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Tsallis thermostatics as a statistical physics of random chains

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In this paper we point out that the generalized statistics of Tsallis-Havrda-Charvát can be conveniently used as a conceptual framework for statistical treatment of random chains. In particular, we use the path-integral approach to show that the ensuing partition function can be identified with the partition function of a fluctuating oriented random loop of arbitrary length and shape in a background scalar potential. To put some meat on the bare bones, we illustrate this with two statistical systems: Schultz-Zimm polymer and relativistic particle. Further salient issues such as the projective special linear group PSL(2,$\mathbb{R}$) transformation properties of Tsallis' inverse-temperature parameter and a grand-canonical ensemble of fluctuating random loops related to the Tsallis-Havrda-Charvát statistics are also briefly discussed.

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I. INTRODUCTION

Over the past two decades, Boltzmann-Gibbs (BG) statistical mechanics has undergone an important conceptual shift. While successful in describing stationary systems characterized by ergodicity or metric transitivity, it fails to reproduce the statistical behavior of many real-world systems in biology, astrophysics, geology, and the economic and social sciences. It is symptomatic of such cases that one tries to find refuge in a new paradigm known as "generalized statistics." The notion of generalized statistics refers to statistical systems that are described with broad (or fat-tail) distributions for which the usual central limit theorem (CLT) [1] is inapplicable. Examples include generalized hyperbolic distributions, Meixner distributions, Weibull distributions, and various power-law tail distributions (e.g., Zipf-Pareto, Lévy, Mandelbrot, or Student t distributions). The underlying mathematical foundations and terminology involved are, as a rule, provided by various generalized central limit theorems (GCLTs); be it the CLT of Lévy [1–4] and Gnedenko [5–8] for non-Gaussian stable distributions or diverse CLTs for correlated random variables [9]. In effect, GCLTs represent pertinent frameworks incorporating such crucial theoretical concepts as Lévy stable distributions and ensuing Lévy stochastic processes [3], information-theoretic systems of Rényi [10,11], nonextensive systems of Tsallis-Havrda-Charvát (THC) [12–16], and their various generalizations [9,17,18]. Associated real-world phenomena obeying generalized statistics account for a rich class of statistical processes observed in complex systems ranging from financial markets [3,19,20], physics [9,21], and biology [22,23] to geoscience [24,25].

The aim of this paper is to point out that with the help of the path integral (PI) one can identify interesting new playgrounds for THC generalized statistics. In particular, the statistics in question is represented by distributions that emerge when the maximal-entropy (MaxEnt) prescription is applied to Tsallis and Havrda-Charvát information measures (or entropies). For the sake of comparison, we discuss also the MaxEnt of (closely related) Rényi's entropy. Regardless how narrow this class of distributions may seem, it constitutes an immense wealth of real-world systems ranging from low-dimensional (at the edge of chaos) [26] and high-dimensional (self-organized criticality) [27] nonlinear dissipative systems through well-developed turbulence [28] to long-range magnetic and fluidlike systems [29]. Here we wish to shed yet more light on THC generalized statistics by showing that it also represents a pertinent framework for the statistical physics of random chains.

Our paper is organized as follows: In Sec. II we present some essentials for both Rényi and THC statistics that are needed in the paper. In Sec. III, we reveal and discuss the group structure of the THC "inverse temperature" parameter. The group in question is the Möbius parabolic group, which is a one-parametric subgroup of the projective special linear group PSL(2,$\mathbb{R}$). For the generalized statistics in question we formulate the relevant PI representation for both the density matrix and the partition function. This is done in Sec. IV with the help of Schwinger's trick. In Sec. V we show that the representation of the density matrix obtained naturally arises in the statistical theory of random chains. We illustrate our point with the Schultz-Zimm polymer. Important representatives of random chains are fluctuating particle histories as encountered, for instance, in quantum mechanics or quantum field theory (QFT). Remarkably, when we apply the generalized density matrix obtained to a free *nonrelativistic* particle in D spatial dimensions, we find that it is equivalent to the (canonical Bloch) density matrix for a free *relativistic* particle in D space-time dimensions provided we set $q = 2$ or $q = 0$. This fact is proved in Sec. VI, where also a generalization to a field-theoretical context is discussed. Finally, Sec. VII summarizes our results and discusses possible extensions of the present work. For the reader's convenience the paper is supplemented with two appendixes which clarify some finer technical details. In Appendix A we derive the Schulz-Zimm distribution of chain lengths for relevant parameter values and

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in Appendix B we show how the generalized density matrix obtained can be seamlessly fitted into a computation of the one-loop contribution to the Gibbs free energy in the case of scalar quantum electrodynamics.

II. SOME FUNDAMENTALS OF RÉNYI'S AND THC STATISTICS

A useful conceptual frame that allows us to generate important classes of observed distributions is based on information entropies. Information entropies generally represent measures of uncertainty or ignorance inherent in a distribution describing a given statistical or information-theoretical system. The central role of information entropies is to serve as inference functionals whose extremization subject to certain constraint conditions (known as prior information) yields MaxEnt distributions which often have direct phenomenological relevance. The importance of information entropies as tools for inductive inference (i.e., inference where new information is given in terms of expected values) in statistical physics was emphasized by many authors (see, e.g., references in [30]), and presently MaxEnt approaches belong among the standard techniques from the statistical physics toolkit [9].

Among the many possible information entropies we focus our attention here on two specific cases: first, on Rényi's entropy [10], defined as

$$\mathcal{S}_q^{(R)} = \frac{1}{1-q} \ln \sum_i p_i^q\,, \quad q > 0; \tag{1}$$

and second, on THC entropy [12,13], which has the form

$$\mathcal{S}_q^{(\text{THC})} = \frac{1}{1-q}\left(\sum_i p_i^q - 1\right), \quad q > 0. \tag{2}$$

A discrete distribution $\mathcal{P} = \{p_i\}$ is usually associated with a discrete set of microstates in statistical physics or a set of all transmittable source symbols in information theory. In the limit $q \to 1$, the two entropies coincide with each other, both reducing to the Shannon (or Shannon-Gibbs) entropy

$$S = -\sum_i p_i \ln p_i\,. \tag{3}$$

In this way the parameter q [or, better, $(q-1)$] characterizes the departure from the usual BG statistics or from Shannon's information theory.

Let us remember that in the context of Shannon's information theory it is well known [31] that the laws of equilibrium statistical mechanics can be viewed as *inferences* based entirely on prior information that is given in terms of expectation values of energy, energy and number of particles, energy and volume, energy and angular momentum, etc. In this case Shannon's entropy quantifies the information on the detailed microscopic state (microstate) of the system, which remains "uncommunicated" by a description that is stated solely in terms of thermodynamic state variables (phrased in terms of expectation values). It should also be stressed that the passage from Shannon-Gibbs to Clausius (i.e., thermodynamic) entropy is established only when the relevant MaxEnt distribution is inserted back into S. Only when this MaxEnt prescription is utilized does S turn out to be a thermodynamic *state function* and not a mere functional in a probability space.

In the spirit of the MaxEnt strategy one can formally repeat the aforementioned philosophy also for $\mathcal{S}_q^{(R)}$ and $\mathcal{S}_q^{(\text{THC})}$. It is still an open question, however, to what extent the inferences obtained can be identified with some genuine statistical system. Here we do not wish to dispute the usefulness of ensuing MaxEnt distributions, which clearly serve as excellent fitting distributions in a number of contexts in complex dynamical systems [9,20,21]. Instead, we wish to point out that there are circumstances (discussed shortly) where the ensuing generalized statistical systems (and not just the MaxEnt distributions) can be clearly identified with the real-world equilibrium statistical systems.

For the sake of simplicity we proceed here only with the analog of canonical ensembles, where the prior information is characterized by a fixed energy expectation value (i.e., the internal energy). The corresponding MaxEnt distributions for $\mathcal{S}_q^{(R)}$ and $\mathcal{S}_q^{(\text{THC})}$ can be obtained by extremization of the associated inference function,

$$L_q^{(R;\text{THC})}(\mathcal{P}) = \mathcal{S}_q^{(R;\text{THC})} - \alpha \sum_i p_i - \beta \langle H \rangle_r\,, \tag{4}$$

where α and β are the Lagrange multipliers, the latter being the analog of the inverse temperature in natural units. The subscript r denotes the expectation value with respect to the weights $P_i(r) \equiv p_i^r / \sum_j p_j^r$ so that

$$\langle H \rangle_r = \sum_i P_i(r) E_i\,. \tag{5}$$

In practice it is common to choose only two values of r that represent two different modes of application [32]. In information theory one typically uses the linear mean $\langle H \rangle_1 = \sum_i p_i E_i$, which corresponds to $r = 1$, while in nonextensive thermodynamics it is customary to utilize a nonlinear q mean $\langle H \rangle_q$ which represents $r = q$. In the latter case $P_i(q)$ is called the *escort* or *zooming* distribution—terminology that has its origin in chaotic dynamics [33].

In principle, we could keep the value of r general when employing the MaxEnt procedure. If this is done, say for Rényi's entropy, then the MaxEnt distribution arises from the condition $\delta L_q^{(R)}(\mathcal{P})/\delta p_i = 0$. For generic r this leads to a higher-order trinomial equation for p_k, namely,

$$a X_k^{(q-1)/(r-1)} - b_k X_k - 1 = 0, \tag{6}$$

when $r \neq 1$. In the special case where $r = 1$ we have

$$a X_k - b_k - 1 = 0. \tag{7}$$

Here, $X_k = p_k^{r-1}$ in Eq. (6) and $X_k = p_k^{q-1}$ in Eq. (7). Parameters involved read

$$a = \frac{1}{Z_q} \equiv \frac{1}{\sum_l p_l^q}\,,$$

$$b_k = \beta \frac{r\,(1-q)[E_k - \langle H \rangle_r]}{q Z_r}\,. \tag{8}$$

Equations (6) are generally not suitable for physical considerations because they do not provide *unique* and *real* solutions. There are only two cases in which the equations obtained are linear and thus yield a unique p_k, namely, when $r = 1$ and

$r = q$. We note in passing that for special values of r and q a solution of (6) of the form

$$p_k = p_k(Z_r, Z_q, \beta, \langle H\rangle_r; E_k) \tag{9}$$

can be found. With two constraints at hand we could eliminate Z_q or Z_r and rewrite β in terms of $\langle H\rangle_r$ (or, more physically, $\langle H\rangle_r$ in terms of β). The resulting p_k would be, however, badly self-referential because either Z_r or Z_q would not be eliminated. Again, cases $r = 1$ and $r = q$ play a special role here since only in these two cases will the vexing self-referentiality be eliminated. This in turn provides a mathematical (although not physical) backing for the two aforementioned choices of r used in the literature. With this proviso the associated MaxEnt distribution for Rényi's entropy reads

$$p_k^R = Z_R^{-1}\left[1 - \beta_r^R(1+q-2r)\Delta_r E_k\right]^{1/(1+q-2r)}. \tag{10}$$

Here $\Delta_r E_k = E_k - \langle H\rangle_r$, and Z_R is the normalization constant (basically the partition function). The term $\beta_r^R = \beta r/q$ is the inverse "temperature" of the system. By the same token one obtains for the THC case

$$p_k^{\text{THC}} = Z_{\text{THC}}^{-1}\left[1 - \beta_r^{\text{THC}}(1+q-2r)\Delta_r E_k\right]^{1/(1+q-2r)}, \tag{11}$$

with the inverse temperature $\beta_r^{\text{THC}} = \beta_r^R / \sum_i (p_i^{\text{THC}})^q$. Similarly as before, this holds only when $r = 1$ or $r = q$. So, in contrast to (10), the THC MaxEnt distributions are self-referential, i.e., β_r^{THC} depends on the properties of the distribution itself. See, e.g., Refs. [34–37] for related discussions. Distributions of the form (10) and (12) are known as $(2r - q)$ Gaussian or Tsallis (thermostatistics) distributions and they indeed appear in one form or another in numerous real-world statistical systems [13]. For historical reasons the case $r = 1$ in Eq. (12) is also known as Bashkirov's first version of thermostatistics, while the case $r = q$ is called Tsallis' third version of thermostatistics [38].

For future convenience we note that the MaxEnt distributions (10) and (12) can be viewed as $\mathfrak{q}$-deformed versions of the usual Gibbs-Boltzmann statistical distributions. For instance, using the Box-Cox $\mathfrak{q}$ exponential [39], i.e.,

$$e_{\{\mathfrak{q}\}}^{x} \equiv [1 + (1-\mathfrak{q})x]^{1/(1-\mathfrak{q})}, \tag{12}$$

the resulting $(2r - q)$ Gaussian distributions can be expressed in the succinct form

$$p(E_i) = Z^{-1} e_{\{2r-q\}}^{-\beta E_i}. \tag{13}$$

Here β represents the corresponding inverse temperature and Z is the normalization. Since $e_{\{1\}}^x = e^x$, it is clear that Tsallis' thermostatistics distribution approaches in the limit $q \to 2r - 1$ the standard Maxwell-Boltzmann distribution of equilibrium statistical thermodynamics.

III. MORE ABOUT β_r^R AND β_r^{THC}

Though distributions (10) and (12) have almost the same form, the main difference is in the self-referentiality of β_r^{THC}. We now illustrate that this subtle fact might have nontrivial (in fact measurable) consequences. To this end we first recall the notable fact from the Boltzmann-Gibbs statistics, namely, that the MaxEnt distribution is invariant under a constant energy shift, $E_i \to E_i + \Delta$. Actually, this result is not the consequence of a particular form of the BG entropy as one could think, but rather it directly results from a Legendre-transform structure and linear form of constraints [40]. With this proviso, the invariance under a constant energy shift is independent of the specific form of entropy and is also valid for q-escort constraints. This fact can also be directly observed in distributions (10) and (12), where the shift factor Δ does not appear in the spectrum-shifted distribution because the term $(E_i - \langle H\rangle_r)$ is manifestly Δ independent:

$$(E_i - \langle H\rangle_r) \xrightarrow{+\Delta} [(E_i + \Delta) - \langle H + \Delta\rangle_r] = (E_i - \langle H\rangle_r). \tag{14}$$

Second, in the case of the BG distribution we can observe yet another important property, namely, that it does not depend on $\langle H\rangle \equiv \langle H\rangle_1$ *explicitly*. This follows from a simple factorization property:

$$p_i^{\text{BG}} = \frac{\exp[-\beta(E_i - \langle H\rangle)]}{\sum_j \exp[-\beta(E_j - \langle H\rangle)]} = \frac{\exp(-\beta E_i)}{\sum_j \exp(-\beta E_j)}. \tag{15}$$

We recall that the dependence on $\langle H\rangle$ is *implicitly* contained in β, because the distribution still has to fulfill the original constraint [41]

$$\sum E_i \; p_i^{\text{BG}}(\beta, E_i) = \langle H\rangle, \tag{16}$$

which (in principle) allows us to resolve β in terms of $\langle H\rangle$. Let us remember that in the BG statistics it is important to have the partition function without an explicit dependence on $\langle H\rangle$. Explicit $\langle H\rangle$ dependence would, for instance, obscure the connection between the partition function and the Helmholtz free energy. In addition, by combining the constant-energy shift invariance of p_i^{BG} with its lack of an explicit $\langle H\rangle$ dependence, one easily obtains the experimentally supported fact that the zero-point energy has no effect on the BG statistical thermodynamics and thus can be safely set to 0.

The question arises, to what extent is the *desired* absence of an explicit $\langle H\rangle$ in p_i^{BG} inherited by distributions (10) and (12). Contrary to the case with the constant-energy shift $E_i \to E_i + \Delta$, the factorization-like property for $\langle H\rangle_r$ is less trivial in Eqs. (10) and (12) and in fact it can be achieved only in special circumstances.

To see what is involved, let us assume that the second MaxEnt constraint, (5), is not yet enforced, so that both the Lagrange multiplier β and $\langle H\rangle_r$ are still independent. Within this framework any $\langle H\rangle_r$ appearing in the considered distributions is explicit and the required lack of explicit dependence on $\langle H\rangle_r$ can be formulated as a form invariance of the distribution under the shift of $\langle H\rangle_r$. The MaxEnt constraint can then be safely imposed at the very end of our reasonings.

For p_i^{BG} the form invariance under the shift

$$\langle H\rangle \to \widetilde{\langle H\rangle} = \langle H\rangle + \mathfrak{b} \tag{17}$$

(with $\mathfrak{b} \in \mathbb{R}$) is a simple consequence of the identity

$$\begin{aligned} p_i^{\text{BG}} &= \frac{\exp[-\beta(E_i - \langle H\rangle)]}{\sum_j \exp[-\beta(E_j - \langle H\rangle)]} \\ &= \frac{\exp[-\beta(E_i - \widetilde{\langle H\rangle})]}{\sum_j \exp[-\beta(E_j - \widetilde{\langle H\rangle})]}. \end{aligned} \tag{18}$$

In this case β is manifestly invariant under the shift represented by Eq. (17).

For THC MaxEnt distributions, (12), we might recall that $\beta_r^{\mathrm{THC}} = \beta_r^R / Z_{\mathrm{THC}}(\beta, q, r, \langle H\rangle_r) = \beta_r^{\mathrm{THC}}(\beta, q, r, \langle H\rangle_r)$. Since we ultimately consider what happens under $\langle H\rangle_r$ translations, we can write only $\beta_r^{\mathrm{THC}} = \beta_r^{\mathrm{THC}}(\langle H\rangle_r)$. For further convenience we denote the shifted mean value $\langle \widetilde{H}\rangle_r = \langle H\rangle_r + \mathfrak{b}$ simply as $\mathfrak{b}$. We also set $\beta_r^{\mathrm{THC}} = \beta$ and $Z_{\mathrm{THC}} = Z$.

In the case of p_k^{THC}, the form invariance under arbitrary shift in $\langle H\rangle_r$ can be achieved by compensating for the shift by the appropriately redefined/transformed form of β_r^{THC}. Although this fact is known (cf., e.g., Ref. [34]), it deserves further qualifications. Our subsequent derivation thus follows a route less traveled but more suitable to our needs.

The aforementioned transformation properties of β_r^{THC} can be read off from the identity

$$\begin{aligned}&[1 - \beta(\mathfrak{b}_1)(1+q-2r)(E_i + \mathfrak{b}_1)]\\ &\quad = [1 - \beta(\mathfrak{b}_2)(1+q-2r)(E_i + \mathfrak{b}_2)]c(\mathfrak{b}_1,\mathfrak{b}_2;q,r),\end{aligned} \tag{19}$$

where $c(\mathfrak{b}_1,\mathfrak{b}_2;q,r)$ is an energy-spectrum-independent constant. From Eq. (19) $c(\mathfrak{b}_1,\mathfrak{b}_2;q,r)$ must fulfill two simultaneous equations:

$$\begin{aligned}&c(\mathfrak{b}_1,\mathfrak{b}_2;q,r) = \beta(\mathfrak{b}_1)/\beta(\mathfrak{b}_2),\\ &c(\mathfrak{b}_1,\mathfrak{b}_2;q,r) = 1 - \beta(\mathfrak{b}_1)(1+q-2r)(\mathfrak{b}_1 - \mathfrak{b}_2).\end{aligned} \tag{20}$$

This implies that (19) is satisfied only when "inverse temperatures" with different $\mathfrak{b}$ values obey the *linear fractional transformations*

$$\begin{aligned}\beta(\mathfrak{b}_2) &= \frac{\beta(\mathfrak{b}_1)}{1 - \beta(\mathfrak{b}_1)(1+q-2r)(\mathfrak{b}_1 - \mathfrak{b}_2)},\\ \beta(\mathfrak{b}_1) &= \frac{\beta(\mathfrak{b}_2)}{1 - \beta(\mathfrak{b}_2)(1+q-2r)(\mathfrak{b}_2 - \mathfrak{b}_1)}.\end{aligned} \tag{21}$$

Equation (19) together with Eq. (21) implies that the THC MaxEnt distribution is (form) invariant under the shift of $\langle H\rangle_r$, namely,

$$\begin{aligned}&\frac{(1 - \beta(\mathfrak{b}_1)(1+q-2r)(E_i + \mathfrak{b}_1))^{1/(1+q-2r)}}{Z(\beta(\mathfrak{b}_1))}\\ &\quad = \frac{(1 - \beta(\mathfrak{b}_2)(1+q-2r)(E_i + \mathfrak{b}_2))^{1/(1+q-2r)}}{Z(\beta(\mathfrak{b}_2))}.\end{aligned} \tag{22}$$

Within this framework we can choose to work directly with E_i rather than with ΔE_i.

Let us remark that an analogous line of thoughts does not hold for the Rényi MaxEnt distributions, (10). Indeed, consider, for instance, the Rényian analog of Eq. (21) and set $\mathfrak{b}_1 = \langle H\rangle_r$. In this case we have

$$\beta_r^R = \frac{\beta_r^R(\mathfrak{b}_2)}{1 - \beta_r^R(\mathfrak{b}_2)(1+q-2r)(\mathfrak{b}_2 - \mathfrak{b}_1)}. \tag{23}$$

By assuming that $\beta_r^R(\mathfrak{b}_2)$ is $\mathfrak{b}_1$ independent (otherwise p_k^R would be $\langle H\rangle_r$ shift dependent) we can differentiate both sides of (23) with respect to $\mathfrak{b}_1$, which yields the identity

$$0 = \left[\beta_r^R(\mathfrak{b}_2)\right]^2 (1+q-2r). \tag{24}$$

Since $\mathfrak{b}_2$ is by assumption arbitrary, this can be fulfilled only when $1+q = 2r$. For the r's at hand the latter is equivalent to $q = 1$, which in turn sends p_k^R back to the BG distribution. The crux of course is that β_r^R (unlike β_r^{THC}) is not $\langle H\rangle_r$ dependent and thus it does not have means to compensate for the shift in $\langle H\rangle_r$.

Let us now turn back to p_k^{THC} and observe that transformations (21) constitute the one-parameter subgroup of the projective special linear group PSL(2,$\mathbb{R}$). This can be seen by realizing that PSL(2,$\mathbb{R}$) is the quotient group SL(2,$\mathbb{R}$)/$\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm\mathbf{1}\}$ ($\mathbf{1}$ stands for a 2×2 unit matrix). The special linear group SL(2,$\mathbb{R}$) is represented (in its fundamental representation) by 2×2 real matrices with determinant 1 that act on a two-dimensional vector space as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad ad - cb = 1. \tag{25}$$

If we now identify

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm\begin{pmatrix} 1 & 0 \\ (2r-1-q)\mathfrak{b} & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \tag{26}$$

and set $\beta(\mathfrak{b}_i) = u_1(\mathfrak{b}_i)/u_2(\mathfrak{b}_i)$, we find that $\beta(\mathfrak{b}_i)$ transforms as (21). The parameter $\mathfrak{b}$ in (26) is the required value of the shift between the initial and the final configuration.

Alternatively, we might identify

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm\begin{pmatrix} 1 & (1+q-2r)\mathfrak{b} \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \tag{27}$$

and set $\beta(\mathfrak{b}_i) = u_2(\mathfrak{b}_i)/u_1(\mathfrak{b}_i)$. In this case we get the inverse transformation in terms of the original parameter $\mathfrak{b}$. Both ensuing linear fractional transformations are, of course, equivalent, which is on the level of (26) and (27), reflected by the fact that the respective matrices satisfy the equivalence relation (similarity transformation), mediated by the matrices

$$\begin{aligned}\mathbb{S}_{++} = \mathbb{S}_{--} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\\ \mathbb{S}_{+-} = \mathbb{S}_{-+} &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.\end{aligned} \tag{28}$$

For this reason it suffices to consider only one type of transformations, say (26).

From (26) and the group of transformations (21) we see that the relation is not one-to-one since any two matrices $\mathbb{A}$ and $-\mathbb{A}$ from (26) correspond to the same transformation in (21). Consequently, we have a *homomorphism* from a subgroup, (26), of SL(2,$\mathbb{R}$) [known as $E(1)^{\pm}$] onto the group of linear fractional transformations, (21), with kernel $\{\pm\mathbf{1}\} = \mathbb{Z}_2$, and thus $E(1) = E(1)^{\pm}/\mathbb{Z}_2$ is isomorphic to the group of transformations (21). $E(1)$ is known as the *parabolic subgroup* of the projective group PSL(2,$\mathbb{R}$) or, equivalently, as the (real) *Möbius parabolic group*. A crucial point in this context is that

$$\beta(0) = \frac{\beta(\mathfrak{b})}{1 - \beta(\mathfrak{b})(1+q-2r)\mathfrak{b}} \quad \text{for} \quad \forall \mathfrak{b} \in \mathbb{R} \tag{29}$$

represents an invariant under the above group of Möbius parabolic transformations: the *Casimir invariant*. In the following we denote the invariant quantity $[\beta(0)(2r-q-1)]^{-1}$ as μ. This invariant quantity represents a natural candidate for an *observable* or a *state variable*. In fact, in a number of cases it seems reasonable to identify μ directly with the statistical temperature. This step is, however, conceptually quite delicate because the thermodynamical temperature (i.e., the temperature obtained via the Clausius or Caratheodory type of entropy-temperature duality) and statistical temperature (i.e., the temperature that appears as the Lagrange multiplier in MaxEnt distributions) are *a priori* not related in Rényi and THC statistics (unlike in Shannon-entropy-based statistics). In the context of the THC statistics above this issue has been discussed by several authors [40,42–45]. The strategy usually employed is to design (or engineer) the rules of thermodynamics (e.g., the first law of thermodynamics or Clausius theorem) or Jaynes' MaxEnt reasoning so that both temperatures coincide. In Secs. V and VI we discuss the role of μ from a different point of view. In particular, we see that it is often more natural and conceptually less controversial to identify μ with observables other than the temperature.

The PSL(2,$\mathbb{R}$) transformation properties of $\beta(\mathfrak{b})$ are also nontrivially reflected in the scaling behavior of the partition function $Z(\beta(\mathfrak{b}))$. In fact, from (19) and (20) we already know that

$$\begin{aligned}&[1-\beta(\mathfrak{b}_2)(1+q-2r)(\mathfrak{b}_1-\mathfrak{b}_2)]^{1/(2r-q-1)}Z(\beta(\mathfrak{b}_2))\\ &\quad = Z(\beta(\mathfrak{b}_1)).\end{aligned} \tag{30}$$

Comparing this with (21) we see that

$$Z(\beta(\mathfrak{b}_1)) = \left[\frac{\beta(\mathfrak{b}_1)}{\beta(\mathfrak{b}_2)}\right]^{1/(2r-q-1)} Z(\beta(\mathfrak{b}_2)). \tag{31}$$

This might be equivalently rewritten as

$$Z(\lambda\beta(\mathfrak{b})) = \lambda^{1/(2r-q-1)}Z(\beta(\mathfrak{b})), \tag{32}$$

which shows that $Z(\beta(\mathfrak{b}))$ is a homogeneous function of degree $1/(2r-q-1)$ in $\beta(\mathfrak{b})$. The solution can be clearly written in the form

$$Z(\beta(\mathfrak{b})) = \beta(\mathfrak{b})^{1/(2r-q-1)}z(\mu), \tag{33}$$

where $z(\mu)$ is some function which depends on the invariant quantity μ (other state variables and coefficients q and r are suppressed). This type of a powerlike scaling law is characteristic for ensembles of fluctuating lines [20,46]. In this respect, PIs provide the most natural tool for studying the statistical fluctuations of linelike statistical systems [47].

IV. PATH-INTEGRAL REPRESENTATION OF THE THC DENSITY MATRIX

For our further reasonings it is convenient to consider a quantum mechanical setting in which the probability distribution is represented by the density operator. In this case, the (unnormalized) density operator associated with the THC MaxEnt distribution reads

$$\hat{\rho}(\beta(\mathfrak{b})) = [1-\beta(\mathfrak{b})(1+q-2r)(\hat{H}+\mathfrak{b})]^{1/(1+q-2r)}. \tag{34}$$

Let us now assume that $\hat{H}$ is the first-quantized Hamiltonian and that $\beta(\mathfrak{b})$ transforms according to the $E(1) \subset$ PSL(2,$\mathbb{R}$) group, so that one can compensate for the change in $\mathfrak{b}$ by appropriately redefining $\beta(\mathfrak{b})$. For $q < 2r-1$ one can rewrite $\hat{\rho}(\beta(\mathfrak{b}))$ with the help of Schwinger's trick [20,48] as

$$\hat{\rho}(\beta(\mathfrak{b})) = \frac{1}{\Gamma(1/\varepsilon)}\int_0^\infty \frac{dt}{t}t^{1/\varepsilon}e^{-t}e^{-\varepsilon\beta(\mathfrak{b})t(\hat{H}+\mathfrak{b})}, \tag{35}$$

with $0<\varepsilon = 2r-q-1$. This allows us to phrase the configuration-space density matrix $\rho(x_a,x_b;\beta(\mathfrak{b})) \equiv \langle x_b|\hat{\rho}(\beta(\mathfrak{b}))|x_a\rangle$ in the PI form

$$\begin{aligned}\rho(x_a,x_b;\beta(\mathfrak{b})) &= \frac{1}{\Gamma(1/\varepsilon)}\int_0^\infty \frac{dt}{t}t^{1/\varepsilon}e^{-t[1+\varepsilon\beta(\mathfrak{b})\mathfrak{b}]}\\ &\quad\times \int_{x(0)=x_a}^{x(\beta(t))=x_b}\mathcal{D}x\int \mathcal{D}p\, e^{\int_0^{\beta(t)}d\tau(ip\dot{x}-H)}.\end{aligned} \tag{36}$$

Here $\beta(t) \equiv \varepsilon\beta(\mathfrak{b})t$. In the case where the potential-energy term in H does not contain time derivatives of x, the momenta can be integrated out, leaving behind the usual configuration-space path integral with the Euclidean action. In particular,

$$\int_0^{\beta(t)} d\tau(ip\dot{x}-H) \mapsto -S_e = -\int_0^{\beta(t)} d\tau L_e(\dot{x},x). \tag{37}$$

Here S_e is the Euclidean action. For a Hamiltonian of the standard form $H(p,x) = p^2/2m + V(x)$ we would get the corresponding Euclidean Lagrangian in the form $L_e(\dot{x},x) = m\dot{x}^2/2 + V(x)$. Because of the plus sign in front of $V(x)$, the $L_e(\dot{x},x)$ is often denoted $H(\dot{x},x)$.

By changing the variable $\beta(t) \mapsto \beta$ so that $\beta = \varepsilon\beta(\mathfrak{b})t$, we can cast (36) in the form

$$\begin{aligned}\rho(x_a,x_b;\beta(\mathfrak{b})) &= \frac{1}{\Gamma(1/\varepsilon)\,[\varepsilon\beta(\mathfrak{b})]^{1/\varepsilon}}\int_0^\infty \frac{d\beta}{\beta}\beta^{1/\varepsilon}e^{-\beta\mu}\\ &\quad\times\int_{x(0)=x_a}^{x(\beta)=x_b}\mathcal{D}x\int\mathcal{D}p\, e^{\int_0^{\beta}d\tau(ip\dot{x}-H)}\\ &= \left[\frac{\beta(0)}{\beta(\mathfrak{b})}\right]^{1/\varepsilon}\int_0^\infty d\beta f_{\mu,1/\varepsilon}(\beta)\\ &\quad\times\int_{x(0)=x_a}^{x(\beta)=x_b}\mathcal{D}x\int\mathcal{D}p\, e^{\int_0^{\beta}d\tau(ip\dot{x}-H)}.\end{aligned} \tag{38}$$

The smearing function,

$$f_{\alpha,\nu}(x) = \frac{1}{\Gamma(\nu)}\alpha^\nu x^{\nu-1}e^{-\alpha x}; \quad \int_0^\infty dx f_{\alpha,\nu}(x) = 1, \tag{39}$$

is the gamma probability density function (PDF) [1]. So, save for the multiplicative prefactor, the density matrix for the THC MaxEnt distribution can be viewed as the Gibbsian density matrix weighted (or smeared) with the gamma distribution. As expected, the $\mathfrak{b}$ dependence entirely disappeared from the PI expression in (38) and it was replaced by the dependence on the invariant quantity μ.

We note in passing that for large ν the following asymptotical behavior holds:

$$\begin{aligned}f_{\alpha,\nu}(x) &\approx \sqrt{\frac{\alpha}{2\pi x}}\left(\frac{\alpha x}{\nu}\right)^{\nu-1/2}e^{-\nu(\alpha x/\nu-1)}\\ &\approx \delta\left(x-\frac{\nu}{\alpha}\right).\end{aligned} \tag{40}$$

So for $q \to 2r-1$ the β integration disappears and the position-space density matrix, (38), approaches the familiar PI representation of the (nonrelativistic) Bloch density matrix known from the BG statistics [47].

From (38) it follows that the corresponding partition function can be written as

$$Z(\beta(\mathfrak{b})) = \int_{-\infty}^{\infty} dx \rho(x,x;\beta(\mathfrak{b})) = \left[\frac{\beta(0)}{\beta(\mathfrak{b})}\right]^{1/\varepsilon} \int_0^{\infty} d\beta f_{\mu,1/\varepsilon}(\beta) \oint \mathcal{D}x \times \int \mathcal{D}p e^{\int_0^{\beta} d\tau (ip\dot{x} - H)}, \tag{41}$$

where the measure of integration is defined as

$$\oint \mathcal{D}x \cdots = \int_{-\infty}^{\infty} dx(0) \int_{x(0)=x(\beta)} \mathcal{D}x \cdots . \tag{42}$$

Unfortunately, the partition function, (41), overcounts the number of physical configurations. This is because the translations $\tau \mapsto \tau + \text{const.}$ do not change the parametrization space, which is now a circle. This extra freedom does not allow us to fix the starting point x on a loop uniquely, and in fact all choices are equivalent. The rules of statistical physics prescribe that in the partition function all equivalent configurations must be counted only once if the theory is to make sense. Since for a loop of length β we have β possibilities for a choice of the starting point we must insert the extra factor $1/\beta$ in Z to ensure that loops with different starting points $x(\tau)$ count as one loop.

So by defining

$$z(\mu) \equiv \frac{1}{(\varepsilon\mu)^{1/\varepsilon}} \int_0^{\infty} \frac{d\beta}{\beta} f_{\mu,1/\varepsilon}(\beta) \times \oint \mathcal{D}x \int \mathcal{D}p e^{\int_0^{\beta} d\tau (ip\dot{x} - H)}, \tag{43}$$

we may write the correct partition function as

$$Z(\beta(\mathfrak{b})) = \beta(\mathfrak{b})^{-1/\varepsilon} z(\mu), \tag{44}$$

which we can recognize as the partition function of a single, fluctuating, oriented random loop of arbitrary length and shape [46,49,50], and correspondingly, (38) represents the density matrix of an open random chain with end points x_a and x_b embedded in the loop. In agreement with (43), the loop lengths are distributed according to the gamma PDF, (39), and the chain interacts with a background scalar potential $V(x(\tau))$. In this connection we should also remark that the integration parameter β need not be related to the inverse temperature. Explicit examples of this fact are illustrated in two subsequent sections. As anticipated, the form (44) coincides with formula (33).

Frequently one is not interested in studying the behavior of a single, fluctuating, closed random chain but wants to consider grand-canonical ensembles of these. In this case one can promote the above THC statistics into a grand-canonical picture by exponentiating the single-closed-loop partition function, (44), so that the grand-canonical partition function reads

$$Z_G = e^Z = 1 + Z + \frac{1}{2!}Z^2 + \frac{1}{3!}Z^3 + \cdots . \tag{45}$$

This expansion comprises the no-loop, one-loop, two-loop, etc., contributions of mutually noninteracting loops. The combinatorial factor $1/N!$ accounts for the indistinguishability of loops. On account of (45), one may thus alternatively view Z_G as the partition function of a *loop gas*.

Similarly to the BG statistics, one should first multiply the canonical partition function Z by an arbitrary parameter $\mathcal{M}^{2(1/\varepsilon - 1)}$ with the dimension $2(1/\varepsilon - 1)$ in mass units to make Z dimensionless before it is inserted into (45). (Here and throughout, $\hbar = c = 1$.) With this proviso the partition function Z_G can be written as

$$\begin{aligned} Z_G &= \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{k=1}^{N} \left[\int_0^{\infty} \frac{d\beta_k}{\beta_k \Gamma(1/\varepsilon)} (\mathcal{M}^2\beta_k)^{1/\varepsilon - 1} e^{-\beta_k \mu} \oint \mathcal{D}x(\beta_k) \int \mathcal{D}p(\beta_k)\right] \exp\left[\sum_{k=1}^{N} \int_0^{\beta_k} d\tau_k (ip(\tau_k)\dot{x}(\tau_k) - H(\tau_k))\right] \\ &= \exp\left[\int_0^{\infty} \frac{d\beta}{\beta\Gamma(1/\varepsilon)} (\mathcal{M}^2\beta)^{1/\varepsilon - 1} e^{-\beta\mu/\mathcal{M}^2}\, \text{Tr}(e^{-\beta\hat{H}})\right] = \exp\left[\frac{\mathcal{M}^{2s}}{s}\, \zeta_{[H+\mu]}(s)\right] = \exp\left[\frac{\zeta_{[(H+\mu)/\mathcal{M}^2]}(s)}{s}\right], \end{aligned} \tag{46}$$

where we have set $s = (1/\varepsilon - 1)$, defined the $\zeta_{[H+\mu]}$ function as

$$\zeta_{[H+\mu]}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} d\beta \beta^{s-1} \text{Tr}(e^{-\beta(\hat{H}+\mu)}), \tag{47}$$

and used the scaling relation

$$\zeta_{[(H+\mu)/\mathcal{M}^2]}(s) = \mathcal{M}^{2s} \zeta_{[(H+\mu)]}(s). \tag{48}$$

Note, also, that the multiplicative factors $[\beta(0)/\beta(\mathfrak{b})]^{1/\varepsilon}$ and $\mu^{1/\varepsilon}$ were assimilated into the parameter $\mathcal{M}$.

With explicit representations, (46), at hand one can now employ various techniques and methodologies used in the PI calculus to evaluate Z_G. This can be done either numerically (e.g., via PI Monte Carlo or molecular dynamics simulations), in the framework of approximative schemes (e.g., variational approaches or ergodic approximations [47,51]), or via analytic perturbation schemes [20,46,52]. Apart from innate PI methods one can also employ operatorial approaches, such as Schwinger's perturbation expansion [53] for Tr $e^{-\beta\hat{H}}$. We do not dwell on these issues here, but instead we briefly mention another important perturbation treatment, namely, the (spectral) ζ-function expansion, which is particularly pertinent in the framework of the THC statistics. The latter corresponds to an expansion of Z_G around $\varepsilon = 1$, or, equivalently, around $q = 0$ (for $r = 1$) or $q = 2$ (for $r = q$), and has a close connection with quantum field theory. In particular, from the last identity in (46) we can easily write the expansion

$$\begin{aligned} Z_G &= e^{[\frac{1}{s}\zeta_{[(H+\mu)/\mathcal{M}^2]}(0) + \zeta'_{[(H+\mu)/\mathcal{M}^2]}(0) + \frac{s}{2}\zeta''_{[(H+\mu)/\mathcal{M}^2]}(0) + \cdots]} \\ &= e^{[\zeta'_{[H+\mu]}(0) + \ln(M^2)\zeta_{[H+\mu]}(0) + \frac{s}{2}\zeta''_{[(H+\mu)/\mathcal{M}^2]}(0) + \cdots]} \\ &= \det[M^2(\hat{H} + \mu)^{-1}] e^{[\frac{s}{2e}\zeta''_{[(H+\mu)/M^2]}(0)\ + \cdots]}. \end{aligned} \tag{49}$$

Here we have introduced the dimensionful factor $M^2 = \mathcal{M}^2 e^{1/s}$ and employed the identity

$$e^{\zeta'_{[H+\mu]}(0)+\ln(M^2)\zeta_{[H+\mu]}(0)} = \det[M^2(\hat{H}+\mu)^{-1}] \qquad (50)$$

[cf. Eq. (B7) in Appendix B]. So, the leading contribution (in s) to Z_G is easily recognized as the partition function of the complex scalar field theory [48], namely,

$$\mathcal{Z} = \det[M^2(\hat{H}+\mu)^{-1}] = \int \mathcal{D}\phi^*\mathcal{D}\phi e^{-S[\phi^*,\phi]},$$

$$\begin{aligned} S[\phi^*,\phi] &= \int d^D\boldsymbol{x}\phi^*(\boldsymbol{x})(\hat{H}+\mu)\phi(\boldsymbol{x}) \\ &= \int d^D\boldsymbol{x}\phi^*(\boldsymbol{x})\left(-\frac{1}{2m}\nabla^2+\mu+V(\boldsymbol{x})\right)\phi(\boldsymbol{x}). \end{aligned} \qquad (51)$$

The parameter M is related in QFT to a normalization constant for $\mathcal{Z}$. In higher perturbation orders in (49) it serves to absorb infinities arising from the behavior of β integrals at small β. As usual in QFT, such a short-distance behavior (originally $[\beta] = [\mathrm{kg}^{-2}] = [\mathrm{m}^2]$) can be systematically dealt with via the renormalization procedure.

By using the vector notation $\boldsymbol{x}$ we emphasize the validity of our reasoning also beyond $D = 1$. The field-theoretic partition functions, (51), typically appear in the framework of effective field theories, in which case

$$V(\boldsymbol{x}) = \left.\frac{\partial^2 U(\phi^*(\boldsymbol{x}),\phi(\boldsymbol{x}))}{\partial\phi^*\boldsymbol{x}\partial\phi(\boldsymbol{x})}\right|_{\phi(\boldsymbol{x})=\phi_c(\boldsymbol{x}),\phi^*(\boldsymbol{x})=\phi_c^*(\boldsymbol{x})}, \qquad (52)$$

where U is the original field potential and $\phi_c(\boldsymbol{x})$ together with $\phi_c^*(\boldsymbol{x})$ is a classical solution of the inceptive (i.e, noneffective) field theory. An example of this type of behavior is illustrated in Appendix B.

From the aforementioned we see that the leading term in (49) corresponds to the QFT representation of the loop gas in a background potential $V(\boldsymbol{x})$. It should be noted that because $d(\beta^{s-1}/\Gamma(s))/ds = 1/\beta + \mathcal{O}(s)$, the loop lengths in this QFT representation are distributed according to the exponential PDF (one-sided Laplace distribution) $e^{-\mu\beta}\mu$. The subleading terms in Z_G, which are characterized by higher-order derivatives of $\zeta_{[(H+\mu)/M^2]}(s)$, describe corrections to the exponential loop-length distribution in terms of powers of s while keeping $\mathrm{Tr}\ e^{-\beta\hat{H}}$ untouched. The latter is nothing but a variant of the Gram-Charlier expansion [54] of the THC-statistics-related gamma PDF around the exponential distribution.

V. EXAMPLE I: SCHULZ-ZIMM MODEL OF A POLYMER

As already mentioned, an important field of application of the foregoing formulas, (38) and (44), lies in the theory of random chains. A random chain of length N is a sequence $(\boldsymbol{x}_0,\ldots,\boldsymbol{x}_N)$ of $N+1$ points in a D-dimensional Euclidean space. Each step $\Delta\boldsymbol{x}_n \equiv \boldsymbol{x}_n - \boldsymbol{x}_{n-1}$ $(n = 1,\ldots,N)$, i.e., bond connecting points $\boldsymbol{x}_{n-1}$ and $\boldsymbol{x}_n$, is a random variable of fixed length $|\Delta\boldsymbol{x}_n| = a$. Random chains are used to describe linear molecular chains (polymers) [55] as well as other linelike objects including vortex and defect lines in condensed matter systems [46,56], fluctuating price histories in financial markets [20,57,58], and fluctuating particle histories in quantum mechanics [20,51]. In this section, we confine ourself to polymer chains.

Polymers are chemical compounds consisting of a large number of monomer units that are linked together by chemical bonds. Examples include DNA, proteins, cellulose, sugars, and rubber. There is a natural framework for modeling polymers in probability theory and statistical physics [20,46,55]: a polymer chain is modeled by a random path with a probability distribution that is Gibbsian; more specifically, one defines an energy functional on polymer configurations such that the higher the energy of the configuration, the less likely its appearance.

Motivated by different physical phenomena, a variety of polymer models have been proposed and studied in the probability and statistical physics literature. A large class of polymers behaves approximately as *ideal random chains*, in which case the links $\Delta\boldsymbol{x}_n$ are independent, identically distributed random variables, taking values uniformly over a sphere of radius a. The parameter a is known as the bond length of the random chain. The probability distribution of the end-to-end vector $\boldsymbol{R} \equiv \boldsymbol{x}_N - \boldsymbol{x}_0$ for an ideal chain of length N can thus be written as [20]

$$\begin{aligned} P_N(\boldsymbol{R}) = \prod_{n=1}^{N}&\left[\int d\Delta\boldsymbol{x}_n \frac{1}{S_D a^{D-1}}\delta(|\Delta\boldsymbol{x}_n| - a)\right] \\ &\times \delta^{(D)}\left(\boldsymbol{R} - \sum_{n=1}^{N}\Delta\boldsymbol{x}_n\right), \end{aligned} \qquad (53)$$

with $S_D = 2\pi^{D/2}/\Gamma(D/2)$ being the surface of a unit sphere in D-dimensional space. In the limit of large N, $P_N(\boldsymbol{R})$ can be approximated (as a consequence of the central limit theorem [1,8,9]) by the Gaussian distribution

$$P_N(\boldsymbol{R}) \approx \left(\frac{D}{2\pi a L}\right)^{\frac{D}{2}} \exp\left(-\frac{D\boldsymbol{R}^2}{2aL}\right) \equiv P_L(\boldsymbol{R}). \qquad (54)$$

In the following we use the actual polymer length $L = Na$ instead of diverging N. Relation (54) may be understood as the propagator of a free nonrelativistic particle of mass $m = D/a$, with time continued to an imaginary value $-iL$, i.e.,

$$P_L(\boldsymbol{R}) = \langle\boldsymbol{x}_N,t_N|\boldsymbol{x}_0,t_0\rangle|_{t_N-t_0=-iL}. \qquad (55)$$

The corresponding PI representation of (54) is also known in the polymer literature as Edwards' integral [55] and reads

$$P_L(\boldsymbol{R}) = \int_{\boldsymbol{x}(0)=\boldsymbol{0}}^{\boldsymbol{x}(L)=\boldsymbol{R}} \mathcal{D}\boldsymbol{x} \exp\left(-\int_0^L H(\dot{\boldsymbol{x}}(\tau))d\tau\right), \qquad (56)$$

where

$$H(\dot{\boldsymbol{x}}) = \frac{D}{2a}\dot{\boldsymbol{x}}^2 \qquad (57)$$

plays the role of the energy density of the polymer conformation $\boldsymbol{x}(\tau)$.

In real polymers, the bonds usually do not allow for an equal probability of all spherical angles because the chains are stiff. To account for stiffness, one may use the coarse-graining trick and increase, for sufficiently long chains, the bond length a in (57) to the effective bond length a_{eff}, so that chain segments of length a_{eff} behave as freely rotating. Alternatively, one can include in the Lagrangian a bending energy, which typically involves the square of the second derivative $\ddot{\boldsymbol{x}}(\tau)$ (see, e.g., [59]).

So far we have considered polymers of a fixed length L (or a fixed degree of polymerization N). In real solutions, however, various chain lengths are found, with their distribution depending on the nature of the polymerization reaction. For linear addition polymerization with termination, where identical monomers are added one at a time to the reactive end of a growing chain until the process is terminated, the *Schulz-Zimm* (or *Schulz*) PDF is typically used [60–62] (for further details see Appendix A). It has the form of the gamma distribution, (39), where $\nu/\alpha = \langle L\rangle \equiv \bar{L}_n$ represents the averaged polymer length (known as the number-average molecular weight) and

$$\frac{\nu+1}{\alpha} = \frac{\langle L^2\rangle}{\langle L\rangle} \equiv \bar{L}_w \tag{58}$$

denotes the so-called weight-average molecular weight. The relative fluctuation (variance)

$$\frac{\mathrm{Var}(L)}{\langle L\rangle} = \frac{\bar{L}_w}{\bar{L}_n} - 1 = \frac{1}{\nu} \tag{59}$$

provides an operational meaning to the parameter ν. The fraction $\bar{L}_w/\bar{L}_n$ is known as a polydispersity index (PDI) and it quantifies the spread of the distribution of molecular lengths. The Schulz-Zimm PDF is used primarily because it has a simple functional form that allows us to interpolate between two key chain formation mechanisms: *disproportionation*, described by the distribution with PDI $= 2$; and *combination*, described by the distribution with PDI $= 3/2$. For more details, see Appendix A.

If the molecular chain can have any length with distribution $\omega(L)$, the length distribution of the end-to-end vector of an ideal polymer is then obtained via *marginalization* over the *nuisance* parameter L, i.e.,

$$P(\boldsymbol{R}) = \int_0^\infty dL\,\omega(L)\int_{\boldsymbol{x}(0)=\boldsymbol{0}}^{\boldsymbol{x}(L)=\boldsymbol{R}} \mathcal{D}\boldsymbol{x}\\ \times \exp\left(-\int_0^L H(\dot{\boldsymbol{x}}(\tau))d\tau\right). \tag{60}$$

In the case of the Schulz-Zimm length-smearing distribution $f_{\alpha,\nu}(L)$ the above marginal distribution $P(\boldsymbol{R})$ coincides with the Tsallis density matrix, (38). The multiplicative factor in (38) is assimilated in the normalization of $P(\boldsymbol{R})$. The role of the smearing parameter β is then played by the chain length L, the parameter $\nu = 1/\varepsilon$, and $\alpha = \mu = [\varepsilon\beta(0)]^{-1}$. With this we have $q = 2 -$ PDI (for $r = 1$) and $q =$ PDI (for $r = q$). The role of Tsallis' $E(1) \subset \mathrm{PSL}(2,\mathbb{R})$ invariant $\beta(0)$ is played by the mean chain length $\bar{L}_n$ (for both $r = 1$ and $r = q$), and the ensuing μ can be identified, according to Appendix A, with w_T/a, where w_T is the probability that a new monomer cannot be added to a polymer chain (the chain is inactive).

Let us, finally, add a few comments. First, by modifying the law of the random walk, (53) [or, alternatively, (56)], by introducing an appropriate energy functional, more realistic features can be introduced to account for the interaction between different monomers and the interaction between the polymer and the environment. Furthermore, randomness (i.e., disorder) can be incorporated into such interactions to model impurities.

Second, by rewriting (60) in its phase-space representation [cf. Eq. (36)] we obtain, after a simple time re-scaling, the identity

$$\begin{aligned} P(\boldsymbol{R}) &= \int_0^\infty \omega(L)\int_{\boldsymbol{x}(0)=\boldsymbol{0}}^{\boldsymbol{x}(1)=\boldsymbol{R}} \mathcal{D}\boldsymbol{x}\mathcal{D}\boldsymbol{p}\\ &\quad\times \exp\left\{\int_0^1 d\tau[i\,\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - LH(\boldsymbol{x},\boldsymbol{p})]\right\}\\ &= \int_{\boldsymbol{x}(0)=\boldsymbol{0}}^{\boldsymbol{x}(1)=\boldsymbol{R}} \mathcal{D}\boldsymbol{x}\mathcal{D}\boldsymbol{p}\exp\left\{i\int_0^1 d\tau\,\boldsymbol{p}\cdot\dot{\boldsymbol{x}}\right\}\\ &\quad\times \exp\left\{-\int_0^1 d\tau[\nu\log(H(\boldsymbol{x},\boldsymbol{p})/\mu+1)]\right\}\\ &\equiv \wp(\boldsymbol{R},1|\boldsymbol{0},0). \end{aligned} \tag{61}$$

Here H is the relevant (time-independent) Hamiltonian and the last PI is defined by time slicing in the postpoint form. The middle identity in (61) was derived in Ref. [63].

With the representation, (61), at hand one can now employ the Feynman-Kac formula (see, e.g., Refs. [20,47]), which allows one to view the PI representation, (61), as a solution of a diffusionlike equation—the so-called (forward) Kramers-Moyal equation

$$\partial_\tau \wp(\boldsymbol{x},\tau|\boldsymbol{0},0) = \mathbb{L}_x\,\wp(\boldsymbol{x},\tau|\boldsymbol{0},0), \tag{62}$$

with the initial condition $\wp(\boldsymbol{x},0|\boldsymbol{0},0) = \delta(\boldsymbol{x})$. The Kramers-Moyal operator $\mathbb{L}_x$ has the form [63]

$$\mathbb{L}_x = \sum_n^\infty (-1)^n \frac{\partial^n}{\partial x_{j_1}\dots x_{j_n}} D^{(n)}_{j_1,\dots,j_n}(\boldsymbol{x}) \tag{63}$$

(the summation over the indices $j_1,\dots,j_n$ is implicitly understood). Coefficients $D^{(n)}_{j_1,\dots,j_n}(\boldsymbol{x},t)$ are the rescaled nth moments of the short-time transition probabilities [63], namely,

$$\begin{aligned} &D^{(n)}_{j_1,\dots,j_n}(\boldsymbol{x})\\ &\quad= \lim_{\tau\to 0}\frac{1}{n!\tau}\int_{\mathbb{R}^D} d\boldsymbol{y}\prod_{i=1}^n (y_{j_i} - x_{j_i})\wp(\boldsymbol{y}-\boldsymbol{x},\tau|\boldsymbol{0},0)\\ &\quad= \lim_{\tau\to 0}\frac{1}{n!\tau}\int_{\mathbb{R}^D} d\boldsymbol{y}\prod_{i=1}^n (y_{j_i} - x_{j_i})\langle\boldsymbol{y}|e^{-\tau\nu\log(\hat{H}/\mu+1)}|\boldsymbol{x}\rangle. \end{aligned} \tag{64}$$

In the last identity we have employed Dirac's bra-ket notation. To calculate the short-time transitional probability in (65) one can employ some of the standard PI perturbation approaches [20]. When the Kramers-Moyal equation is truncated after the second order, one obtains a conventional (forward) Fokker-Planck equation for the transitional probability $\wp(\boldsymbol{x},\tau|\boldsymbol{0},0)$. When H is identified with (57), then the ensuing Fokker-Planck equation reduces to the Schultz-Zimm master equation.

VI. EXAMPLE II: CONNECTION WITH RELATIVISTIC PARTICLES

In a sense the simplest representative of the THC statistics-based random chains is fluctuating relativistic particle orbits. In particular, the density matrix, (38), together with the partition function, (44), can be identified with the density matrix and partition function for a free spinless relativistic particle, respectively, provided we use a suitable S_e and set $q = 2r - 2$.

The simplest place to start is the Polyakov-type action for a free spinless particle, which reads [20,64–66]

$$\begin{aligned} &S[x,\eta;\tau_1,\tau_2] \\ &\quad = -\tfrac{1}{2}\int_{\tau_1}^{\tau_2} d\tau\big(\eta^{-1}(\tau)\,\dot{x}^{\mu}(\tau)\dot{x}_{\mu}(\tau) + \eta(\tau)\,m_0^2\big). \end{aligned} \tag{65}$$

Here η represents the square root of the world-line metric (i.e., einbein) and τ is a label time (be it the proper time, affine parameter, etc.) parametrizing the world line. We have chosen the Lorentz signature in D dimensions to be $(+,-,-,\ldots,-)$. The action, (65), is invariant under reparametrizations of the label time, i.e.,

$$\begin{aligned} &\tau \mapsto \bar{\tau} = f(\tau) \Rightarrow \dot{x}_{\mu}\dot{x}^{\mu} \mapsto \dot{\bar{x}}_{\mu}\dot{\bar{x}}^{\mu} = \frac{\dot{x}_{\mu}\dot{x}^{\mu}}{\dot{f}^2}, \\ &d\tau \mapsto d\bar{\tau} = d\tau\,\dot{f}, \quad \eta \mapsto \bar{\eta} = \frac{\eta}{\dot{f}}, \end{aligned} \tag{66}$$

with $f(\tau)$ fulfilling the conditions $f(\tau_1) = \tau_1$, $f(\tau_2) = \tau_2$, and $\dot{f} > 0$. The result is $S = \bar{S}$. The transition amplitude from $x^{\mu}(\tau_1) = x_1^{\mu}$ to $x^{\mu}(\tau_2) = x_2^{\mu}$ can then be written as

$$\langle x_2,\tau_2|x_1,\tau_1\rangle = \int_{x(\tau_1)=x_1}^{x(\tau_2)=x_2}\mathcal{D}x\int\mathcal{D}\eta e^{iS}. \tag{67}$$

This path integral is, however, not quite right. It contains an enormous overcounting, because the configurations (η,x) and $(\bar{\eta},\bar{x})$, which are related to one another by the reparametrization transformation, (66), represent the same physical configuration. If we define the space of all einbeins η as Σ and the D-dimensional Minkowski space as $\mathbb{R}_{\mathrm{M}}^{\mathrm{D}}$, then the true space of physical configurations is not $\Sigma \times \mathbb{R}_{\mathrm{M}}^{\mathrm{D}}$ but rather the factor space $(\Sigma \times \mathbb{R}_{\mathrm{M}}^{\mathrm{D}})/G$, with G representing the reparametrization group. At least locally we can always write that $\Sigma \times \mathbb{R}_{\mathrm{M}}^{\mathrm{D}} \sim ((\Sigma \times \mathbb{R}_{\mathrm{M}}^{\mathrm{D}})/G) \times G$. Thus

$$\mathcal{D}\eta\mathcal{D}x = d\mu\big((\Sigma \times \mathbb{R}_{\mathrm{M}}^{\mathrm{D}})/G\big)d\mu(G). \tag{68}$$

Here $d\mu(G)$ represents the measure on the reparametrization group. It can be shown [56] that

$$\begin{aligned} \mathcal{D}\eta\mathcal{D}x &= \frac{dL\mathcal{D}x}{\sqrt{L}}\sqrt{\det{}_R\left(-\frac{d^2}{d\tau^2}\right)}\,d\mu(G) \\ &= \mathcal{N}\,dL\mathcal{D}x\,d\mu(G). \end{aligned} \tag{69}$$

(The subscript R indicates the regularized determinant.) The variable L corresponds to the actual length of the world line, i.e.,

$$L = \int_{\tau_1}^{\tau_2} d\tau\eta(\tau). \tag{70}$$

After factorizing out the volume of the reparametrization group we obtain the true transition amplitude, (67), which now reads

$$\begin{aligned} \langle x_2,\tau_2|x_1,\tau_1\rangle &= \int d\mu\big((\Sigma \times \mathbb{R}_{\mathrm{M}}^{\mathrm{D}})/G\big)e^{iS} \\ &= \mathcal{N}\int_0^{\infty} dLe^{-im_0^2L/2}\int_{x(\tau_1)=x_1}^{x(\tau_2)=x_2}\mathcal{D}xe^{i\tilde{S}}, \\ \tilde{S}[x;\tau_1,\tau_2] &= -\frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau\dot{x}^{\mu}(\tau)\dot{x}_{\mu}(\tau). \end{aligned} \tag{71}$$

Equation (71) is the so-called world-line representation of Green's function for the Klein-Gordon equation, and formally it may be obtained also via the Feynman-Fock fifth parameter approach [51]. Result (71) can be naturally related to the density matrix by the substitution $\tau \mapsto -i\mathrm{t}$ and $L \mapsto -i\beta$. In this case we arrive at the density matrix

$$\begin{aligned} \rho(x_a,x_b;\beta) &= \mathcal{N}\int_0^{\infty} d\beta e^{-m_0^2\beta/2}\int_{x(0)=x_1}^{x(\beta)=x_2}\mathcal{D}xe^{-\tilde{S}_e}, \\ \tilde{S}_e[x,\beta] &= \frac{1}{2}\int_0^{\beta} d\mathrm{t}(\dot{\mathbf{x}}(\mathrm{t})\cdot\dot{\mathbf{x}}(\mathrm{t}) + \dot{x}_0(\mathrm{t})\dot{x}_0(\mathrm{t}))\,, \end{aligned} \tag{72}$$

with $\tilde{S}_e$ representing the ensuing Euclidean action.

Comparing (72) with (36) we see that Tsallis' density matrix for a free *nonrelativistic* particle in D spatial dimensions is equivalent to the (canonical) density matrix for a free *relativistic* particle in D space-time dimensions provided we identify $2r - q = 2$ and $m_0^2/2 = \mu$. Because $\mathrm{PSL}(2,\mathbb{R}) \simeq SO^{+}(1,2) \subset SO^{+}(1,D-1)$ [here $SO^{+}(1,D-1)$ represents the restricted Lorentz group in D dimensions], we have that μ is a Lorentz invariant in D dimensions (since m_0^2 is) and hence it is automatically invariant also under subgroup $\mathrm{PSL}(2,\mathbb{R})$. This ensures that (72) agrees with the THC density matrix.

With the density matrix at hand we can construct the corresponding one-particle partition function Z. Once again, we have to be careful and insert the extra factor $1/\beta$ in the PI measure to avoid overcounting loops with different starting points $\boldsymbol{x}(\mathrm{t}) = (x_0(\mathrm{t}),\mathbf{x}(\mathrm{t}))$.

The THC density matrix, (72), was constructed on the premise that S (and the ensuing $\tilde{S}_e$) describes a single particle. Of course, the single-particle relativistic quantum theory is logically untenable, since a multiparticle production is allowed whenever a particle reaches the threshold energy for pair production. So, strictly speaking, representation (72) holds only when the energy-momentum involved is lower than the particle's resting mass. In addition, Leutwyler's no-interaction theorem [67] prohibits interaction for any finite number of particles in the context of relativistic mechanics. To get around the no-interaction theorem it is essential to have an infinite number of degrees of freedom to describe interaction. The latter is typically achieved via local quantum field theories.

Despite the aforesaid shortcomings, it should be stressed that the PIs for a single relativistic particle, (71) and (72), represent a key building block in QFT. In fact, QFT, in general, can be viewed as a grand-canonical ensemble of fluctuating particle histories (world lines) where a Feynman diagrammatic representation of quantum fields depicts directly the pictures of the world lines in a grand-canonical ensemble. In particular, the partition function for quantized relativistic fields can be fully rephrased in terms of single-particle relativistic PIs. This, the so-called "world-line quantization" of particle physics, is epitomized, e.g., in Feynman's world-line representation of the one-loop effective action in quantum electrodynamics [68], in the Strassler and Bern-Kosower "string-inspired" approaches to QFT [69,70], and in disorder field theory [46].

Let us, finally, mention that the similar analysis we have just done for a spinless relativistic particle can be straightforwardly generalized to any spinning relativistic particle with a nonzero resting mass (massive Rarita-Schwinger particle) [65]. It is also trivial to extend our approach to account for charged relativistic particles that are coupled to the external electromagnetic field [65]. This can be done as usual by the minimal substitution (via the covariant derivative). To put some meat on the bare bones, we use in Appendix B the world-line quantization to calculate the Gibbs free energy of the charged scalar quantum field in the background electromagnetic potential. At present, it seems that massless particles do not fit easily into the outlined THC statistical scenario.

VII. CONCLUSIONS

In this article we have introduced a class of statistical processes in which Tsallis' thermostatistics finds its natural conceptual playground. The processes in question are characterized by the position-space density matrix, which is invariant under group $E(1) \subset \mathrm{PSL}(2,\mathbb{R})$ (i.e., the group of Möbius parabolic transformations) of the THC "inverse-temperature" parameter. We have seen that such behavior is dictated by purely thermodynamic considerations (the first law of thermodynamics alongside the Legendre structure) and supported by the self-referentiality of the underlying THC distribution, (12). In contrast with the THC MaxEnt distribution, the closely related Rényi MaxEnt distribution, (10), though of superficially identical form, is not self-referential and consequently cannot compensate for the shift in $\langle H\rangle_r$ by consistently redefining β. In this connection we should perhaps mention one terminological nuisance that is sometimes used, namely, that Student's t distribution (which is key in a sampling theory) constitutes Tsallis' statistics. This is certainly not the case. The THC MaxEnt distribution resembles Student's t distribution only superficially. This is because the THC entropy maximizer is badly self-referential in a temperature parameter. On the other hand, Student's t distribution can be directly identified with Rényi's MaxEnt distribution where the temperature parameter is a constant.

We have shown that statistical systems that fit the above pattern of behavior can be identified with certain types of random chains in a background scalar potential. This can be seen particularly clearly when the associated density matrices are formulated in path-integral language. In this case the ensuing partition function coincides with the partition function of a fluctuating random loop of arbitrary length, while the density matrix itself describes an open random chain with end points embedded in the loop. A specific point of the THC statistics [namely, its $E(1)$ symmetry] is that the loop lengths are distributed according to the gamma PDF. As an illustration of the issues involved we have presented a treatment of two simple statistical systems, namely, an ensemble of fluctuating polymer chains in a Schulz-Zimm approximation and an ensemble of relativistic particle orbits formulated in the framework of relativistic quantum mechanics.

The PI representation of the THC density matrix also serves as a convenient starting point for various generalizations. In particular, it provides a natural passage from a single THC statistics-based fluctuating random loop to grand-canonical ensembles of these. We have discussed the basic inner workings of this procedure and highlighted its connection to the spectral ζ-function expansion and ensuing QFT representation of the loop gas. A closely related computation of the Gibbs free energy in scalar quantum electrodynamics is presented in Appendix B. Particularly in the latter case we observed yet another role of the THC parameter, namely, it can be identified with a regulator in the ζ-function regularization of functional determinants.

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APPENDIX A

Here we provide a brief derivation of the Schulz-Zimm distribution of chain lengths for values of the parameter $\nu = 1,2$. We adopt the effective picture of Kamide and Dobashi [71], in which there is a certain probability w_P of adding a monomer to the growing chain and a probability $w_T = 1 - w_P$ that the termination reaction occurs such that monomers can no longer be added, i.e., the chain is dead. The reader may find a detailed description of the chemical processes involved, e.g., in Ref. [72]. A chain composed of n monomers ($n \geqslant 1$) is thus formed with the probability $p(n) = w_P^{n-1} w_T$. There are two common processes of termination: disproportionation, where two growing chains meet to deactivate their reactive centers but do not combine; and combination, where two growing chains of lengths n and m combine to form a dead chain of length $n + m$.

In the case of termination by disproportionation, the distribution of chain lengths is given simply by $p_D(n) = p(n) = w_P^{n-1} w_T$. In the continuum limit, $L = na$ (a is the bond length) and we use the approximation $w_P \approx 1$, i.e., $w_T \ll 1$, in which long chains are likely to be formed. The probability density $P_D(L)$ of finding a chain of length L is determined from the identity

$$p_D(n) = \int_{(n-1)a}^{na} dL\, P_D(L). \tag{A1}$$

This yields the PDF in the form

$$P_D(L) = \frac{w_T}{a} \exp\left(-\frac{w_T}{a} L\right). \tag{A2}$$

In the case of termination by combination, a chain of n monomers arises from the combination of two growing chains with lengths m and $n - m$ ($1 \leqslant m \leqslant n - 1$). The distribution of chain lengths is then given by

$$p_C(n) = \sum_{m=1}^{n-1} p(m)p(n-m) = w_T^2(n-1)w_P^{n-2}, \tag{A3}$$

and the corresponding density function is given, in analogy with (A1), as

$$P_C(L) = L\frac{w_T^2}{a^2}\exp\left(-\frac{w_T}{a}L\right). \tag{A4}$$

By comparing (A2) and (A4) with the gamma PDF, (39), we can identify

$$P_D(L) = f_{w_T/a,1}(L) \tag{A5}$$

and

$$P_C(L) = f_{w_T/a,2}(L). \tag{A6}$$

The THC distribution parameters q and the Casimir invariant $\beta(0)$ of the end-to-end vector distribution, (60), can thus be set as follows: termination by disproportionation corresponds to $q_D = 2$ (for $r = q$) and

$$\beta_D(0) = \frac{a}{w_T}; \tag{A7}$$

termination by combination, to $q_C = 3/2$ (for $r = q$) and

$$\beta_C(0) = \frac{2a}{w_T}. \tag{A8}$$

Combination and disproportionation are two competitive processes which do not occur to the same extent for all polymers [71,72].

APPENDIX B

The uses of the path-integral representations (38) and (44) are not restricted to the calculation of probability densities or partition functions. These formulas can also be directly used to calculate the one-loop effective action, i.e., Gibbs free energy, in QFT. The one-loop effective action $\Gamma^{(1)}$ is defined as [48,49,73]

$$\Gamma^{(1)} = -\frac{1}{2}\ln\det\left(\frac{\delta^2 S_e}{\delta\phi_i\delta\phi_j}\right)\bigg|_{\phi_k=\phi_{k,c}}. \tag{B1}$$

Here S_e denotes the Euclidean field-theory action, $\{\phi_k\}$ represents the constituent multiplet of scalar fields, and $\phi_{k,c}$ is a solution of the *classical* field equations of motion. For instance, for a relativistic complex massive scalar field that is minimally coupled to a background electromagnetic field and self-interacts via potential $U(\phi_k)$, the Euclidean Lagrange density reads [48]

$$\mathcal{L}_e = \frac{1}{2}(\phi^*,\phi)\begin{bmatrix}(\hat{p}_\mu + eA_\mu)^2 + m^2 + U_{\phi^*\phi}(\phi_{k,c}) & U_{\phi^*\phi^*}(\phi_{k,c}) \\ U_{\phi\phi}(\phi_{k,c}) & (\hat{p}_\mu - eA_\mu)^2 + m^2 + U_{\phi\phi^*}(\phi_{k,c})\end{bmatrix}\begin{pmatrix}\phi \\ \phi^*\end{pmatrix}, \tag{B2}$$

with $\hat{p}_\mu = -i\partial_\mu$ and $U_{\phi^*\phi} = \partial^2 U/\partial\phi^*\partial\phi = U_{\phi\phi^*}$, etc. All scalar products are understood with respect to the Euclidean metric $\delta_{\mu\nu}$. With the help of the *Schur complement* technique for calculation of determinants of partitioned matrices [74], the ensuing one-loop contribution to the Gibbs free energy reads

$$\begin{aligned}\Gamma^{(1)}[A] = &-\tfrac{1}{2}\ln\det[(\hat{p}+eA)^2 + m^2 + U_{\phi^*\phi}] \\ &-\tfrac{1}{2}\ln\det[(\hat{p}-eA)^2 + m^2 + U_{\phi\phi^*} - U_{\phi\phi}\mathcal{G}U_{\phi^*\phi^*}],\end{aligned} \tag{B3}$$

where $\mathcal{G} = [(\hat{p}+eA)^2 + m^2 + U_{\phi^*\phi}(\phi_{k,c})]^{-1}$ denotes the corresponding Green's function of the charged scalar particle in the classical background fields A_μ and $U_{\phi^*\phi}(\phi_{k,c})$.

To illustrate the connection with the THC statistics we consider for simplicity the situation with $U = 0$. In this case we have

$$\Gamma^{(1)}[A] = -\ln\det[(\hat{p}+eA)^2 + m^2]. \tag{B4}$$

Note that the global factor $1/2$ has disappeared, because

$$\begin{aligned}\det[(\hat{p}-eA)^2 + m^2] &= \det[\mathcal{C}((\hat{p}-eA)^2 + m^2)\mathcal{C}^{-1}] \\ &= \det[(\hat{p}+eA)^2 + m^2],\end{aligned} \tag{B5}$$

where the unitary operator $\mathcal{C}$ represents the charge conjugation operator, i.e., $\mathcal{C}\phi\mathcal{C}^{-1} = \phi^*, \mathcal{C}\phi^*\mathcal{C}^{-1} = \phi$ and $\mathcal{C}A^\mu\mathcal{C}^{-1} = -A^\mu$.

To calculate the functional determinant in (B4) we utilize the method of the so-called ζ-function regularization [75]. The strategy is as follows; we denote $\hat{\mathcal{A}} \equiv (\hat{p}+eA)^2 + m^2$ and define the (spectral) ζ function $\zeta_{\mathcal{A}}(s)$ through the spectrum $\{\lambda_n\}$ of $\hat{\mathcal{A}}$ as

$$\zeta_{\mathcal{A}}(s) = \sum_n \frac{1}{\lambda_n^s}. \tag{B6}$$

With this the determinant in (B3) can be calculated as

$$\det\hat{\mathcal{A}} = \lim_{s\to 0} e^{-\zeta'_{\mathcal{A}}(s)}. \tag{B7}$$

Since the spectrum is typically not known, we can utilize, instead of the defining relation, (B6), the *heat kernel* method to to compute $\zeta_{\mathcal{A}}(s)$. The heat kernel $\mathcal{G}_{\mathcal{A}}(x,y,\beta)$ of the operator $\hat{\mathcal{A}}$ is the fundamental solution of the heat-transfer-type equation,

$$\hat{\mathcal{A}}_x\mathcal{G}_{\mathcal{A}}(x,y,\beta) = -\frac{\partial}{\partial\beta}\mathcal{G}_{\mathcal{A}}(x,y,t), \tag{B8}$$

with the Cauchy condition $\mathcal{G}_{\mathcal{A}}(x,y,\beta = 0) = \delta(x-y)$.

By writing the heat kernel in Dirac's bra-ket notation,

$$\mathcal{G}_{\mathcal{A}}(x,y,\beta) = \theta(\beta)\langle x|\exp(-\beta\hat{\mathcal{A}})|y\rangle, \tag{B9}$$

and using the identity

$$\frac{1}{\lambda_i^s} = \frac{1}{\Gamma(s)}\int_0^\infty d\beta\beta^{s-1}e^{-\beta\lambda_i}, \tag{B10}$$

valid for $\mathrm{Re}(s) > 0$, we see that $\zeta_{\mathcal{A}}(s)$ can be alternatively written as the Mellin transform of the trace of the heat kernel $\mathcal{G}_{\mathcal{A}}(x,y,\beta)$, namely,

$$\zeta_{\mathcal{A}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\beta\beta^{s-1}\int_{-\infty}^{\infty} dx\mathcal{G}_{\mathcal{A}}(x,x,\beta). \tag{B11}$$

Here the parameter β is known as Schwinger's proper time parameter. By employing (B7) we obtain

$$\ln\det\hat{\mathcal{A}} = -\lim_{s\to 0}\frac{d}{ds}\int_0^\infty \frac{d\beta\beta^{s-1}}{\Gamma(s)}\int_{-\infty}^{\infty} dx\mathcal{G}_{\mathcal{A}}(x,x,\beta) \tag{B12}$$

or, equivalently [cf. (B3)],

$$\begin{aligned}\Gamma^{(1)}[A] &= \ln\det\hat{\mathcal{A}}/M^2\\ &= \lim_{s\to 0}\frac{d}{ds}\int_0^\infty \frac{d\beta\beta^{s-1}}{\Gamma(s)}\mathrm{Tr}[e^{-\beta((\hat{p}+eA)^2+m^2)/M^2}].\end{aligned} \tag{B13}$$

Here we have introduced a factor M with the dimension of mass to maintain the argument of $\ln(\ldots)$ dimensionless. In this case also β is dimensionless.

The path integral enters when the functional trace $\mathrm{Tr}(\ldots)$ is rewritten in the position-space representation, i.e.,

$$\mathrm{Tr}(\ldots) = \int_{-\infty}^{\infty} dx\langle x|\ldots|x\rangle, \tag{B14}$$

and when the PI representation,

$$\begin{aligned}&\langle x|e^{-\beta(\hat{p}+eA)^2}|x\rangle\\ &= \int_{q(0)=q(\beta)=x}\mathcal{D}q\exp\left[-\int_0^\beta d\tau(\dot{q}^2/4+ie\dot{q}A)\right],\end{aligned} \tag{B15}$$

is utilized. With this we can finally write

$$\begin{aligned}\Gamma^{(1)}[A] = \lim_{s\to 0}\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^\infty \frac{d\beta}{\beta}(M^2\beta)^s e^{-\beta m^2}\\ \times\oint\mathcal{D}q\exp\left[-S_e[q,\dot{q}]\right],\end{aligned} \tag{B16}$$

where $S_e = \int_0^\beta d\tau(\dot{q}^2/4 + ie\dot{q}A(q))$ is the corresponding quantum-mechanical Euclidean action. Again, the scalar products $\dot{q}_\mu\dot{q}^\mu$ and $\dot{q}_\mu A^\mu(q)$ are with respect to the Euclidean metric $\delta_{\mu\nu}$.

Since the β integral is generally not absolutely convergent, one cannot naively interchange derivation and limitation with integration. So, strictly speaking, one must first evaluate the integral with the regulator $s > 0$ and only at the end perform the differentiation in $s \to 0$. Fortunately, for many operators $\hat{\mathcal{A}}$ one can analytically extend $\zeta_{\mathcal{A}}(s)$ to a meromorphic function which is regular at $s = 0$. This in turn allows us to analytically continue the path integral with $s > 0$ to the path integral with the would-be $1/\beta$ term [formally $d(M^{2s}\beta^{s-1}/\Gamma(s))/ds = 1/\beta + \mathcal{O}(s)$]. In fact, the latter path integral typically leads to the so-called Schwinger determinant, which is an unregularized (infinite) expression; thus it is necessary to provide some regulation scheme to obtain a well-defined result.

The form, (B16), in turn, allows us to pinpoint yet another interesting rôle of the THC parameter, namely, $1/\varepsilon - 1$ can be identified with a regulator s in the ζ-function regularization, which, as we have just seen, is used in regularization of functional determinants [such as (B3)] in QFT.

In principle, one can proceed with the outlined THC statistics even to higher-loop orders in QFT calculations of the Gibbs free energy by using the so-called world-line path-integral representation. We do not dwell on this approach here. The interested reader may consult, e.g., Ref. [73].

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