the classical rate of activation. As noise intensity is decreased below a threshold, tunneling transitions prevail. The quantum noise could be characterized by the temperature of the thermal bath and by the coupling strength of the system to the environment. In the absence of driving, at sufficiently high noise intensity the damping effects are so strong that quantum coherence is completely suppressed by incoherent tunneling transitions. We try to find the coherent tunneling transitions between adjacent sites on the border of the sublattice and the perfect lattice. We define the coherent tunneling rate (CTR) as the norm of the off-diagonal element of the density matrix between two adjacent sites on the border of the sublattice and the perfect lattice ($\rho_{m,m+1}$). First we look at the coherent tunneling transitions (CTTs) in systems in which noise decreases the diffusion rate. Here in the absence of any noise, the CTR is fluctuating in all time periods [see Fig. 12(a)]. As the noise increases, the fluctuation in the CTR becomes damped sooner [see Figs. 9(b)–9(d)], and by increasing the noise intensity the coherency will be killed very soon; this is the usual decoherence effect of the white noise. However, in the systems in which noise enhances the diffusion rate or delocalizes a completely localized system, it has a different role. In the triangular potential, increasing the noise intensity has various effects on the CTT (see Fig. 13). In the absence of

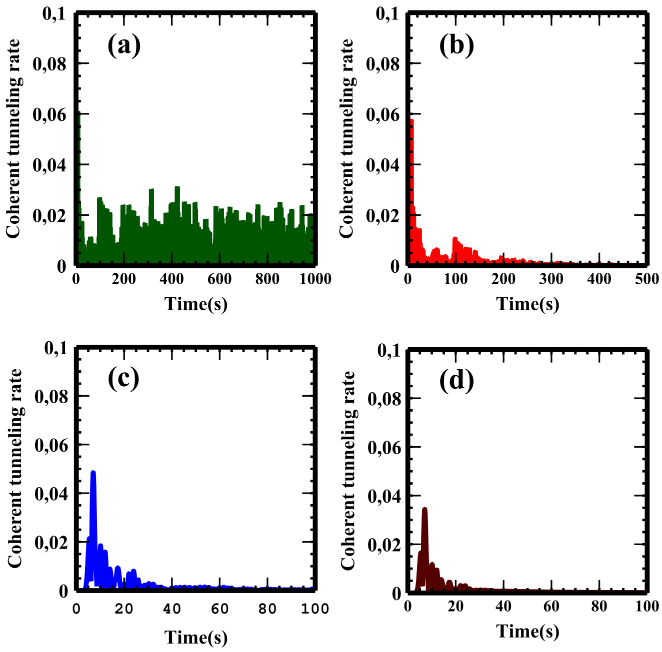

FIG. 12. Coherent tunneling rate between adjacent sites in the border of the sublattice and a perfect lattice for a constant potential with a sublattice potential intensity $V = 1.0$ (with $L = 10$). The plots from top left to bottom right show cases with noise intensity $\Gamma = 0$, 0.01, 0.04, and 0.1, respectively. All quantities are dimensionless.

any noise, the CTR is zero. As noise increases, the CTR shifts to later time steps, and the duration of the coherency increases significantly.

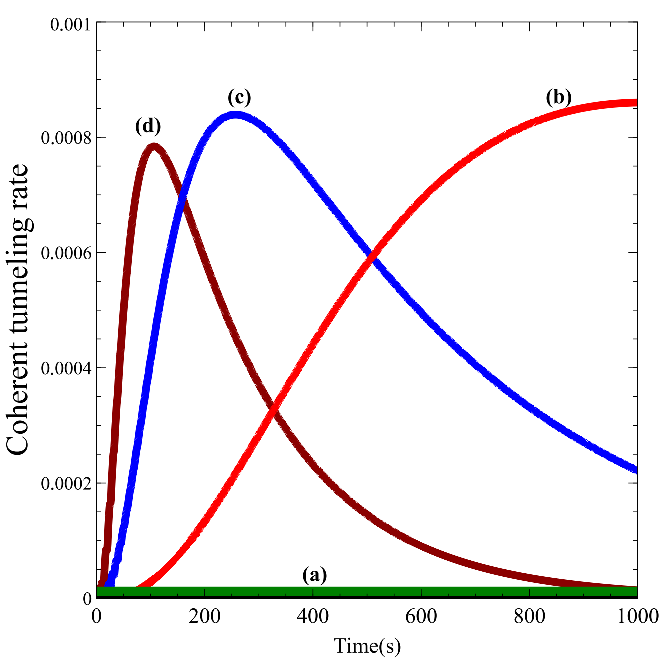

FIG. 13. Same as Fig. 12 for a sublattice with a triangular potential (with $L = 10$ and a larger perfect lattice). The green line (a), red line (b), blue line (c), and brown line (d) represent noise intensity $\Gamma = 0$, 0.01, 0.04, and 0.1, respectively.

VII. CALCULATION METHOD

For a tight-binding Hamiltonian with N sites, the number of elements in the density matrix is N^2, and solving the master equation by numerical integration involves use of superoperators of size $N^2 \times N^2$. We used a sparse matrix format in which not all of the N^4 elements need to be stored in the memory. However, the time required to evolve a quantum system according to the master equation still increases rapidly as a function of the system size. In particular, the amount of RAM that is needed becomes larger and larger. Consequently, solving the master equation is practical only for relatively small systems: $N \lesssim 1000$. We used Scipy [50] (an open source Python library used for scientific computing and technical computing) for integrating and parsing the matrices. With regard to the precision in the value of the exponents, we should mention that in all the cases, the standard error of coefficients for the regression is less than 0.002.

VIII. DISCUSSION

To explain the appearance of superballistic diffusion, a simple probabilistic model called the point-source model was introduced in Ref. [4]. The disordered sublattice can be replaced by a point source that radiates the probability with a constant velocity v, simulating the dynamics of the perfect lattice. At $t = 0$, all of the probability is trapped inside the point source, and as time passes, the probability of finding the particle in the sublattice decays exponentially as $P_L(t) = \exp(-\lambda t^2)$, where λ is the probability decay rate. When t is small, Zhang *et al.* argued that the exponential decay of the probability of finding the particle in the sublattice can be approximated as $P_L(t) = \exp(-\lambda t^\alpha) \approx 1 - \lambda t^\alpha$, where $\gamma \approx \alpha + 2$ [6]. If $\alpha = 0$, there is no decay, If $\alpha = 1$, then an exact exponential decay occurs. In their own article, they mentioned a big difference between an analytic calculation ($\gamma \approx 4.3$) and a numerical one ($\gamma \approx 4.7$).

Recently, Nguyen *et al.* showed that the nonlinear fitting extension of the point-source model in the form that was introduced in Ref. [6] is unable to explain super-ballistic diffusion when the diffusion rate is faster than cubic, so there is no correct explanation for super-ballistic exponents when the diffusion rate is faster than cubic [51]. Therefore, a comprehensive interpretation is needed of super-ballistic exponents with a diffusion rate that is faster than cubic.

For a long time, noise was considered to be only a source of disorder, or a nuisance to be avoided. This view has been changing due to several phenomena that illustrate the constructive facets of noise. Among them, the most widely studied is the phenomenon of stochastic resonance, where the addition of noise to a system enhances its response to a periodic force [52–58]. This counterintuitive aspect of noise has been found in a wide variety of situations, including bistable [59] and monostable [60] systems, nondynamical elements with [61] and without [62] threshold, pattern-forming systems [63], and state-dependent noise in the presence of a stationary fluctuating input [64].

In this work, we studied the effect of white noise on the diffusion of a wave packet in a 1D lattice. White noise has different roles in these lattices. Usually it suppresses diffusion, but in some cases not only does it not suppress diffusion, it will actually improve it. There is a special case in which a system was localized, and the white noise delocalized it and even changed it to a superballistic diffusion regime. For the white noise effect on these tight-binding Hamiltonians, there is no numerical or analytic explanation nor an interpretation available. The role of the suppression of the white noise could be interpreted by increasing the energy band mismatch between the sublattice and the rest of the uniform lattice, and via the structure of the underlying eigenstates. White noise leads to frequent changes in the quantum system's instantaneous eigenstate. In Ref. [52], the interplay between externally added noise and the intrinsic noise of the system that relaxes fast toward a stationary state was analyzed theoretically. It was found that increasing the intensity of the external noise can reduce the total noise of the system. The reduction in output noise is due to the fact that the system is driven into states with lower intrinsic noise, where the confinement effort is effectively greater. Here the intrinsic noise of the system is the noise within the Hamiltonian eigenstates, as in Refs. [65,66], and the external noise is the induced white noise.

The addition of white noise enhances the diffusion rate in some cases and suppresses it in others. We believe that when noise has a counterintuitive aspect (enhancement), there is some stochastic resonance between the matter-wave states and the white noise. Therefore, whenever this resonance occurs, the noise enhances the diffusion; otherwise it suppresses it. The type of potential and its intensity affects the occurrence of stochastic resonance. We have assessed many different cases showing that the potential type, its intensity, and the noise intensity together have some correlation with stochastic resonance. The potential should be nested within a large enough perfect lattice, and its intensity should be between certain values (with respect to the noise intensity), but its structure could vary. In this paper, we used the triangular potential to present super-ballistic diffusion. However, even with a fixed-valued potential, we observed the localization-to-delocalization transition, which still exhibits sub-ballistic diffusion.

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