Eigenvector statistics of the product of Ginibre matrices

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We develop a method to calculate left-right eigenvector correlations of the product of m independent $N \times N$ complex Ginibre matrices. For illustration, we present explicit analytical results for the vector overlap for a couple of examples for small m and N. We conjecture that the integrated overlap between left and right eigenvectors is given by the formula $O = 1 + (m/2)(N - 1)$ and support this conjecture by analytical and numerical calculations. We derive an analytical expression for the limiting correlation density as $N \to \infty$ for the product of Ginibre matrices as well as for the product of elliptic matrices. In the latter case, we find that the correlation function is independent of the eccentricities of the elliptic laws.

I. INTRODUCTION

Products of random matrices have continuously attracted attention since the 1960s [1–5]. They are of relevance in many fields of mathematics, physics, and engineering, including dynamical systems [2,6], disordered systems [7–9], statistical mechanics [10], quantum mechanics [11], quantum transport and mesoscopic systems [12,13], hidden Markov models [14], image processing [15], quantum chromodynamics [16], wireless telecommunication [17,18], quantitative finance [19–21], and many others [22]. Recently, enormous progress has been made in the understanding of macroscopic [23–44] and microscopic [45–66] statistics of eigenvalues and singular values as well as of Lyapunov spectra for products of random matrices [67–77]. In contrast, not much has been learned about the eigenvector statistics of the products of random matrices so far. In this paper, we address this problem by considering a correlation function for eigenvectors of the product of Ginibre matrices. More precisely, we study the overlap between left and right eigenvectors for finite N and for $N \to \infty$. In the first part of the paper, we adapt ideas developed in Refs. [78,79] to the product of random matrices by using the generalized Schur decomposition [45] for finite N, while in the second part we combine the generalized Green function method [81–84] with linearization (subordination) [10,28,37] to derive the limiting law for the overlap for $N \to \infty$.

II. DEFINITIONS

Consider a diagonalizable matrix X over the field of complex numbers. Let $\{\Lambda_\alpha\}$ be the eigenvalues of X. The corresponding left eigenvectors $\langle L_\alpha|$ and right eigenvectors $|R_\alpha\rangle$ satisfy the relations

$$X|R_\alpha\rangle = \Lambda_\alpha|R_\alpha\rangle, \quad \langle L_\alpha|X = \langle L_\alpha|\Lambda_\alpha. \tag{1}$$

Note that the Hermitian conjugate of the second equation has the form $X^\dagger|L_\alpha\rangle = \bar{\Lambda}_\alpha|L_\alpha\rangle$, where the symbol "bar" denotes the complex conjugation of Λ_α. The eigenvectors fulfill the biorthogonality and closure relations in the form

$$\langle L_\alpha|R_\beta\rangle = \delta_{\alpha\beta}, \quad \sum_\alpha |L_\alpha\rangle\langle R_\alpha| = 1. \tag{2}$$

The two relations are invariant with respect to the scale transformation

$$|R_\alpha\rangle \to c_\alpha|R_\alpha\rangle, \quad \langle L_\alpha| \to \langle L_\alpha|c_\alpha^{-1}, \tag{3}$$

with arbitrary nonzero coefficients c_α's. According to Refs. [78,79], an overlap of the left and right eigenvectors is defined in the following way:

$$O_{\alpha\beta} = \langle L_\alpha|L_\beta\rangle\langle R_\beta|R_\alpha\rangle. \tag{4}$$

By construction, the quantity $O_{\alpha\beta}$ is invariant with respect to the scale transformation given by Eq. (3) and consequently does not depend on the vector normalizations.

If X is a random matrix, one defines averages over the ensemble

$$\langle O_{\alpha\beta}\rangle = \int d\mu(X)O_{\alpha\beta}, \tag{5}$$

where $d\mu(X)$ is the probability measure for the random matrix in question. The dependence of $O_{\alpha\beta}$ on X is suppressed in the notation. We use this notation throughout the paper also for other observables that depend on random matrices. The global diagonal overlap averaged over the ensemble is given by

$$O = \left\langle \frac{1}{N} \sum_{\alpha=1}^{N} O_{\alpha\alpha} \right\rangle, \tag{6}$$

while the global off-diagonal one is expressed by the formula

$$O_{\text{off}} = \left\langle \frac{2}{N(N-1)} \sum_{\alpha<\beta} O_{\alpha\beta} \right\rangle. \tag{7}$$

We are interested here in unitarily invariant random matrices for which the probability measure is invariant with respect

to the similarity transformation $X \to UXU^{-1}$, where U is a unitary matrix. In particular, this invariance implies that $\langle O_{\alpha\alpha} \rangle = \langle O_{11} \rangle$ and $\langle O_{\alpha\beta} \rangle = \langle O_{12} \rangle$ for any α and β. It follows that

$$O = \langle O_{11} \rangle, \quad O_{\text{off}} = \langle O_{12} \rangle. \tag{8}$$

We can also define the local diagonal overlap density by the formula

$$O(z) = \left\langle \frac{1}{N} \sum_{\alpha=1}^{N} O_{\alpha\alpha} \delta(z - \Lambda_\alpha) \right\rangle = \langle O_{11} \delta(z - \Lambda_1) \rangle, \tag{9}$$

and the off-diagonal one by

$$O_{\text{off}}(z,w) = \left\langle \frac{2}{N(N-1)} \sum_{\alpha<\beta} O_{\alpha\beta} \delta(z - \Lambda_\alpha)\delta(w - \Lambda_\beta) \right\rangle$$

$$= \langle O_{12}\delta(z - \Lambda_1)\delta(w - \Lambda_2) \rangle. \tag{10}$$

The symbol $\delta(z)$ denotes the Dirac δ function on the complex plane. Clearly, the diagonal global overlap is equal to the integrated overlap density given by Eq. (9), i.e.,

$$O = \int d^2z \, O(z). \tag{11}$$

III. PRODUCT OF GINIBRE MATRICES

Consider the product

$$X = X_1 X_2 \cdots X_m \tag{12}$$

of m independent identically distributed $N \times N$ Ginibre random matrices [85] with complex entries. The probability measure factorizes and can be written as a product of measures for individual Ginibre matrices

$$d\mu(X) \equiv d\mu(X_1, X_2, \ldots, X_m)$$

$$= d\mu(X_1)d\mu(X_2)\cdots d\mu(X_m), \tag{13}$$

each of which is given by

$$d\mu(X_i) = (\pi\sigma^2)^{-N^2} e^{-\frac{1}{\sigma^2}\text{Tr}X_i X_i^\dagger} DX_i, \tag{14}$$

where σ is a scale parameter, and $DX_i = \prod_{\alpha\beta} d\text{Re}X_{i,\alpha\beta}d\text{Im}X_{i,\alpha\beta}$. According to Eq. (9), the local diagonal overlap density can be calculated with respect to the measure $d\mu(X)$ in the following way

$$O(z) = \int d\mu(X)O_{11}\delta(z - \Lambda_1), \tag{15}$$

where Λ_α's correspond to the eigenvalues of the product X in Eq. (12). An analogous formula holds for the off-diagonal density. In the calculations we set $\sigma = 1$. One can easily transform the result to other values of σ in Eq. (14) by using the formula

$$O_\sigma(z) = \frac{1}{\sigma^{2m}} O_{\sigma=1}\left(\frac{z}{\sigma^m}\right), \tag{16}$$

which merely corresponds to the scale transformation of all Ginibre matrices $X_i \longrightarrow \sigma X_i$ in the product Eq. (12). Later, when discussing the limiting laws for $N \to \infty$ we will choose $\sigma = N^{-1/2}$. This choice of the scale parameter σ will ensure the existence of the limiting eigenvalue density on a compact support being the unit disk in the complex plane.

IV. CALCULATIONS OF THE OVERLAP FOR FINITE N

In order to calculate the global left-right vector overlap, defined by Eq. (4), for the product of Ginibre matrices in Eq. (12), we will change the parametrization of the matrices X_i's using the generalized Schur decomposition [45],

$$X_i = U_{i-1}\tau_i U_i^\dagger, \tag{17}$$

for $i = 1, \ldots, m$, where U_i are unitary matrices from the unitary group $U(N)$, and τ_i are upper triangular matrices of size $N \times N$. We use a cyclic indexing $U_i \equiv U_{m+i}$, in particular $U_0 \equiv U_m$. Sometimes it is convenient to express each τ_i as a sum of a diagonal matrix λ_i and a strictly upper triangular one t_i, namely,

$$\tau_i = \lambda_i + t_i = \begin{pmatrix} \lambda_{i,1} & t_{i,12} & t_{i,13} & \cdots & & t_{i,1N} \\ 0 & \lambda_{i,2} & t_{i,23} & \cdots & & t_{i,2N} \\ & & & \ddots & & \\ 0 & 0 & 0 & \lambda_{i,N-1} & t_{i,N-1N} \\ 0 & 0 & 0 & \cdots & & \lambda_{i,N} \end{pmatrix}. \tag{18}$$

In this representation, the product X is unitarily equivalent to a matrix $\mathcal{T}$, that is $X = U_m \mathcal{T} U_m^\dagger$, where

$$\mathcal{T} = \tau_1 \tau_2 \cdots \tau_m. \tag{19}$$

The matrix $\mathcal{T}$ has also an upper triangular form,

$$\mathcal{T} = \Lambda + T = \begin{pmatrix} \Lambda_1 & T_{12} & T_{13} & \cdots & & T_{1N} \\ 0 & \Lambda_2 & T_{23} & \cdots & & T_{2N} \\ & & & \ddots & & \\ 0 & 0 & 0 & \Lambda_{N-1} & T_{N-1N} \\ 0 & 0 & 0 & \cdots & & \Lambda_N \end{pmatrix}. \tag{20}$$

The diagonal elements of $\mathcal{T}$ are given by

$$\mathcal{T}_\alpha \equiv \Lambda_\alpha = \lambda_{1,\alpha}\lambda_{2,\alpha}\cdots\lambda_{m,\alpha}, \tag{21}$$

and the off-diagonal ones by

$$\mathcal{T}_{\alpha\nu} = \sum_{\alpha \leqslant \beta \leqslant \cdots \leqslant \nu} \tau_{1,\alpha\beta}\tau_{2,\beta\gamma}\cdots\tau_{m,\mu\nu}. \tag{22}$$

Any instance of $\tau_{i,\alpha\alpha}$ with two identical Greek indices can be replaced by $\lambda_{i,\alpha}$ and of $\tau_{i,\alpha\beta}$ with two different Greek indices by $t_{i,\alpha\beta}$ in the last formula. One can also express the integration measure in terms of U's, λ's, and t's. Since one is interested in invariant observables, the U's can be integrated out. For the scale parameter $\sigma = 1$, one gets [45]

$$d\mu(\lambda,t) = Z^{-1}|\Delta(\mathbf{\Lambda})|^2 \prod_{i,\alpha} e^{-|\lambda_{i,\alpha}|^2} d^2\lambda_{i,\alpha}$$

$$\times \prod_{j,\beta<\gamma} \frac{1}{\pi} e^{-|t_{j,\beta\gamma}|^2} d^2 t_{j,\beta\gamma}, \tag{23}$$

where the normalization factor Z is given by the formula

$$Z = N![\pi^N 1!2!\cdots(N-1)!]^m, \tag{24}$$

and the Vandermonde determinant $\Delta(\mathbf{\Lambda})$ for the product $X = X_1 X_2 \cdots X_m$ has the form

$$
\Delta(\mathbf{\Lambda}) = \prod_{\alpha < \beta} (\lambda_{1,\alpha}\lambda_{2,\alpha} \cdots \lambda_{m,\alpha} - \lambda_{1,\beta}\lambda_{2,\beta} \cdots \lambda_{m,\beta})
$$

$$
= \prod_{\alpha < \beta} (\Lambda_\alpha - \Lambda_\beta). \tag{25}
$$

The square of the determinant in Eq. (23) comes from the Jacobian of the transformation Eq. (17).

The next step is to express the observables in terms of $t's$ and $\lambda's$. For example, to calculate the diagonal overlap density [cf. Eq. (15)], we have to find $O_{11} = O_{11}(t,\lambda)$ and to integrate over t's and λ's with the Dirac δ constraint,

$$
O(z) = \int d\mu(\lambda,t) O_{11}(t,\lambda)\delta(z - \Lambda_1), \tag{26}
$$

while for the global overlap $O = \int d\mu(\lambda,t) O_{11}(t,\lambda)$. The measure $d\mu(\lambda,t)$, Eq. (23), factorizes $d\mu(\lambda,t) = d\mu(\lambda)d\mu(t)$. One can first integrate over t's. This is a Gaussian integral and can be easily performed. After this integration, only the dependence on λ's is left,

$$
O_{11}(\lambda) = \int d\mu(t) O_{11}(t,\lambda), \tag{27}
$$

where $d\mu(t)$ is a normalized Gaussian measure equal to the t-dependent piece of $d\mu(\lambda,t)$, Eq. (23). The last step is to integrate over λ's with the measure given by Eq. (23),

$$
O(z) = Z^{-1} \int d\mu(\lambda) |\Delta(\mathbf{\Lambda})|^2 e^{-\sum_{i,\alpha} |\lambda_{i,\alpha}|^2} O_{11}(\lambda)\delta(z - \Lambda_1), \tag{28}
$$

where as before Λ_α's stand for $\Lambda_\alpha = \lambda_{1,\alpha}\lambda_{2,\alpha} \cdots \lambda_{m,\alpha}$. We will do this below. First we have to find the function $O_{11}(t,\lambda)$. This can be done as follows. We choose the basis in which the product matrix X is equal to $\mathcal{T}$. Such a basis exists since the two matrices are unitarily equivalent. In this basis, the first right eigenvector $|R_1\rangle$ is represented as a column vector with "1" in the position 1 and 0's elsewhere: $|R_1\rangle = (1,0,0,\ldots)^T$. The vector is written here as transpose of a row vector to save space. Denote the elements of the first left eigenvector $\langle L_1| = (B_1, B_2, \ldots)$. The eigenvalue equation $\langle L_1|\mathcal{T} = \langle L_1|\Lambda_1$ leads to the following recursion relation for B_β's [78,79]:

$$
B_\beta = \frac{1}{\Lambda_1 - \Lambda_\beta} \sum_{\alpha=1}^{\beta-1} B_\alpha T_{\alpha\beta}. \tag{29}
$$

The recursion is initiated by $B_1 = 1$ as follows from the biorthogonality relation Eq. (2). One finds

$$
B_1 = 1,
$$
$$
B_2 = \frac{T_{12}}{\Lambda_1 - \Lambda_2},
$$
$$
B_3 = \frac{T_{13}}{\Lambda_1 - \Lambda_3} + \frac{T_{12}T_{23}}{(\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_3)},
$$
$$
B_4 = \frac{T_{14}}{\Lambda_1 - \Lambda_4} + \frac{T_{12}T_{24}}{(\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_4)}
$$
$$
+ \frac{T_{13}T_{34}}{(\Lambda_1 - \Lambda_3)(\Lambda_1 - \Lambda_4)}
$$
$$
+ \frac{T_{12}T_{23}T_{34}}{(\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_3)(\Lambda_1 - \Lambda_4)}, \text{etc.} \tag{30}
$$

The element O_{11} of the overlap matrix is related to B's as

$$
O_{11} = \sum_{\alpha=1}^{N} |B_\alpha|^2, \tag{31}
$$

and B's depend on t's and λ's through T's and Λ's. Combining Eqs. (30) and (31) with Eq. (26) we obtain an explicit form of the integral over t's and λ's, which can be done. We will give a couple of examples below.

V. EXAMPLES

Let us first illustrate the calculations for $N = 2, m = 2$, and $\sigma = 1$—that is for the product of two 2×2 Ginibre matrices. First, we express T_{12} in terms of t's and λ's as follows:

$$
\mathcal{T} = \begin{pmatrix} \lambda_{1,1} & t_{1,12} \\ 0 & \lambda_{1,2} \end{pmatrix}\begin{pmatrix} \lambda_{2,1} & t_{2,12} \\ 0 & \lambda_{2,2} \end{pmatrix} = \begin{pmatrix} \Lambda_1 & T_{12} \\ 0 & \Lambda_2 \end{pmatrix}. \tag{32}
$$

This gives $T_{12} = \lambda_{1,1}t_{2,12} + t_{1,12}\lambda_{2,2}$ and $\Lambda_\alpha = \lambda_{1,\alpha}\lambda_{2,\alpha}$ for $\alpha = 1,2$. Thus, we have

$$
O_{11}(t,\lambda) = 1 + \frac{|T_{12}|^2}{|\Lambda_1 - \Lambda_2|^2} = 1 + \frac{|\lambda_{1,1}t_{2,12} + t_{1,12}\lambda_{2,2}|^2}{|\lambda_{1,1}\lambda_{2,1} - \lambda_{1,2}\lambda_{2,2}|^2}. \tag{33}
$$

According to Eq. (27), the integration over t's leads to the following result:

$$
O_{11}(\lambda) = 1 + \frac{|\lambda_{1,1}|^2 + |\lambda_{2,2}|^2}{|\lambda_{1,1}\lambda_{2,1} - \lambda_{1,2}\lambda_{2,2}|^2}. \tag{34}
$$

Now we have to compute the integral over λ's given by Eq. (28), namely

$$
O(z) = \frac{1}{2\pi^4} \int (|\lambda_{1,1}\lambda_{2,1} - \lambda_{1,2}\lambda_{2,2}|^2 + |\lambda_{1,1}|^2
$$
$$
+ |\lambda_{2,2}|^2)\delta(z - \lambda_{1,1}\lambda_{2,1})\prod_{i,\alpha} e^{-|\lambda_{i,\alpha}|^2}d^2\lambda_{i,\alpha}. \tag{35}
$$

We first integrate over the λ's that do not appear in the Dirac δ, that is $\lambda_{1,2}$ and $\lambda_{2,2}$. These integrals are in general of the Gaussian type combined with a power function, i.e., $\int d^2z |z|^{2k} \exp(-|z|^2) = \pi k!$. As a result of the integration, we obtain

$$
O(z) = \frac{1}{2\pi^2} \int (|z|^2 + 2 + |\lambda_{1,1}|^2)
$$
$$
\times \delta(z - \lambda_{1,1}\lambda_{2,1})e^{-|\lambda_{1,1}|^2 - |\lambda_{2,1}|^2}d^2\lambda_{1,1}d^2\lambda_{2,1}. \tag{36}
$$

Now we integrate over $\lambda_{2,1}$. We use the scaling property of the Dirac δ, $\delta[a(z - z_0)] = (1/|a|^2)\delta(z - z_0)$, to get

$$
O(z) = \frac{1}{2\pi^2} \int \frac{|z|^2 + 2 + |\lambda_{1,1}|^2}{|\lambda_{1,1}|^2}
$$
$$
\times \exp\left(-|\lambda_{1,1}|^2 - \frac{|z|^2}{|\lambda_{1,1}|^2}\right)d^2\lambda_{1,1}. \tag{37}
$$

The integral over $\lambda_{1,1}$ can be conveniently done in polar coordinates, $\lambda_{1,1} = \sqrt{x}\exp(i\phi)$,

$$
O(z) = \frac{1}{2\pi} \int_0^\infty \frac{|z|^2 + 2 + x}{x}\exp\left(-x - \frac{|z|^2}{x}\right)dx, \tag{38}
$$

yielding

$$O(z) = \frac{1}{\pi}[(2 + |z|^2)K_0(2|z|) + |z|K_1(2|z|)], \quad (39)$$

where K_ν denotes the modified Bessel function of the second kind. The global overlap is

$$O = \int d^2z \, O(z) = 2. \quad (40)$$

The overlap density depends on the modulus $|z|$. It is convenient to represent this quantity as a radial function in the variable $r = |z|$,

$$O_{\rm rad}(r) = 2\pi r \, O(r). \quad (41)$$

Clearly $O_{\rm rad}(r)dr$ is equal to the overlap density integrated over the annulus $r \leqslant |z| \leqslant r + dr$. In our case, we have

$$O_{\rm rad}(r) = 2r(2 + r^2)K_0(2r) + 2r^2 K_1(2r). \quad (42)$$

In principle, one may repeat the calculation for any N and m. All integrals except those over the λ's appearing in the argument of the Dirac δ, i.e., $\delta(z - \lambda_{1,1} \cdots \lambda_{1,m})$, are Gaussian and can be done explicitly. The integrals over λ's from the Dirac δ generate instead Meijer G-functions due to the multiplicative constraint [80]. Let us illustrate it for the product of three 2×2 Ginibre matrices. The calculation goes as before. The element T_{12} of the $\mathcal{T}$ matrix is

$$T_{12} = \lambda_{1,1}\lambda_{2,1}t_{3,12} + \lambda_{1,1}t_{2,12}\lambda_{3,2} + t_{1,12}\lambda_{2,2}\lambda_{3,2}, \quad (43)$$

and the diagonal elements are $\Lambda_1 = \lambda_{1,1}\lambda_{2,1}\lambda_{3,1}$, $\Lambda_2 = \lambda_{1,2}\lambda_{2,2}\lambda_{3,2}$. Hence, the counterpart of Eq. (33) is

$$O_{11}(\lambda,t) = 1 + \frac{|\lambda_{1,1}\lambda_{2,1}t_{3,12} + \lambda_{1,1}t_{2,12}\lambda_{3,2} + t_{1,12}\lambda_{2,2}\lambda_{3,2}|^2}{|\lambda_{1,1}\lambda_{2,1}\lambda_{3,1} - \lambda_{1,2}\lambda_{2,2}\lambda_{3,2}|^2}. \quad (44)$$

Integrating over t's we get

$$O_{11}(\lambda) = 1 + \frac{|\lambda_{1,1}\lambda_{2,1}|^2 + |\lambda_{1,1}\lambda_{3,2}|^2 + |\lambda_{2,2}\lambda_{3,2}|^2}{|\lambda_{1,1}\lambda_{2,1}\lambda_{3,1} - \lambda_{1,2}\lambda_{2,2}\lambda_{3,2}|^2}, \quad (45)$$

and over the λ's (except those in the Dirac δ),

$$O(z) = \frac{1}{2\pi^6}\int (|z|^2 + 2 + |\lambda_{1,1}\lambda_{2,1}|^2 + |\lambda_{1,1}|^2)\delta(z - \lambda_{1,1}\lambda_{2,1}\lambda_{3,1})$$
$$\times e^{-|\lambda_{1,1}|^2 - |\lambda_{2,1}|^2 - |\lambda_{3,1}|^2} d^2\lambda_{1,1} d^2\lambda_{2,1} d^2\lambda_{3,1}. \quad (46)$$

Next, we integrate over $\lambda_{3,1}$ and use polar coordinates for $\lambda_{1,1} = \sqrt{x_1}\exp(i\phi_1)$ and $\lambda_{2,1} = \sqrt{x_2}\exp(i\phi_2)$. We eventually obtain

$$O(z) = \frac{1}{2\pi}\int_0^\infty \int_0^\infty \frac{|z|^2 + 2 + x_1 x_2 + x_1}{x_1 x_2}$$
$$\times \exp\left(-x_1 - x_2 - \frac{|z|^2}{x_1 x_2}\right) dx_1 dx_2, \quad (47)$$

which yields the radial function

$$O_{\rm rad}(r) = r^2 G_{03}^{30}\left(\begin{smallmatrix} - \\ -\frac{1}{2},\frac{1}{2},\frac{1}{2} \end{smallmatrix}\middle| r^2\right) + 2r G_{03}^{30}\left(\begin{smallmatrix} - \\ 0,0,0 \end{smallmatrix}\middle| r^2\right)$$
$$+ r G_{03}^{30}\left(\begin{smallmatrix} - \\ 0,0,1 \end{smallmatrix}\middle| r^2\right) + r G_{03}^{30}\left(\begin{smallmatrix} - \\ 1,1,1 \end{smallmatrix}\middle| r^2\right). \quad (48)$$

One finds that the global overlap for $N = 2$ and $m = 3$ is

$$O = \int d^2z \, O(z) = \int_0^\infty O_{\rm rad}(r)dr = \frac{5}{2}. \quad (49)$$

One may repeat the calculations for larger N and larger m. The integrals one has to do are elementary but the bookkeeping gets involved and the calculations become tedious. For example, for $N = 3$ and $m = 2$ one has to sum three terms depending on the coefficients B_1, B_2, and B_3 as follows from Eq. (30), which depend on λ's and t's through Λ's and T's: $T_{12} = \lambda_{1,1}t_{2,12} + t_{1,12}\lambda_{2,2}$, $T_{13} = \lambda_{1,1}t_{2,13} + t_{1,12}t_{2,23} + t_{1,13}\lambda_{2,3}$, and $T_{23} = \lambda_{1,2}t_{2,23} + t_{1,23}\lambda_{2,3}$. Integrals over t's can be done in an algebraic way using the Wick theorem and the following two-point functions,

$$\langle t_{i,\alpha\beta}\bar{t}_{j,\mu\nu}\rangle_t = \delta_{ij}\delta_{\alpha\mu}\delta_{\beta\nu}, \quad \langle t_{i,\alpha\beta}t_{j,\mu\nu}\rangle_t = 0, \quad (50)$$

where the symbol $\langle t_{i,\alpha\beta}\bar{t}_{j,\mu\nu}\rangle_t$ is to be understood as follows:

$$\langle t_{i,\alpha\beta}\bar{t}_{j,\mu\nu}\rangle_t = \int t_{i,\alpha\beta}\bar{t}_{j,\mu\nu}\prod_{k,\eta<\gamma}\frac{1}{\pi}e^{-|t_{k,\eta\gamma}|^2}d^2 t_{k,\eta\gamma}. \quad (51)$$

We skip the calculations and give the final results, which read

$$O_{\rm rad}(r) = \frac{1}{3}r(r^4 + 8r^2 + 12)K_0(2r) + \frac{1}{3}(2r^4 + 8r^2)K_1(2r) \quad (52)$$

and

$$O = \int O_{\rm rad}(r)dr = 3. \quad (53)$$

In Figs. 1, 2, and 3, we show the theoretical predictions for the radial profile of the overlap densities and the corresponding histograms from Monte Carlo simulations for $N = 2, m = 2$ [cf. Eq. (42)], $N = 2, m = 3$ [cf. Eq. (48)], and $N = 3, m = 2$ [cf. Eq. (52)], respectively. We see that the Monte Carlo data follow the theoretical curves.

VI. CONJECTURE

The calculations of the global density are slightly easier because there is no Dirac δ, $\delta(z - \Lambda_1)$, in the integrand. They
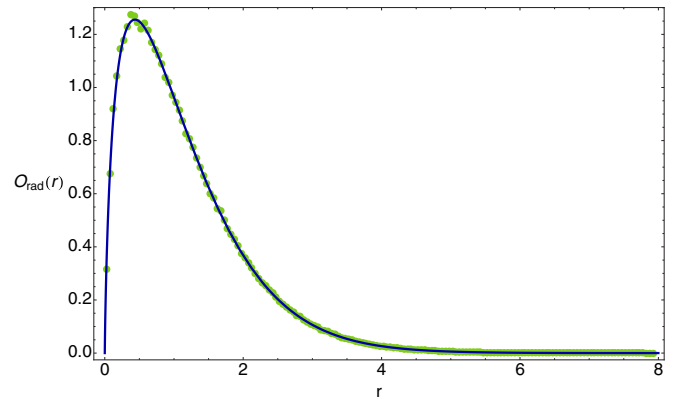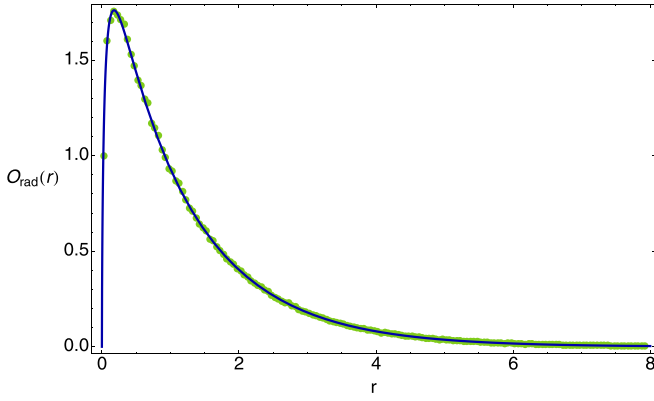

FIG. 1. Overlap density for $N = 2$ and $m = 2$: theoretical prediction given by Eq. (42) (solid line) and numerical histogram (points) generated in Monte Carlo simulations of 10^6 products of two 2×2 Ginibre matrices.

FIG. 2. Overlap density for $N = 2$ and $m = 3$: theoretical prediction given by Eq. (48) (solid line) and numerical histogram (points) generated in Monte Carlo simulations of 10^6 products of three 2×2 Ginibre matrices.

are particularly simple for $N = 2$. In this case,

$$T_{12} = \sum_{k=1}^{m} t_{k,12} \prod_{j=1}^{k-1} \lambda_{j,1} \prod_{j=k+1}^{m} \lambda_{j,2}, \qquad (54)$$

and after inserting this into Eq. (33) and integrating the t's, one obtains

$$O = 1 + \frac{1}{2\pi^{2m}} \sum_{k=1}^{m} \int \prod_{j=1}^{k-1} |\lambda_{j,1}|^2$$

$$\times \prod_{j=k+1}^{m} |\lambda_{j,2}|^2 \prod_{i=1}^{m} e^{-|\lambda_{i,1}|^2 - |\lambda_{i,2}|^2} d^2\lambda_{i,1} d^2\lambda_{i,2}. \quad (55)$$

Each integral over λ is either of the form $\int |z|^2 \exp(-|z|^2) d^2z = \pi$ or $\int \exp(-|z|^2) d^2z = \pi$, so all together the integration over λ's gives the factor π^{2m}, which cancels the prefactor π^{-2m}, yielding

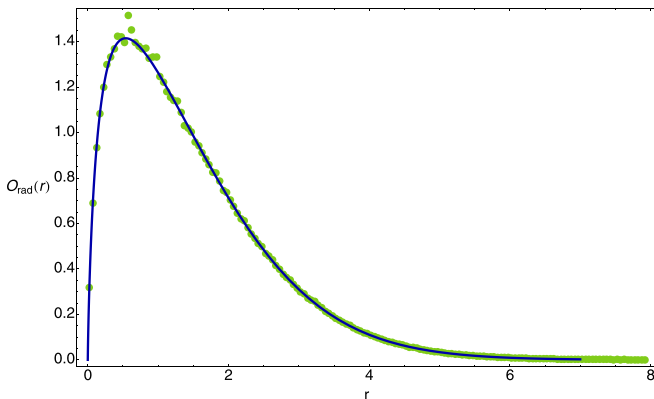$$O = 1 + \frac{m}{2}. \qquad (56)$$

FIG. 3. Overlap density for $N = 3$ and $m = 2$: theoretical prediction given by Eq. (52) (solid line) and numerical histogram (points) generated in Monte Carlo simulations of 10^6 products of two 3×3 Ginibre matrices.
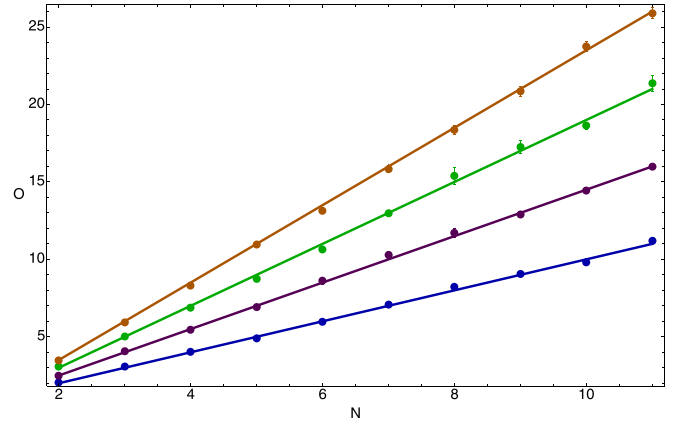

FIG. 4. Conjectured form of the overlap Eq. (59) (solid lines) for $m = 2, 3, 4, 5$ and $N = 2, \ldots, 11$ and numerical histograms (points) generated in Monte Carlo simulations, each for 10^4 instances.

Now, consider the case $m = 1$ for any N. This case was discussed in Ref. [79]. As follows from the discussion presented in this paper, one can cast the overlap into the form of the following multidimensional integral,

$$O = \frac{1}{Z} \int \prod_{\alpha=1}^{N-1} \left(1 + \frac{1}{|\lambda_N - \lambda_\alpha|^2} \right) |\Delta(\lambda)|^2 \prod_{\alpha=1}^{N} e^{-|\lambda_\alpha|^2} d^2\lambda_\alpha, \tag{57}$$

where $Z = \pi^N 1! 2! \cdots N!$ [cf. Eq. (24)]. What remains to do is to compute this integral. We do this in Appendix A, where we show that the integral yields

$$O = 1 + \tfrac{1}{2}(N - 1). \qquad (58)$$

The results given by Eqs. (56) and (58) suggest that O grows linearly with m and N, hence it is tempting to conjecture that for any m and N the global overlap is given by the formula

$$O = 1 + \frac{m}{2}(N - 1). \qquad (59)$$

The result given by Eq. (53) is in agreement with this formula and Monte Carlo simulations fully corroborate this conjecture as shown in Fig. 4.

VII. LARGE N LIMIT

We now consider the limit $N \to \infty$. We set the width parameter $\sigma^2 = 1/N$ in the measure Eq. (14). The limit $N \to \infty$ has to be taken carefully since we expect $O_N(z)$ to grow with N as it results from Eq. (59). In order to explicitly indicate the size dependence of $O(z)$ on N here we exceptionally added the subscript N to $O(z) = O_N(z)$, which is implicit in the remaining part of the paper. It is convenient to define the growth rate of the overlap density as

$$o_N(z) = \frac{O_N(z)}{N}. \qquad (60)$$

It depends on N but is expected to approach a N-independent function $o(z)$: $o_N(z) \to o(z)$ for $N \to \infty$. As follows from Eq. (59), $\int d^2z\, o(z) = m/2$.

In the calculations, we shall use the method [28] that was previously employed to calculate the limiting eigenvalue

density,

$$\rho(z) = \lim_{N \to \infty} \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(z - \Lambda_j) \right\rangle. \tag{61}$$

The method is based on the generalized Green function [81–83],

$$\widehat{G}(z,\epsilon) = \left\langle \begin{pmatrix} z\mathbb{1}_N - X & \epsilon\mathbb{1}_N \\ -\bar{\epsilon}\mathbb{1}_N & \bar{z}\mathbb{1}_N - X^\dagger \end{pmatrix}^{-1} \right\rangle, \tag{62}$$

which consists of $N \times N$ blocks $G_{\alpha\beta}$,

$$\widehat{G}(z,\epsilon) = \begin{pmatrix} G_{11}(z,\epsilon) & G_{12}(z,\epsilon) \\ G_{21}(z,\epsilon) & G_{22}(z,\epsilon) \end{pmatrix}. \tag{63}$$

For clarity, the symbol "hat" is reserved for matrices with a superimposed block structure. By defining the block-trace $\mathrm{Tr_b}$ as a matrix of traces of individual blocks,

$$\mathrm{Tr_b}\widehat{G} = \begin{pmatrix} \mathrm{Tr}\,G_{11} & \mathrm{Tr}\,G_{12} \\ \mathrm{Tr}\,G_{21} & \mathrm{Tr}\,G_{22} \end{pmatrix}, \tag{64}$$

one can project the $2N \times 2N$ matrix $\widehat{G}$ onto a 2×2 matrix $\widehat{g}$,

$$\widehat{g}(z) = \begin{pmatrix} g_{11}(z) & g_{12}(z) \\ g_{21}(z) & g_{22}(z) \end{pmatrix} = \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \mathrm{Tr_b}\widehat{G}(z,\epsilon). \tag{65}$$

The elements of this matrix are related to each other, $g_{22}(z) = \bar{g}_{11}(z)$ and $g_{21}(z) = -\bar{g}_{12}(z)$ [44], so we have

$$\widehat{g}(z) = \begin{pmatrix} g(z) & \gamma(z) \\ -\bar{\gamma}(z) & \bar{g}(z) \end{pmatrix}. \tag{66}$$

In the large N limit, the eigenvalue density is related to the diagonal element [81–83],

$$\rho(z) = \frac{1}{\pi} \frac{\partial g(z)}{\partial \bar{z}}, \tag{67}$$

and the growth rate of the overlap to the off-diagonal one [84],

$$o(z) = \frac{1}{\pi} |\gamma(z)|^2. \tag{68}$$

For large N, the leading contribution to the overlap grows linearly with N: $O_N(z) \sim No(z)$.

Equations (67) and (68) are general and can be applied to any random matrix provided the Green function $\widehat{g}(z)$ can be calculated. So the goal is now to calculate the Green function for the problem at hand. To this end, we use the planar diagrams enumeration technique [86–88].

VIII. DYSON-SCHWINGER EQUATIONS

The enumeration of planar Feynman diagrams is a method to derive the large N limit for matrix models [86–88]. The method is based on a field-theoretical representation of multidimensional integrals in terms of Feynman diagrams. One is interested in calculating the Green function,

$$\widehat{G}_{AB} = \langle (\widehat{Q} - \widehat{X})_{AB}^{-1} \rangle, \tag{69}$$

where $\widehat{Q}$ is a constant matrix and $\widehat{X}$ is the random matrix that is averaged over. Matrix indices are denoted by A and B in the last equation. In this approach, the Green function plays the role of generating function for connected two-point Feynman

diagrams. The contributions from nonplanar diagrams are suppressed at least as $1/N$ in the large N limit, so for $N \to \infty$ only planar diagrams survive in the counting. One can write a set of equations that relate the Green function $\widehat{G}_{AB}$ to a generating function $\widehat{\Sigma}_{AB}$ for one-line irreducible diagrams. Such equations are known in the field-theoretical literature as Dyson-Schwinger equations. Here, we are interested only in Gaussian random matrices. In this case, the Dyson-Schwinger equations assume a simple form in the planar limit $N \to \infty$ [28],

$$\widehat{G}_{AB} = (\widehat{Q} - \widehat{\Sigma})_{AB}^{-1},$$
$$\widehat{\Sigma}_{AD} = \sum_{BC} \widehat{P}_{AB,CD}\widehat{G}_{BC}, \tag{70}$$

where $\widehat{P}_{AB,CD}$ represents the propagator,

$$\widehat{P}_{AB,CD} = \langle \widehat{X}_{AB}\widehat{X}_{CD} \rangle. \tag{71}$$

The matrix $\widehat{Q}_{AB}$ and the propagator $\widehat{P}_{AB,CD}$ are inputs to be injected into these equations, while $\widehat{G}_{AB}$ and $\widehat{\Sigma}_{AB}$ are unknown functions to be determined for the given inputs. In other words, one has first to specify what $\widehat{Q}$ and $\widehat{P}$ are, and then, using these equations, one can find the Green function $\widehat{G}$, from there $\widehat{g}$ and finally the eigenvalue density $\rho(z)$ [cf. Eq. (67)] and the overlap growth rate $o(z)$ [cf. Eq. (68)].

IX. SINGLE GINIBRE MATRIX

In this section, we review the calculations [82,84] for a single Ginibre matrix [85]. In the next section, we will then show how to generalize the method to the product of Ginibre matrices [28].

As mentioned before, first one has to identify the matrix $\widehat{Q}$ and to calculate the propagator $\widehat{P}_{AB,CD}$. The Green function Eq. (62) reads

$$\widehat{G}(z,\epsilon) = \langle (\widehat{Q} - \widehat{X})^{-1} \rangle, \tag{72}$$

with

$$\widehat{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix} \tag{73}$$

and

$$\widehat{Q} = \widehat{q} \otimes \mathbb{1}_N, \tag{74}$$

where

$$\widehat{q} = \begin{pmatrix} z & \epsilon \\ -\bar{\epsilon} & \bar{z} \end{pmatrix}. \tag{75}$$

The symbol $\otimes$ denotes the Kronecker product. The blocks of the matrix $\widehat{X}$ can be identified with the Ginibre matrix and its Hermitian conjugate: $\widehat{X}_{11} = X$, $\widehat{X}_{22} = X^\dagger$, and $\widehat{X}_{12} = \widehat{X}_{21} = 0$, respectively. In order to calculate the propagator, we recall that the two-point correlations for the Ginibre matrix Eq. (14) with $\sigma^2 = 1/N$ are

$$\langle X_{ab}X_{cd}^\dagger \rangle = \int d\mu(X) X_{ab}X_{cd}^\dagger = \frac{1}{N}\delta_{ad}\delta_{bc} \tag{76}$$

and

$$\langle X_{ab}X_{cd} \rangle = \langle X_{ab}^\dagger X_{cd}^\dagger \rangle = 0. \tag{77}$$

Since all matrices have a block structure, it is convenient to separately write index positions of the blocks and positions of elements inside the blocks, and to split matrix indices into pairs of indices $A = (\alpha,a)$, $B = (\beta,b)$, $C = (\gamma,c)$, $D = (\delta,d)$, etc., with the Greek indices referring to the positions of the blocks, and small Latin indices to the positions within each block. The Greek indices run over the range 1 to 2 and the small Latin indices over the range 1 to N. The dimension of the matrices is $2N \times 2N$. This block structure is also inherited by the propagators. Using the identification $\widehat{X}_{11} \leftrightarrow X$, $\widehat{X}_{22} \leftrightarrow X^\dagger$, along with Eqs. (76) and (77), we see that the propagator factorizes into the interblock part (in Greek indices) and intrablock part (in Latin indices),

$$\widehat{P}_{AB,CD} = \widehat{p}_{\alpha\beta,\gamma\delta} \frac{1}{N} \delta_{ad}\delta_{bc}. \tag{78}$$

The only nontrivial elements of the interblock part are $\widehat{p}_{11,22} = \widehat{p}_{22,11} = 1$. All remaining elements vanish: $\widehat{p}_{\alpha\beta,\gamma\delta} = 0$. Since both the propagator Eq. (78) and the matrix $\widehat{Q}_{AB} = q_{\alpha\beta}\delta_{ab}$ are proportional to the Kronecker δ's in Latin indices, this implies that the matrices $\widehat{G}$ and $\widehat{\Sigma}$, being the solution of the Dyson-Schwinger Eqs. (70), are also proportional to the Kronecker δ in the intrablock indices,

$$\widehat{G}_{AB} = \widehat{g}_{\alpha\beta}\delta_{ab}, \quad \widehat{\Sigma}_{AB} = \widehat{\sigma}_{\alpha\beta}\delta_{ab}. \tag{79}$$

Alternatively, one can write $\widehat{G} = \widehat{g} \otimes \mathbb{1}$ and $\widehat{\Sigma} = \widehat{\sigma} \otimes \mathbb{1}$. Therefore, one can reduce the Dyson-Schwinger Eq. (70) to equations for interblock elements (in Greek indices)

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \left(\begin{pmatrix} z & \epsilon \\ -\bar{\epsilon} & \bar{z} \end{pmatrix} - \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right)^{-1},$$
$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0 & g_{12} \\ g_{21} & 0 \end{pmatrix}. \tag{80}$$

In the second equation, we used that $\widehat{p}_{11,22} = \widehat{p}_{22,11} = 1$ and $\widehat{p}_{\alpha\beta,\gamma\delta} = 0$ for other combinations of indices. The limit $N \to \infty$ has already been taken in these equations, since they count contributions of planar diagrams. Now we can take the limit $\epsilon \to 0$ [cf. Eq. (65)]. This merely corresponds to setting $\epsilon = 0$. Eliminating the $\{\sigma_{\alpha\beta}\}$, we get

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} z & -g_{12} \\ -g_{21} & \bar{z} \end{pmatrix}^{-1}. \tag{81}$$

Setting $g = g_{11} = \bar{g}_{22}$ and $\gamma = g_{12} = -\bar{g}_{21}$, we obtain

$$\begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix} = \begin{pmatrix} z & -\gamma \\ \bar{\gamma} & \bar{z} \end{pmatrix}^{-1} \equiv \frac{1}{|z|^2 + |\gamma|^2} \begin{pmatrix} \bar{z} & \gamma \\ -\bar{\gamma} & z \end{pmatrix}. \tag{82}$$

The solution reads

$$g(z) = \frac{1}{z}, \quad \gamma(z) = 0, \quad \text{for } |z| \geqslant 1, \tag{83}$$

and

$$g(z) = \bar{z}, \quad |\gamma(z)| = \sqrt{1 - |z|^2}, \quad \text{for } |z| \leqslant 1. \tag{84}$$

The solution for $\gamma(z)$ inside the unit circle is given up to the phase, but this is sufficient for our purposes since the correlations density $o(z)$ given by Eq. (68) depends only on

the modulus of $\gamma(z)$. Using Eqs. (67) and (68), one eventually finds

$$\rho(z) = \frac{1}{\pi} \chi_D(z) \tag{85}$$

and

$$o(z) = \frac{1}{\pi} (1 - |z|^2) \chi_D(z), \tag{86}$$

where χ_D is an indicator function for the unit disk, $\chi_D(z) = 1$ for $|z| \leqslant 1$ and $\chi_D(z) = 0$ for $|z| > 1$.

X. PRODUCT OF TWO GINIBRE MATRICES

In this section, we generalize the approach from the previous section to the product of two Ginibre matrices [28]. The integration measure for the product $X = X_1 X_2$ of independent Ginibre matrices X_1 and X_2 is the product of individual integration measures $d\mu(X_1)d\mu(X_2)$ given by Eq. (14). According to Eq. (76) the only nonvanishing two-point correlations are

$$\langle X_{1,ab} X_{1,cd}^\dagger \rangle = \langle X_{2,ab} X_{2,cd}^\dagger \rangle = \frac{1}{N} \delta_{ad}\delta_{bc}. \tag{87}$$

The Green function Eq. (62) for the product reads

$$\widehat{G}(z,\epsilon) = \left\langle \begin{pmatrix} z\mathbb{1}_N - X_1 X_2 & \epsilon\mathbb{1}_N \\ -\bar{\epsilon}\mathbb{1}_N & \bar{z}\mathbb{1}_N - X_2^\dagger X_1^\dagger \end{pmatrix}^{-1} \right\rangle. \tag{88}$$

This form is difficult to handle because the product of Gaussian matrices $X_1 X_2$ is not Gaussian. One can, however, linearize the problem by considering a block matrix of dimensions $2N \times 2N$,

$$R = \begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix}, \tag{89}$$

which is Gaussian. We call it root matrix because its square,

$$R^2 = \begin{pmatrix} X_1 X_2 & 0 \\ 0 & X_2 X_1 \end{pmatrix}, \tag{90}$$

reproduces two copies of the product, $X_1 X_2$ and $X_2 X_1$. The two copies have identical eigenvalues. The Green function for the root matrix is

$$\widehat{G}(z,\epsilon) = \left\langle \begin{pmatrix} z\mathbb{1}_N - R & \epsilon\mathbb{1}_N \\ -\bar{\epsilon}\mathbb{1}_N & \bar{z}\mathbb{1}_N - R^\dagger \end{pmatrix}^{-1} \right\rangle, \tag{91}$$

which is actually a $4N \times 4N$ block matrix,

$$\widehat{G}(z,\epsilon) = \langle (\widehat{q} \otimes \mathbb{1}_N - \widehat{R})^{-1} \rangle, \tag{92}$$

where

$$\widehat{q} = \begin{pmatrix} z & 0 & \epsilon & 0 \\ 0 & z & 0 & \epsilon \\ -\bar{\epsilon} & 0 & \bar{z} & 0 \\ 0 & -\bar{\epsilon} & 0 & \bar{z} \end{pmatrix} \xrightarrow{\epsilon \to 0} \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & \bar{z} & 0 \\ 0 & 0 & 0 & \bar{z} \end{pmatrix} \tag{93}$$

and

$$\widehat{R} = \begin{pmatrix} 0 & X_1 & 0 & 0 \\ X_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_2^\dagger \\ 0 & 0 & X_1^\dagger & 0 \end{pmatrix}. \tag{94}$$

In this representation, the resolvent Eq. (92) has the standard form in which $\widehat{R}$ is linear in the random matrices X's. Indexing blocks of $\widehat{R}$ by $\widehat{R}_{\alpha\beta}$, with $\alpha = 1, \ldots, 4$ and $\beta = 1, \ldots, 4$, we have $\widehat{R}_{12} = X_1$, $\widehat{R}_{21} = X_2$, $\widehat{R}_{34} = X_2^\dagger$, $\widehat{R}_{43} = X_1^\dagger$. As follows from Eq. (87), the block $\widehat{R}_{12}$ is correlated with $\widehat{R}_{43}$ and $\widehat{R}_{21}$ with $\widehat{R}_{34}$, so the propagator

$$\widehat{P}_{AB,CD} = \widehat{p}_{\alpha\beta,\gamma\delta}\frac{1}{N}\delta_{ad}\delta_{bc} \qquad (95)$$

has the following nonzero elements, $\widehat{p}_{12,43} = \widehat{p}_{43,12} = \widehat{p}_{21,34} = \widehat{p}_{34,21} = 1$. All other elements of $\widehat{p}_{\alpha\beta,\gamma\delta} = 0$. The situation is completely analogous to that discussed in the previous section, except that now the problem has dimensions 4×4 in interblock indices. The intrablock correlations are the same as before—that is they are proportional to $(1/N)\delta_{ad}\delta_{bc}$— so the solution has the diagonal form proportional to the Kronecker δ in Latin indices Eq. (79). The Dyson-Schwinger Eqs. (70) for the interblock elements of the Green function of the root matrix read for $\epsilon \to 0$

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

$$= \begin{pmatrix} z - \sigma_{11} & -\sigma_{12} & -\sigma_{13} & -\sigma_{14} \\ -\sigma_{21} & z - \sigma_{22} & -\sigma_{23} & -\sigma_{24} \\ -\sigma_{31} & -\sigma_{32} & \bar{z} - \sigma_{33} & -\sigma_{34} \\ -\sigma_{41} & -\sigma_{42} & -\sigma_{43} & \bar{z} - \sigma_{44} \end{pmatrix}^{-1} \qquad (96)$$

and

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & g_{24} & 0 \\ 0 & 0 & 0 & g_{13} \\ g_{42} & 0 & 0 & 0 \\ 0 & g_{31} & 0 & 0 \end{pmatrix}. \qquad (97)$$

In the second equation, we used the propagator structure: $\widehat{p}_{12,43} = \widehat{p}_{43,12} = \widehat{p}_{21,34} = \widehat{p}_{34,21} = 1$ and $\widehat{p}_{\alpha\beta,\gamma\delta} = 0$, otherwise. Inserting $\{\sigma_{\alpha\beta}\}$ into the first equation, we get

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

$$= \begin{pmatrix} z & 0 & -g_{24} & 0 \\ 0 & z & 0 & -g_{13} \\ -g_{42} & 0 & \bar{z} & 0 \\ 0 & -g_{31} & 0 & \bar{z} \end{pmatrix}^{-1}. \qquad (98)$$

It is convenient to solve this equation by defining matrices $\tilde{g}$ and $\tilde{\sigma}$ unitarily equivalent to $\widehat{g}$ and $\widehat{\sigma}$: $\tilde{g} = P\widehat{g}P^{-1}$ and $\tilde{\sigma} = P\widehat{\sigma}P^{-1}$, where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (99)$$

The effect of the similarity transformation is equivalent to permutation of indices of the corresponding matrices: $g_{\alpha\beta} = \tilde{g}_{\pi(\alpha)\pi(\beta)}\sigma_{\alpha\beta} = \tilde{\sigma}_{\pi(\alpha)\pi(\beta)}$ with $\pi : (1,2,3,4) \to (1,3,2,4)$. After

this transformation, Eq. (98) is equivalent to

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} & \tilde{g}_{14} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} & \tilde{g}_{24} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{41} & \tilde{g}_{42} & \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix}$$

$$= \begin{pmatrix} z & -\tilde{g}_{34} & 0 & 0 \\ -\tilde{g}_{43} & \bar{z} & 0 & 0 \\ 0 & 0 & z & -\tilde{g}_{12} \\ 0 & 0 & -\tilde{g}_{21} & \bar{z} \end{pmatrix}^{-1}. \qquad (100)$$

The matrix $\tilde{g}$ is a block matrix made of 2×2 blocks. The off-diagonal blocks are zero while the diagonal ones fulfill the following equations:

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{34} \\ -\tilde{g}_{43} & \bar{z} \end{pmatrix}^{-1} \qquad (101)$$

and

$$\begin{pmatrix} \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{12} \\ -\tilde{g}_{21} & \bar{z} \end{pmatrix}^{-1}. \qquad (102)$$

The two equations admit only a symmetric solution

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix}, \qquad (103)$$

being a solution of

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{12} \\ -\tilde{g}_{21} & \bar{z} \end{pmatrix}^{-1}. \qquad (104)$$

The last equation is exactly the same as for a single Ginibre matrix Eq. (81), so the solution eventually reads

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} & \tilde{g}_{14} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} & \tilde{g}_{24} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{41} & \tilde{g}_{42} & \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix} = \begin{pmatrix} g & \gamma & 0 & 0 \\ -\bar{\gamma} & \bar{g} & 0 & 0 \\ 0 & 0 & g & \gamma \\ 0 & 0 & -\bar{\gamma} & \bar{g} \end{pmatrix}$$

$$= \mathbb{1}_2 \otimes \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix}, \qquad (105)$$

where g and γ are given by Eqs. (83) and (84). If we permute indices back to the original order $\widehat{g} = P^{-1}\tilde{g}P$, we find

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} g & 0 & \gamma & 0 \\ 0 & g & 0 & \gamma \\ -\bar{\gamma} & 0 & \bar{g} & 0 \\ 0 & -\bar{\gamma} & 0 & \bar{g} \end{pmatrix}$$

$$= \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix} \otimes \mathbb{1}_2. \qquad (106)$$

We see that the Green function for the root matrix consists of two identical blocks equal to the Green function of a single Ginibre matrix. In other words, the Green function of the root matrix behaves exactly as a pair of copies of the Green function of a single Ginibre matrix. The eigenvalue density and the growth rate of correlations between left and right eigenvectors of this matrix are given by Eqs. (85) and (86) as

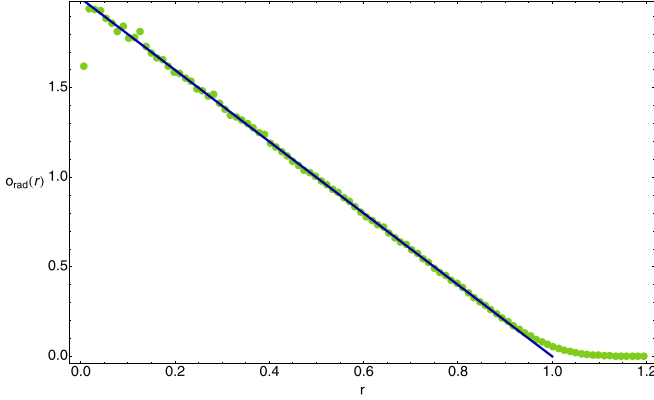$$\rho_R(z) = \frac{1}{\pi}\chi_D(z) \qquad (107)$$

FIG. 5. Triangle law: theoretical prediction for $N \to \infty$, Eq. (113), and numerical histogram (points) generated in Monte Carlo simulations for 10^5 products of two 100×100 Ginibre matrices.

and

$$o_R(z) \sim \frac{1}{\pi}(1 - |z|^2)\chi_D(z). \tag{108}$$

Note that the size of the root matrix is $2N \times 2N$, so the leading term of the overlap behaves for large N as

$$O_R(z) \sim \frac{2N}{\pi}(1 - |z|^2)\chi_D(z). \tag{109}$$

From these expressions, one may derive the corresponding expressions for R^2, which are directly related to the product $X_1 X_2$ as follows from Eq. (90). The eigenvalues of R^2 are related to those of R as $\lambda = \lambda_R^2$, so one can find the densities by the change of variables $z = w^2$: $\rho(z)d^2z = \rho_R(w)d^2w$ and $O(z)d^2z = O_R(w)d^2w$. This gives

$$\rho(z) = \frac{1}{2\pi|z|}\chi_D(z) \tag{110}$$

and

$$O(z) \sim \frac{N}{\pi|z|}(1 - |z|)\chi_D(z), \tag{111}$$

respectively. The result for the eigenvalue density $\rho(z)$ was first found in Ref. [28]. The overlap $O(z)$ is a new result. The product $X_1 X_2$ is of size $N \times N$, so the growth rate is obtained by dividing $O(z)$ by N,

$$o(z) = \lim_{N \to \infty} \frac{O(z)}{N} = \frac{1}{\pi|z|}(1 - |z|)\chi_D(z). \tag{112}$$

The radial profile is obtained from the last expression by setting $r = |z|$ and multiplying the result by $2\pi r$ [cf. Eq. (41)]. This gives a triangle law,

$$o_{\mathrm{rad}}(r) = 2(1 - r)\chi_I(r), \tag{113}$$

where χ_I is an indicator function for the interval $[0,1]$: $\chi_I(r) = 1$ for $r \in [0,1]$ and $\chi_I(r) = 0$ otherwise. This prediction is compared to Monte Carlo data for $N = 100$ in Fig. 5.

As one infers from Fig. 5, there are deviations from the limiting law for finite N. The radial profile drops to zero at the origin and develops a tail going beyond the support of the limiting profile for large r. We study the N-dependence of these effects in Fig. 6. We see that the gap at the origin closes
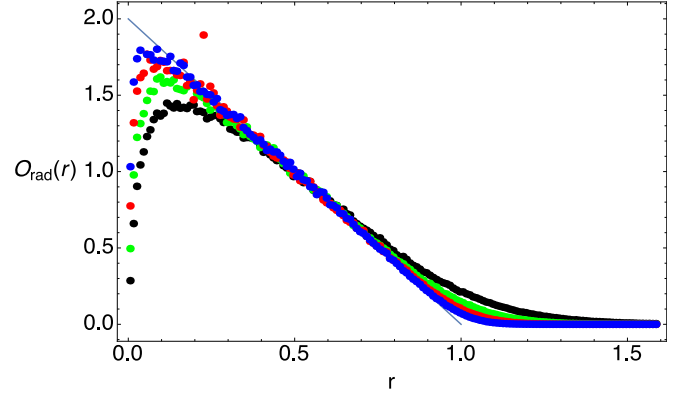

FIG. 6. Size dependence of the finite-size corrections to the triangle law for the product of two Ginibre matrices. Numerical histograms are generated in Monte Carlo simulations for $N = 10, 20, 40, 80$ (black, green, red, blue). Each histogram is produced out of 2×10^5 data points.

in a way characteristic of the hard-edge behavior, while the tail at the edge of the support gets shorter and falls off quicker as N increases. The behavior at the origin can be probably related to the microscopic behavior of the gap probabilities, which are driven by the Bessel kernel and were first studied in the context of QCD [89]. More generally, for the product of m matrices the behavior at the origin is controlled by the hypergeometric kernel [45]. In turn, the tail behavior at the soft edge is described by the error-function type of corrections [28,45].

XI. PRODUCT OF ELLIPTIC GAUSSIAN MATRICES

For completeness, we also consider the product of elliptic matrices defined by the measure [90]

$$d\mu(X) = \frac{1}{Z}\exp\left\{-\frac{1}{\sigma^2(1 - \kappa^2)} \right.$$
$$\left. \times \mathrm{Tr}\left[XX^\dagger - \frac{\kappa}{2}(XX + X^\dagger X^\dagger)\right]\right\}DX. \tag{114}$$

As before, we set $\sigma^2 = 1/N$ and scale it with N while taking the limit $N \to \infty$. The parameter κ belongs to the range $[-1,1]$. It is related to the ellipse eccentricity. For $\kappa = 0$, Eq. (114) reproduces the Ginibre measure. Generically, the support of the eigenvalue density of matrices generated according to the measure given by Eq. (114) is elliptic. When κ approaches 1 (or -1), the support flattens and in the limit $\kappa \to \pm 1$ gets completely squeezed to an interval of the real (or imaginary) axis. The corresponding matrix becomes Hermitian (or anti-Hermitian). The two-point correlations for the elliptic ensemble Eq. (114) are

$$\langle X_{ab} X_{cd}^\dagger \rangle = \langle X_{ab}^\dagger X_{cd} \rangle = \frac{1}{N}\delta_{ad}\delta_{bc} \tag{115}$$

and

$$\langle X_{ab} X_{cd} \rangle = \langle X_{ab}^\dagger X_{cd}^\dagger \rangle = \kappa\frac{1}{N}\delta_{ad}\delta_{bc}. \tag{116}$$

Consider the product $X = X_1 X_2$ of two elliptic matrices X_1 and X_2 with different eccentricity parameters κ_1 and κ_2. As in the previous section, we construct the root matrix Eq. (94),

which is a $4N \times 4N$ matrix. The propagator for the root matrix elements is

$$\widehat{P}_{AB,CD} = \widehat{p}_{\alpha\beta,\gamma\delta} \frac{1}{N} \delta_{ad} \delta_{bc}, \tag{117}$$

where $\widehat{p}_{\alpha\beta,\gamma\delta}$ has now more nonzero elements. In addition to $\widehat{p}_{12,43} = \widehat{p}_{43,12} = \widehat{p}_{21,34} = \widehat{p}_{34,21} = 1$, we have $\widehat{p}_{12,12} = \widehat{p}_{21,21} = \kappa_1$ and $\widehat{p}_{34,34} = \widehat{p}_{43,43} = \kappa_2$, which come from Eq. (116). We can now write the Dyson-Schwinger equations for this propagator. The first equation is identical as that for the product of Ginibre matrices, Eq. (96). The second one differs from the previous one, Eq. (97), since now we have additional nonzero elements coming from the eccentricity parameters κ_1 and κ_2,

$$
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44}
\end{pmatrix}
$$
$$
= \begin{pmatrix}
0 & \kappa_1 g_{21} & g_{24} & 0 \\
\kappa_1 g_{12} & 0 & 0 & g_{13} \\
g_{42} & 0 & 0 & \kappa_2 g_{43} \\
0 & g_{31} & \kappa_2 g_{34} & 0
\end{pmatrix}. \tag{118}
$$

Inserting the $\{\sigma_{\alpha\beta}\}$ into Eq. (96) and permuting indices as in the previous section, we get

$$
\begin{pmatrix}
\tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} & \tilde{g}_{14} \\
\tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} & \tilde{g}_{24} \\
\tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} & \tilde{g}_{34} \\
\tilde{g}_{41} & \tilde{g}_{42} & \tilde{g}_{43} & \tilde{g}_{44}
\end{pmatrix}
$$
$$
= \begin{pmatrix}
z & -\tilde{g}_{34} & -\kappa_1 \tilde{g}_{31} & 0 \\
-\tilde{g}_{43} & \bar{z} & 0 & -\kappa_2 \tilde{g}_{42} \\
-\kappa_1 \tilde{g}_{13} & 0 & z & -\tilde{g}_{12} \\
0 & -\kappa_2 \tilde{g}_{24} & -\tilde{g}_{21} & \bar{z}
\end{pmatrix}^{-1}. \tag{119}
$$

This equation is much more complicated than that for the product of Ginibre matrices, Eq. (105), because the two off-diagonal blocks on the right-hand side are nonzero. However, making the *ansatz* that the off-diagonal blocks of the solution vanish,

$$
\begin{pmatrix}
\tilde{g}_{13} & \tilde{g}_{14} \\
\tilde{g}_{23} & \tilde{g}_{24}
\end{pmatrix} = \begin{pmatrix}
\tilde{g}_{31} & \tilde{g}_{32} \\
\tilde{g}_{41} & \tilde{g}_{42}
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \tag{120}
$$

forces the two remaining blocks to satisfy the very same equation as for the product of Ginibre matrices, Eq. (100),

$$
\begin{pmatrix}
\tilde{g}_{11} & \tilde{g}_{12} & 0 & 0 \\
\tilde{g}_{21} & \tilde{g}_{22} & 0 & 0 \\
0 & 0 & \tilde{g}_{33} & \tilde{g}_{34} \\
0 & 0 & \tilde{g}_{43} & \tilde{g}_{44}
\end{pmatrix}
$$
$$
= \begin{pmatrix}
z & -\tilde{g}_{34} & 0 & 0 \\
-\tilde{g}_{43} & \bar{z} & 0 & 0 \\
0 & 0 & z & -\tilde{g}_{12} \\
0 & 0 & -\tilde{g}_{21} & \bar{z}
\end{pmatrix}^{-1}, \tag{121}
$$

hence the solution is the same as before. This solution is independent of the eccentricity parameters κ_1 and κ_2 and moreover it is always spherically symmetric, even though the two matrices in the product are elliptic. To summarize, in the large N limit the eigenvalue density and the left-right
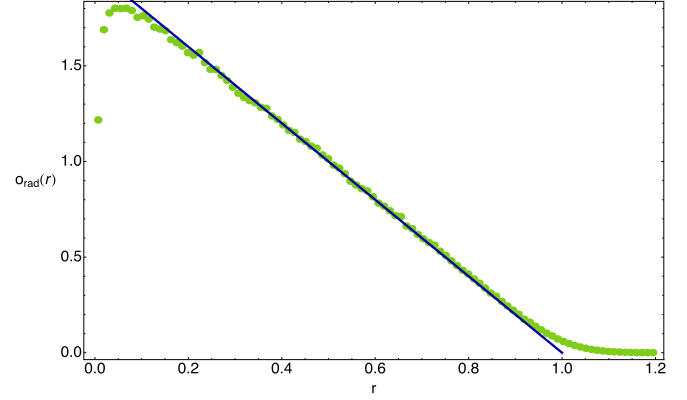

FIG. 7. Triangle law: theoretical prediction for $N \to \infty$, Eq. (113), and numerical histogram (points) generated in Monte Carlo simulations for 10^5 products of Ginibre times GUE matrices of dimensions 100×100.

eigenvector correlations for the product of two elliptic matrices are spherically symmetric, Eq. (110) [28], and the eigenvector correlations are identical as for the product of Ginibre matrices, Eq. (113). This prediction is compared to Monte Carlo data for $N = 100$ in Fig. 7. We see that it also follows the triangle law as for the product of Ginibre matrices. The finite N data exhibit, however, stronger finite-size effects as compared to those for the product of two Ginibre matrices, which manifest as a stronger deviation from the limiting density for small values of r. Compare Figs. 5 and 7. More generally, the limiting profile for $N \to \infty$ is independent of κ_1 and κ_2, while the finite-size corrections do depend on the eccentricities. We checked numerically that the overlap density for the product of elliptic matrices is isotropic (circularly invariant).

XII. PRODUCT OF *M* GINIBRE MATRICES

We now proceed analogously as in Sec. X, where we discussed the product of two Ginibre matrices in the large N limit. The integration measure for the product $X = X_1 X_2 \cdots X_m$ of m independent Ginibre matrices $X_1, X_2, \ldots, X_m$ is the product $d\mu(X_1) d\mu(X_2) \cdots d\mu(X_m)$ of the individual integration measures given by Eq. (14). In turn, the two-point correlations are given by Eq. (76),

$$\langle X_{\mu,ab} X^\dagger_{\nu,cd} \rangle = \frac{1}{N} \delta_{\mu\nu} \delta_{ad} \delta_{bc},$$

$$\langle X_{\mu,ab} X_{\nu,cd} \rangle = \langle X^\dagger_{\mu,ab} X^\dagger_{\nu,cd} \rangle = 0, \tag{122}$$

for $\mu, \nu = 1, \ldots, m$ and $a, b, c, d = 1, \ldots, N$. As in Sec. X, instead of directly applying the Green function technique to the product $X_1 X_2 \cdots X_m$, we apply it to the root matrix R being a block matrix of dimensions $mN \times mN$,

$$
R = \begin{pmatrix}
0 & X_1 & 0 & \ldots & 0 \\
0 & 0 & X_2 & \ldots & 0 \\
 & & & \ddots & \\
0 & 0 & 0 & \ldots & X_{m-1} \\
X_m & 0 & 0 & \ldots & 0
\end{pmatrix}. \tag{123}
$$

The m-th power of the root matrix,

$$R^m = \begin{pmatrix} X_1 X_2 \cdots X_m & 0 & \ldots & 0 \\ 0 & X_2 \cdots X_m X_1 & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & X_m X_1 \cdots X_{m-1} \end{pmatrix}, \tag{124}$$

reproduces m cyclic copies of the product $X_1 X_2 \cdots X_m$, which all have identical eigenvalues. The Green function for the root matrix is a $2mN \times 2mN$ block matrix,

$$\widehat{G}(z,\epsilon) = \langle (\widehat{q} \otimes \mathbb{1}_N - \widehat{R})^{-1} \rangle, \tag{125}$$

where

$$\widehat{q} = \begin{pmatrix} z\mathbb{1}_m & \epsilon\mathbb{1}_m \\ -\bar{\epsilon}\mathbb{1}_m & \bar{z}\mathbb{1}_m \end{pmatrix} \xrightarrow{\epsilon \to 0} \begin{pmatrix} z\mathbb{1}_m & 0 \\ 0 & \bar{z}\mathbb{1}_m \end{pmatrix} \tag{126}$$

and

$$\widehat{R} = \begin{pmatrix} R & 0 \\ 0 & R^\dagger \end{pmatrix} = \begin{pmatrix} 0 & X_1 & 0 & \ldots & 0 & & & & & \\ 0 & 0 & X_2 & \ldots & 0 & & & & & \\ & & \ddots & & & & & 0 & & \\ 0 & 0 & 0 & \ldots & X_{m-1} & & & & & \\ X_m & 0 & 0 & \ldots & 0 & & & & & \\ & & & & & 0 & 0 & \ldots & 0 & X_m^\dagger \\ & & & & & X_1^\dagger & 0 & \ldots & 0 & 0 \\ & & 0 & & & 0 & X_2^\dagger & \ldots & 0 & 0 \\ & & & & & & & \ddots & & \\ & & & & & 0 & 0 & \ldots & X_{m-1}^\dagger & 0 \end{pmatrix}. \tag{127}$$

The resolvent given by Eq. (125) has the standard form with $\widehat{R}$ being linear in X's. We index blocks of $\widehat{R}$ by Greek letters $\widehat{R}_{\alpha\beta}$, with $\alpha,\beta = 1,\ldots,2m$. We have the following equivalence $\widehat{R}_{\alpha,[\alpha]+1} \equiv X_\alpha$ and $\widehat{R}_{m+[\alpha]+1,m+\alpha} \equiv X_\alpha^\dagger$ for $\alpha = 1,\ldots,m$ and $[\alpha] = \alpha$ modulo m. All other blocks are zero. As follows from Eq. (122), we see that the only nonzero two-point correlations are

$$\langle R_{\alpha,[\alpha]+1} R_{m+[\alpha]+1,m+\alpha} \rangle = \langle X_\alpha X_\alpha^\dagger \rangle,$$
$$\langle R_{m+[\alpha]+1,m+\alpha} R_{\alpha,[\alpha]+1} \rangle = \langle X_\alpha^\dagger X_\alpha \rangle, \tag{128}$$

for $\alpha = 1,\ldots,m$. Thus, the propagator has the form

$$\widehat{P}_{AB,CD} = \widehat{p}_{\alpha\beta,\gamma\delta} \frac{1}{N} \delta_{ad}\delta_{bc}, \tag{129}$$

with

$$\widehat{p}_{\alpha,[\alpha]+1;m+[\alpha]+1,m+\alpha} = \widehat{p}_{m+[\alpha]+1,m+\alpha;\alpha,[\alpha]+1} = 1, \tag{130}$$

and $\widehat{p}_{\alpha\beta,\gamma\delta} = 0$, otherwise. The situation is analogous to that discussed in Sec. X, except that now there are $2m \times 2m$ blocks. The intrablock correlations are the same as before, $(1/N)\delta_{ad}\delta_{bc}$, so the solution is given as before as Kronecker product with the Kronecker δ in the intrablock indices $\widehat{G}_{AB} = \widehat{g}_{\alpha\beta}\delta_{ab}$ [cf. Eq. (79)]. The first Dyson-Schwinger Eq. (70) for the interblock elements of the Green function of the root matrix

reads for $\epsilon \to 0$

$$\begin{pmatrix} g_{1,1} & \cdots & g_{1,2m} \\ \vdots & \ddots & \vdots \\ g_{2m,1} & \cdots & g_{2m,2m} \end{pmatrix}$$

$$= \left[\begin{pmatrix} z\mathbb{1}_m & 0 \\ 0 & \bar{z}\mathbb{1}_m \end{pmatrix} - \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,2m} \\ \vdots & \ddots & \vdots \\ \sigma_{2m,1} & \cdots & \sigma_{2m,2m} \end{pmatrix} \right]^{-1}. \tag{131}$$

The second Dyson-Schwinger Eq. (70) yields

$$\sigma_{\alpha,m+\alpha} = g_{[\alpha]+1,m+[\alpha]+1}, \quad \sigma_{m+[\alpha]+1,[\alpha]+1} = g_{m+\alpha,\alpha}, \tag{132}$$

for $\alpha = 1,\ldots,m$, and $\sigma_{\alpha\beta} = 0$ for all other elements of the matrix $\widehat{\sigma}$.

The Dyson-Schwinger equations assume a simple form in a modified basis obtained by permutation of matrix indices, $\alpha \to \pi(\alpha)$, where $\pi(\alpha) = 2\alpha - 1$ and $\pi(\alpha + m) = 2\alpha$ for $\alpha = 1,\ldots,m$. We define $\widehat{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\pi(\alpha)\pi(\beta)}$ and $\widehat{g}_{\alpha\beta} = \tilde{g}_{\pi(\alpha)\pi(\beta)}$. This transformation can be alternatively viewed as a similarity transformation $\widehat{g} = P^{-1}\tilde{g}P$ and $\widehat{\sigma} = P^{-1}\tilde{\sigma}P$, where the elements of the matrix P are $P_{\alpha\beta} = \delta_{\alpha\pi(\beta)}$ and $P_{\alpha\beta}^{-1} = \delta_{\pi(\alpha)\beta}$. Clearly, $\tilde{g}$ and $\widehat{g}$ as well as $\tilde{\sigma}$ and $\widehat{\sigma}$ are unitarily equivalent. Equations (132) are equivalent to

$$\tilde{\sigma}_{2\alpha-1,2\alpha} = \tilde{g}_{(2\alpha+1),(2\alpha+2)}, \quad \sigma_{2\alpha,2\alpha-1} = \tilde{g}_{(2\alpha-2),(2\alpha-3)}, \tag{133}$$

where the function $y = (x)$ on the right-hand side maps the set of integers on the subset $\{1,2,\ldots,2m\}$ in the following

way. Any integer x can be decomposed uniquely as $x = y + 2mk$, where $y \in \{1,2,\ldots,2m\}$ and k is an integer. The function (x) selects y from this decomposition.

$$
\begin{pmatrix}
\tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} & \tilde{g}_{14} & \cdots & \cdots & \cdots \\
\tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} & \tilde{g}_{24} & \cdots & \cdots & \cdots \\
\tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} & \tilde{g}_{34} & \cdots & \cdots & \cdots \\
\tilde{g}_{41} & \tilde{g}_{42} & \tilde{g}_{43} & \tilde{g}_{44} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \tilde{g}_{2m-1,2m-1} & \tilde{g}_{2m,2m-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \tilde{g}_{2m-1,2m} & \tilde{g}_{2m,2m}
\end{pmatrix}
=
\begin{pmatrix}
z & -\tilde{g}_{34} & 0 & 0 & \cdots & 0 & 0 \\
-\tilde{g}_{2m,2m-1} & \bar{z} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & z & -\tilde{g}_{45} & \cdots & 0 & 0 \\
0 & 0 & -\tilde{g}_{12} & \bar{z} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & z & -\tilde{g}_{12} \\
0 & 0 & 0 & 0 & 0 & -\tilde{g}_{2m-2,2m-3} & \bar{z}
\end{pmatrix}^{-1}.
$$
(134)

The matrix $\tilde{g}$ can be viewed as a block matrix made of 2×2 blocks. The off-diagonal blocks are zero and the diagonal ones fulfill the following equations:

$$
\begin{pmatrix}
\tilde{g}_{2\alpha-1,2\alpha-1} & \tilde{g}_{2\alpha-1,2\alpha} \\
\tilde{g}_{2\alpha,2\alpha-1} & \tilde{g}_{2\alpha,2\alpha}
\end{pmatrix}
$$
$$
= \begin{pmatrix}
z & -\tilde{g}_{(2\alpha+1),(2\alpha+2)} \\
-\tilde{g}_{(2\alpha-2),(2\alpha-3)} & \bar{z}
\end{pmatrix}^{-1}, \quad (135)
$$

for $\alpha = 1, \ldots, m$. Making the *ansatz* that the solution should be symmetric—that is $\tilde{g}_{2\alpha-1,2\alpha-1} = g$, $\tilde{g}_{2\alpha,2\alpha} = \bar{g}$, $\tilde{g}_{2\alpha-1,2\alpha} = \gamma$, and $\tilde{g}_{2\alpha,2\alpha-1} = -\bar{\gamma}$, for all $\alpha = 1,\ldots,m$, the last equations reduce to a single one,

$$
\begin{pmatrix}
g & \gamma \\
-\bar{\gamma} & \bar{g}
\end{pmatrix}
=
\begin{pmatrix}
z & -\gamma \\
\bar{\gamma} & \bar{z}
\end{pmatrix}^{-1}, \quad (136)
$$

which is identical as that for a single Ginibre matrix, Eq. (82). Hence, the solution for γ and g is given by Eqs. (83) and (84). This *ansatz* is equivalent to the one we used for $m = 2$ and merely means that the solution should not break the symmetry between different cyclic permutations of Ginibre matrices in the product. Inserting the solution into $\tilde{g}$, we find

$$
\tilde{g} = \mathbb{1}_m \otimes \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix}, \quad (137)
$$

where g and γ are given by Eqs. (83) and (84). Permuting indices back to the original order, $\widehat{g} = P\tilde{g}P^{-1}$,

$$
\widehat{g} = \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix} \otimes \mathbb{1}_m. \quad (138)
$$

Hence, we see that the Green function of the root matrix behaves as m copies of the Green function of a single Ginibre matrix. The eigenvalue density and the growth rate of correlations between left and right eigenvectors of this matrix are identical as Eqs. (85) and (86), namely

$$
\rho_R(z) = \frac{1}{\pi}\chi_D(z) \quad (139)
$$

and

$$
o_R(z) = \frac{1}{\pi}(1 - |z|^2)\chi_D(z). \quad (140)
$$

In particular, $(x) = x$ and $(2m+1) = 1$, $(2m+2) = 2$, $(0) = 2m$, $(-1) = 2m - 1$. Eliminating $\tilde{\sigma}$'s from the Dyson-Schwinger equations, we obtain a compact equation for $\tilde{g}$'s,

The leading term of the overlap is, therefore,

$$
O_R(z) \sim \frac{mN}{\pi}(1 - |z|^2)\chi_D(z). \quad (141)
$$

The eigenvalues λ of R^m are related to those of R as $\lambda = \lambda_R^m$, so by changing variables as $z = w^m$ we can find the corresponding distributions for R^m: $\rho(z)d^2z = \rho_R(w)d^2w$ and $O(z)d^2z = O_R(w)d^2w$. This gives

$$
\rho(z) = \frac{1}{m\pi}|z|^{\frac{2}{m}-2}\chi_D(z) \quad (142)
$$

and

$$
O(z) \sim \frac{N}{\pi}|z|^{\frac{2}{m}-2}\left(1 - |z|^{\frac{2}{m}}\right)\chi_D(z), \quad (143)
$$

respectively. Thus, for large N the growth rate of the overlap for the product $X_1 X_2 \cdots X_m$ is

$$
o(z) = \lim_{N\to\infty}\frac{O(z)}{N} = \frac{1}{\pi}|z|^{\frac{2}{m}-2}\left(1 - |z|^{\frac{2}{m}}\right)\chi_D(z). \quad (144)
$$

The radial profile defined by Eq. (41) is

$$
o_{\text{rad}}(r) = 2r^{\frac{2}{m}-1}\left(1 - r^{\frac{2}{m}}\right)\chi_I(r), \quad (145)
$$

where as before χ_D is the indicator function for the unit disk $|z| \leqslant 1$ and χ_I for the interval $[0,1]$. While finalizing the manuscript, we learned that this result was derived independently in Ref. [91] with the aid of an extension of the Haagerup-Larsen theorem [92,93]. The integrated growth rate is

$$
\int o_{\text{rad}}(r)dr = \frac{m}{2}, \quad (146)
$$

which means that for large N the overlap grows as $O \sim mN/2$ in agreement with Eq. (59). In Fig. 8, we plot the expression given by Eq. (145) for $m = 4$ and compare it to Monte Carlo data for $N = 100$.

So far we have discussed the product of m Ginibre matrices. We could repeat the whole discussion from this section for the product of elliptic matrices, Eq. (114), with arbitrary eccentricity parameters $\kappa_1, \kappa_2, \ldots, \kappa_m$. We would then arrive at an equation for $\tilde{g}$ like Eq. (134) except that the matrix on the right-hand side would now have nonzero nondiagonal 2×2 blocks. These blocks would be made of elements of nondiagonal blocks of $\tilde{g}$ multiplied in some way by κ's. Adopting the ansatz from Sec. XI that all off-diagonal blocks
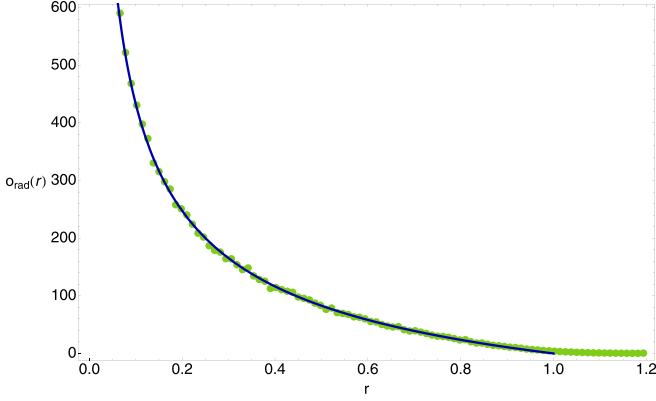
FIG. 8. Limiting overlap density for $m = 4$: theoretical prediction for $N \to \infty$, Eq. (145), and numerical histogram (points) generated in Monte Carlo simulations for 10^5 products of four 100×100 Ginibre matrices.

of $\tilde{g}$ are equal to zero we would reduce this equation to Eq. (134) and get the same result as for the product of m Ginibre matrices.

XIII. CONCLUSIONS

In this paper, we have studied macroscopic and microscopic eigenvector statistics of the product of Ginibre matrices. We have developed analytical methods to calculate the left-right eigenvector overlap for finite N and in the limit $N \to \infty$. The overlap is not only an interesting object from the mathematical point of view but is also of interest for physical problems. In the physics literature, it is known as Petermann factor and is, for example, used as a measure of nonorthogonality of cavity modes in chaotic scattering [94,95]. The off-diagonal overlap has been recently used as a sensitive indicator of nonorthogonality occurring in open systems due to perturbations resulting from shifts of resonance widths [96,97]. It plays also an important role in the description of Dysonian diffusion for non-Hermitian random matrices [98,99].

There are many open problems and potential generalizations of the studies presented in this paper. For example, one may try to extend the studies of the microscopic eigenvector statistics to products of truncated unitary matrices [100], which can also be mapped onto a determinantal point process [51] via generalized Schur decomposition [45]. A great challenge is to determine the microscopic eigenvalue and eigenvector statistics for products of elliptic matrices or to find any nontrivial solvable example of products of random matrices having nonspherical measures.

We have considered complex random matrices here. It would also be interesting to study overlaps for products of real

and quaternionic matrices. They are much more challenging since in these cases the microscopic correlations are driven by Pfaffian point processes rather than determinantal ones. The real and quaternionic ensembles have additional scaling regimes near the real axis, which introduce an additional complication. Moreover, the Schur decomposition, which is at the heart of the method used in this paper, cannot be applied in a straightforward way to real matrices since generically they are not orthogonally similar to upper triangular ones. On the other hand, we believe that the limiting laws for $N \to \infty$ are identical for real and complex ensembles since the underlying Dyson-Schwinger equations are identical in the planar limit ($N \to \infty$).

Concerning the large N limit and macroscopic statistics, it would be interesting to generalize the calculations of the overlap to polynomials of random matrices [37,40,42] and to go beyond isotropic (R-diagonal) matrices [91,93], as well as to better understand the overlap in terms of the quaternionic formalism [44], and finally to calculate the off-diagonal elements of the overlap Eq. (7) using the Bethe-Salpeter equation [79].

APPENDIX: CALCULATION OF THE INTEGRAL EQ. (57)

In this Appendix, we detail the calculation of the integral given by Eq. (57),

$$O = \frac{1}{Z} \int \prod_{\alpha=1}^{N-1} \left(1 + \frac{1}{|\lambda_N - \lambda_\alpha|^2}\right) |\Delta_N(\boldsymbol{\lambda})|^2 \prod_{\alpha=1}^{N} e^{-|\lambda_\alpha|^2} d^2\lambda_\alpha,$$

(A1)

where we have renamed the Vandermonde determinant on N complex variables as $\Delta_N(\boldsymbol{\lambda})$ for convenience.

We can rewrite this as

$$O = \frac{1}{Z} \int \prod_{\alpha=1}^{N-1} \left(\frac{|\lambda_N - \lambda_\alpha|^2 + 1}{|\lambda_N - \lambda_\alpha|^2}\right) |\Delta_N(\boldsymbol{\lambda})|^2 \prod_{\alpha=1}^{N} e^{-|\lambda_\alpha|^2} d^2\lambda_\alpha$$

$$= \frac{1}{Z} \int \prod_{\alpha=1}^{N-1} (|\lambda_N - \lambda_\alpha|^2 + 1) |\Delta_{N-1}(\boldsymbol{\lambda})|^2 \prod_{\alpha=1}^{N} e^{-|\lambda_\alpha|^2} d^2\lambda_\alpha,$$

(A2)

which can be more compactly expressed as

$$O = \frac{1}{Z}(N-1)! \int d^2\lambda_N e^{-|\lambda_N|^2} \det \left(\underbrace{\int d^2z \, e^{-|z|^2} z^{j-1} \bar{z}^{k-1}(|\lambda_N - z|^2 + 1)}_{I_{jk}(\lambda_N)}\right)_{j,k=1,\dots,N-1},$$

(A3)

using the complex version of the Andréief identity [101]. The integral over z yields

$$I_{jk}(\lambda) = \pi[(|\lambda|^2 + 1)(k-1)! + k!]\delta_{j,k} - \pi k!\lambda\delta_{j-1,k}$$
$$- \pi(k-1)!\bar{\lambda}\delta_{j+1,k}. \quad \text{(A4)}$$

This is a tridiagonal matrix. When calculating its determinant, $I_{N-1}(\lambda) = \det[I_{jk}(\lambda)]_{j,k=1,\ldots,N-1}$, it is convenient to pull out a common factor from each column of the matrix,

$$I_{jk}(\lambda) = \pi(k-1)!D_{jk}(\lambda), \quad \text{(A5)}$$

where

$$D_{jk}(\lambda) = (|\lambda|^2 + 1 + k)\delta_{j,k} - k\lambda\delta_{j-1,k} - \bar{\lambda}\delta_{j+1,k}. \quad \text{(A6)}$$

The determinant $I_{N-1}(\lambda)$ can be related to the determinant $D_{N-1}(\lambda) = \det[D_{jk}(\lambda)]_{j,k=1,\ldots,N-1}$ as follows:

$$I_{N-1}(\lambda) = \pi^{N-1}0!1!\cdots(N-2)!D_{N-1}(\lambda). \quad \text{(A7)}$$

Thus, we can rewrite Eq. (A3) as

$$O = \frac{1}{\pi N!}\int d^2\lambda e^{-|\lambda|^2}D_{N-1}(\lambda), \quad \text{(A8)}$$

where we have also replaced the normalization constant by the explicit expression $Z = \pi^N 1!2!\cdots N!$ [cf. Eq. (24)]. It remains to find the determinant $D_n(\lambda)$ for $n = N - 1$. It has the form

$$D_n = \begin{vmatrix} a_1 & b_1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ 0 & & c_{n-1} & a_n \end{vmatrix}, \quad \text{(A9)}$$

with $a_n = |\lambda|^2 + 1 + n$, $b_n = -n\lambda$, $c_n = -\bar{\lambda}$. In general, the sequence $\{D_n\}$ is called *continuant* and satisfies the following

recurrence relation:

$$D_n = a_n D_{n-1} - b_{n-1}c_{n-1}D_{n-2},$$

with initial conditions $D_0 = 1$ and $D_1 = a_1$. In our case the recurrence takes the form

$$D_n = (|\lambda|^2 + 1 + n)D_{n-1} - (n-1)|\lambda|^2 D_{n-2}. \quad \text{(A10)}$$

The sequence $\{D_n\}$ reveals an interesting pattern for small n, which allows us to conjecture that D_n is given in closed form by

$$D_n(\lambda) = \sum_{k=0}^{n} \frac{n!(n+1-k)}{k!}|\lambda|^{2k}. \quad \text{(A11)}$$

One can check by straightforward algebraic manipulations that this polynomial indeed fulfills the recurrence relation Eq. (A10). The Gaussian integral of this polynomial gives a simple result:

$$\int d^2\lambda e^{-|\lambda|^2}D_n(\lambda) = \pi \sum_{k=0}^{n} n!(n+1-k)$$
$$= \pi n! \frac{(n+1)(n+2)}{2}$$
$$= \pi(n+1)!\left(1 + \frac{n}{2}\right), \quad \text{(A12)}$$

which for $n = N - 1$, using Eq. (A8), leads to

$$O = 1 + \tfrac{1}{2}(N-1), \quad \text{(A13)}$$

as claimed in Eq. (58).

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