One-loop diagrams in the random Euclidean matching problem

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The matching problem is a notorious combinatorial optimization problem that has attracted for many years the attention of the statistical physics community. Here we analyze the Euclidean version of the problem, i.e., the optimal matching problem between points randomly distributed on a d-dimensional Euclidean space, where the cost to minimize depends on the points' pairwise distances. Using Mayer's cluster expansion we write a formal expression for the replicated action that is suitable for a saddle point computation. We give the diagrammatic rules for each term of the expansion, and we analyze in detail the one-loop diagrams. A characteristic feature of the theory, when diagrams are perturbatively computed around the mean field part of the action, is the vanishing of the mass at zero momentum. In the non-Euclidean case of uncorrelated costs instead, we predict and numerically verify an anomalous scaling for the sub-sub-leading correction to the asymptotic average cost.

I. INTRODUCTION

Let us consider a complete graph K_N of N vertexes, N even, indexed in $[N] := \{v\}_{v=1,\dots,N}$, and a set of cost coefficients $w_{uv} = w_{vu}$, $1 \leqslant u < v \leqslant N$, in such a way that w_{uv} is associated with the (undirected) edge (u,v) of the graph. The matching problem consists in finding an optimal matching m on the graph K_N. An optimal matching m is a subset of edges of K_N that satisfies two fundamental properties. First, m must be a *perfect* (or admissible) matching, i.e., each vertex of K_N must be adjacent to one, and only one, edge in m. Second, in an optimal matching m, the sum of the costs of the edges in m, also called *matching cost*, is minimal (optimality condition). We can associate an occupation number $m_{uv} \in \{1,0\}$ to each edge (u,v) of the original complete graph, depending on whether it belongs to the matching m, or not. We identify then the matching m with the symmetric matrix $\mathsf{m} := (m_{uv})_{uv}$ that we denote, for the sake of simplicity, by the same symbol. By means of the matrix m, the matching cost can be written as

$$\mathcal{E}_w(\mathsf{m}) := \sum_{u<v}^{N} m_{uv} w_{uv}. \tag{1}$$

We can recast the original optimal matching problem into the following integer programming problem for the matrix m:

$$\underset{\mathsf{m}}{\text{minimize}}\ \mathcal{E}_w(\mathsf{m}), \tag{2}$$

given the constraints

$$\sum_{v=1}^{N} m_{uv} = 1, \quad u = 1,\dots,N, \tag{3a}$$

$$m_{uv} = m_{vu} \quad \forall u,v, \quad m_{vv} = 0 \quad \forall v. \tag{3b}$$

We denote by m^* the optimal matching, and by $\mathcal{E}_w^* := \mathcal{E}_w(\mathsf{m}^*)$ the optimal cost.

The study of the matching problem has a very long tradition in the literature. It is well known that, from the algorithmic point of view, the problem belongs to the P computational complexity class, as Kuhn [1] proved in 1955. Edmonds, Edmonds and Karp [2,3] later extended and improved the original result of Kuhn, showing that, for a matching problem on a generic graph G with V vertices and E edges, the optimal matching can be found in $O(V E \ln E)$ iterations. Matching problems have an important theoretical relevance, but they also appear in many practical applications, such as computer vision [4], control theory [5,6], and pattern matching [7] among many other fields.

Aside with the purely algorithmic aspects of the problem, however, the study of the *typical* properties of the solution of a given optimization problem, with respect to an ensemble of realizations, is of a certain interest. For this reason, in a set of seminal contributions, Orland [8] and Mézard and Parisi [9–12] analyzed *random* matching problems. Their works paved the way to the application of analytical tools from the theory of disordered systems to many other combinatorial optimization problems [13,14]. In most statistical physics literature, the cost coefficients $\{w_{uv}\}_{uv}$ of a random matching problem are taken to be i.i.d. random variables (random link

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approximation). The average optimal cost $\overline{\mathcal{E}_w^*}$ for the random matching problem has been derived, under this assumption, in Ref. [9], in the large N limit. The results in Ref. [9] were later rigorously proved by Aldous [15]. A similar analysis has been performed for the random *bipartite* matching problem (or assignment problem), i.e., the random matching problem defined on a bipartite graph, in which two types of vertexes to be matched appear. A conjecture on the finite size corrections to the average optimal cost for the assignment problem was proposed in Ref. [16] and generalized by Coppersmith and Sorkin [17]. This conjecture was later proved independently by Linusson and Wästlund [18] and Nair *et al.* [19]. In Ref. [20] finite size corrections to the average optimal cost both in the random matching problem and in the random assignment problem were analyzed, using replica techniques. Many other results about random matching problems have been obtained in recent years. In particular, the theory of cavity method [21] has been successfully applied to the study of matching problems in general [22], an example being the evaluation of the number of solutions of the problem on sparse random graphs [23], or the study of the multi-index matching problem [24]. The application of the cavity method (called belief propagation in its algorithmic version) to the assignment problem has been rigorously justified by Bayati *et al.* [25].

Let us discuss now a variation of the random matching problem, in which the cost coefficients $\{w_{uv}\}_{uv}$ appearing in Eq. (1) are correlated random variables, due to an underlying Euclidean structure. In particular, we associate with each vertex v of the complete graph a point $\mathbf{x}_v$ in the d-dimensional unit hypercube $[0,1]^d$. Then we will consider the cost coefficients to be given by

$$w_{uv} := \|\mathbf{x}_u - \mathbf{x}_v\|^p, \quad p > 0. \tag{4}$$

In the expression above, $\| \bullet \|$ is the Euclidean norm. The points $\{\mathbf{x}_v\}_v$ are assumed to be independently and uniformly distributed in $[0,1]^d$, and, as usual, we are interested in the asymptotic limit of the average (over the points' distribution) of the optimal cost. The formulated problem is therefore called a (random) *Euclidean* matching problem (EMP). In Fig. 1 we present a pictorial representation of an instance of the EMP on the unit square. In the bipartite version of the EMP, or Euclidean assignment problem (EAP), two sets of points with the same cardinality are randomly generated in a certain domain, and we ask for the average optimal cost of the matching among them, requiring that points of different type only are matched.

The EMP has been investigated by Mézard and Parisi [12] assuming the Euclidean correlation among the weights as a perturbation to the purely random case. The adopted strategy was to include, in a replica computation, only triangular correlation (i.e., the correlation among three weights), neglecting higher orders. This approach was proved successful, as numerically verified in Ref. [26]. Their work inspired the present contribution and will be therefore discussed more carefully below. Many results have been obtained for the EAP as well. In particular, apart from fundamental geometric properties of the solution [27] and rigorous results on the scaling of the optimal cost [28–30], a successful ansatz for the $p = 2$ case was recently proposed in Refs. [31,32] for the evaluation of the average optimal cost and of the correlation
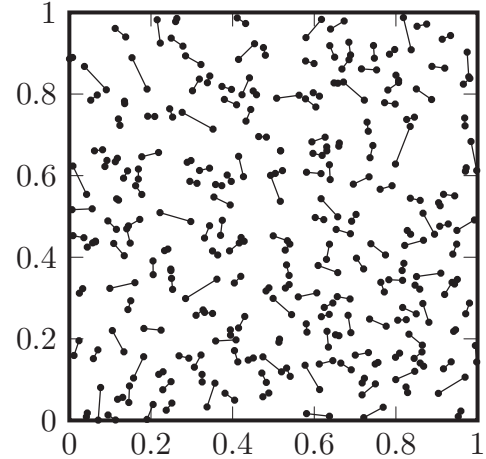

FIG. 1. An example of optimal matching among uniformly distributed points on the unit square, assuming open boundary conditions. The cost matrix is given by the Euclidean distances among the points, i.e., it has the expression given in Eq. (4) with $p = 1$. Observe also that, for $p = 1$ and $d = 2$, no crossing appears between links in the optimal solution, as a consequence of the triangular inequality.

functions of the solution. This ansatz was later justified through a functional approach [33,34]. In the one-dimensional case, in particular, a correspondence between the solution of the problem and a Gaussian stochastic process emerged [34–36].

In the present paper we will consider the EMP on the unit hypercube in d dimensions [37]. We will improve the calculation of Mézard and Parisi [12], going beyond the triangular approximation and including all one-loop, or polygonal, corrections to the pure mean field case. In particular, we will show that polygonal corrections can be written down, after some calculations, in a numerically manageable form. The paper is organized as follows. In Sec. II we set up a replicated formalism for the EMP, dealing with Euclidean correlations through Mayer's cluster expansion. We provide also a set of diagrammatic rules emerging from the theory that allow us to evaluate the contribution of each diagram in the expansion. In Sec. III we proceed performing a saddle point action approximation and, moreover, imposing a replica symmetric assumption. In Sec. IV we focus our attention on a specific class of diagrams appearing in the expansion, e.g., the class of one-loop diagrams, which we call polygons: we treat the polygonal contribution using a (replicated) transfer matrix formalism. In Sec. V the asymptotic cost in the polygonal approximation is computed for different dimensions d and for $p = 1$. We also give some details on the spectral properties of the transfer matrix operators and highlight the presence of a null mass at zero momentum. Finally, in Sec. VI we will show how, from the formal structure of the polygonal series, some nontrivial finite size correction exponents in the random link case can be derived.

II. CLUSTER EXPANSION

As usual in statistical physics' analysis of optimization problems, and following Mézard and Parisi, Mézard and

Parisi [9,12], we shall associate a partition function to a given instance of the EMP. Let us assume that a set of N points $\{\mathbf{x}_v\}_{v=1,\dots,N}$ is given on the d-dimensional unit hypercube $[0,1]^d$, with associated cost matrix elements $w_{uv} := \|\mathbf{x}_u - \mathbf{x}_v\|^p$, with $p > 0$. The points $\{\mathbf{x}_v\}_v$ are supposed uniformly and independently distributed in $[0,1]^d$. We define

$$
Z_w(\beta) := \sum_{\text{matchings } \mathsf{m}} e^{-\beta N^{\frac{p}{d}} \mathcal{E}_w(\mathsf{m})}
$$

$$
= \left(\prod_{v=1}^{N} \int_0^{2\pi} \frac{e^{i\lambda_v} d\lambda_v}{2\pi} \right) \prod_{u<v} [1 + e^{-\beta N^{\frac{p}{d}} w_{uv} - i\lambda_u - i\lambda_v}],
$$

(5)

where the Lagrange multipliers $\{\lambda_v\}_v$ enforce the constraints in Eq. (3). As discussed in Ref. [12], the factor $N^{\frac{p}{d}}$ is necessary in order to have an appropriate large N limit for thermodynamic functions, when the average over points' positions is considered. Denoting by $\overline{\bullet}$ the expectation over the points' positions, the average free energy density of the system is given by

$$
f(\beta) := \lim_{N\to\infty} -\frac{1}{\beta N} \overline{\ln Z_w(\beta)}.
$$

(6)

It is convenient to define a rescaled average optimal cost $\hat{\mathcal{E}}$, so that the following relations hold:

$$
\hat{\mathcal{E}} = \lim_{N\to\infty} \overline{N^{\frac{p}{d}-1} \mathcal{E}_w^*} = \lim_{\beta\to\infty} f(\beta).
$$

(7)

We deal with the average over the disorder using the replica trick

$$
f(\beta) = \lim_{N\to\infty} \lim_{n\to 0} \frac{1 - \overline{Z_w^n(\beta)}}{nN\beta}.
$$

(8)

As usual we will consider an integer number n of replicas during the computation, and then we will perform analytic continuation for $n \downarrow 0$. The average replicated partition function reads

$$
\overline{Z_w^n} = \left(\prod_{a=1}^{n} \prod_{v=1}^{N} \int_0^{2\pi} \frac{e^{i\lambda_v^a} d\lambda_v^a}{2\pi} \right) \overline{\prod_{u<v} (1 + \mu_{uv})},
$$

(9)

where we have introduced the quantity

$$
\mu_{uv} := \sum_{r=1}^{n} e^{-r\beta N^{\frac{p}{d}} w_{uv}} \sum_{1 \leqslant a_1 < \cdots < a_r \leqslant n} e^{-i \sum_{m=1}^{r} \left(\lambda_u^{a_m} + \lambda_v^{a_m} \right)}.
$$

(10)

In the random link matching problem the average

$$
\overline{\prod_{u<v} (1 + \mu_{uv})}
$$

(11)

is easily performed, using the fact that the joint probability distribution of the weights $\{w_{uv}\}_{uv}$ factorizes [9]. In our case, however, this is not true anymore, due to the underlying Euclidean structure. In particular, the function μ_{uv} depends on the vertexes u and v because of both the Euclidean distance $\|\mathbf{x}_u - \mathbf{x}_v\|$, and the two sets of Lagrange multipliers $\{\lambda_u^a\}_a$ and $\{\lambda_v^a\}_a$. The quantity in Eq. (11) can be therefore represented through a diagrammatic expansion, in complete analogy with the classical cluster expansion [38–40] introduced by Mayer

FIG. 2. Biconnected graphs up to four vertexes. The corresponding symmetry factors σ_g are $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{1}{24}$ from left to right.

and Mayer [41], with μ_{uv} playing the role of the Mayer function. In particular, applying the results of Pulvirenti and Tsagkarogiannis [42], we can write

$$
\overline{\prod_{u<v} (1 + \mu_{uv})} \sim \exp \left(\sum_{\substack{\mathsf{g} \subseteq \mathsf{K}_N \\ \text{biconn.}}} \overline{\prod_{e \in \mathsf{g}} u_e} \right).
$$

(12)

The sum on the right-hand side runs over all biconnected subgraphs g of the complete graph K_N. A biconnected graph is a graph that remains connected after the removal of any vertex with all adjacent edges (see Fig. 2). Here and in the following we will denote by $(u,v) \in \mathsf{g}$, or equivalently $e \in \mathsf{g}$, an edge of the graph g. Moreover, we will denote by E_g and V_g, or simply E and V, the number of edges and the number of vertexes in g, respectively. In the mean field approximation only subgraphs with $E = 1$ are considered. The average appearing in the arguments of the sum in Eq. (12) removes the dependencies on the point positions. However, each contribution still depends on the indexes of the vertexes of the specific subgraph through the Lagrange multipliers $\{\lambda_v^a\}_{v,a}$. We introduce therefore a set of order parameters, symmetric under permutations of replica indexes, defined by

$$
Q_{a_1,\dots,a_k} := \frac{1}{N} \sum_{v=1}^{N} \exp\left(-i \sum_{j=1}^{k} \lambda_v^{a_j} \right), \quad 1 \leqslant k \leqslant n, \quad (13)
$$

and the associated Lagrange multipliers $\hat{Q}_{a_1,\dots,a_k}$. In the large N limit, for a given subgraph g, there are approximately $N^V \sigma_\mathsf{g}$ subgraphs in K_N isomorphic to g, σ_g^{-1} being the number of automorphisms of g. With these considerations in mind, and after some simple manipulations, we can write

$$
\sum_{\substack{\mathsf{g} \subseteq \mathsf{K}_N \\ \text{biconn.}}} \overline{\prod_{e \in \mathsf{g}} u_e} = -n\beta N \sum_{\mathsf{g}}' S_\mathsf{g}[\beta, Q].
$$

(14)

The primed sum runs over all biconnected graphs with vertices labeled in $[V]$, for $2 \leqslant V \leqslant N$, considered up to an automorphism. In Eq. (14) we have collected a factor $-n\beta N$ for later convenience. The contribution of each graph is given by

$$
-n\beta S_\mathsf{g}[\beta, Q] := \sigma_\mathsf{g} \sum_{\{\mathbf{a}^e\}_e} \overline{\prod_{e \in \mathsf{g}} e^{-\beta |\mathbf{a}^e| w_e}}^{\mathsf{g}} \prod_{v=1}^{V} Q_{\mathbf{a}(v)} \delta_{\mathbf{a}(v)}.
$$

(15)

Here, for each edge $e = (u,v)$, we have a sum over all nonempty subsets $\mathbf{a}^e := \{a_k^e\}_k \subseteq [n]$, whose cardinalities are denoted by $|\mathbf{a}^e|$. We have also defined

$$
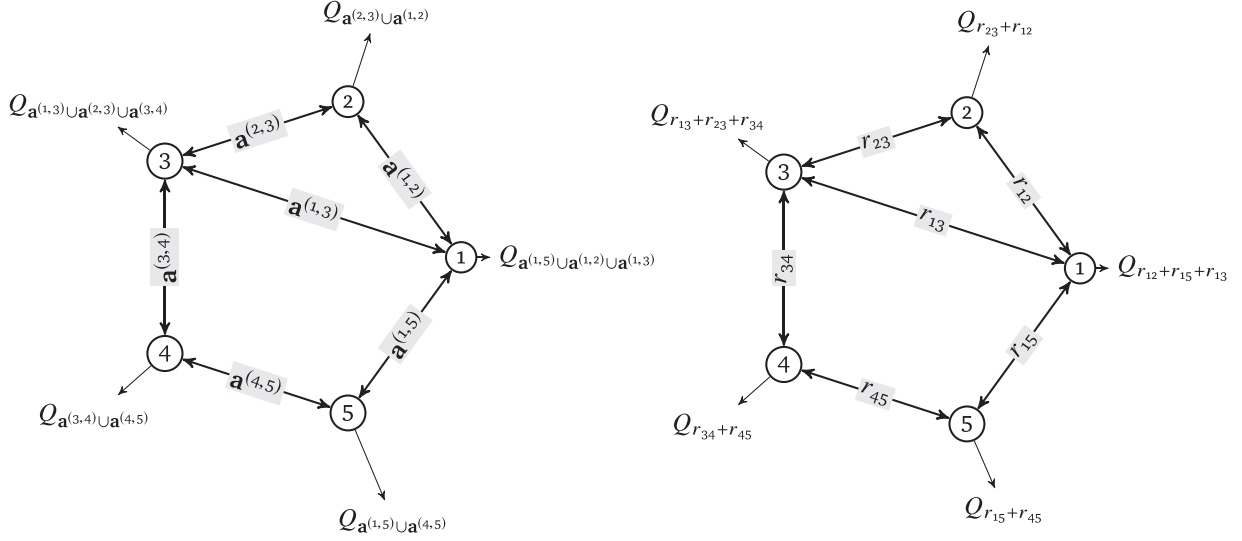\mathbf{a}(v) := \bigcup_{u \in \partial v} \mathbf{a}^{(u,v)},
$$

(16)

FIG. 3. Pictorial representation of the construction of the replica indexes before and after the replica symmetric assumption for a biconnected graph.

union over the set ∂v of the vertexes adjacent to v in g (see Fig. 3). The indicator function $\delta_{\mathbf{a}(v)}$ takes value one if the incident edges of v have distinct replica indexes, zero otherwise. This implies

$$r_e := |\mathbf{a}(v)| = \sum_{u \in \partial v} |\mathbf{a}^{(u,v)}|. \quad (17)$$

Finally, the average in Eq. (12) must be performed using the joint costs' distribution for a graph g,

$$\rho_{\mathrm{g}}(\{w_e\}_e) = \left(\prod_{u=1}^{V} \int_{\mathbb{R}^d} d^d x_u \right) \delta(\mathbf{x}_1)$$
$$\times \prod_{(u,v) \in \mathrm{g}} \delta(w_{uv} - \|\mathbf{x}_u - \mathbf{x}_v\|^p). \quad (18)$$

Let us make, now, a final remark. The strategy of Ref. [12] was to perform the explicit computation of $\rho_{\mathrm{K}_3}(w_{12}, w_{23}, w_{31})$ for the triangular graph $\mathrm{g} \equiv \mathrm{K}_3$. Since this procedure is not easily generalizable, we can adopt a different approach. We can assign a momentum to each edge in the graph g, writing Eq. (15) in the Fourier space as

$$-n\beta S_{\mathrm{g}}[\beta, Q]$$

$$= \frac{\sigma_{\mathrm{g}}}{(2\pi)^{d(E-V+1)}} \prod_{e \in \mathrm{g}} \left[\sum_{\mathbf{a}^e} \int d^d k_e \; g_{|\mathbf{a}^e|}(k_e) \right]$$

$$\times \prod_{v=1}^{V} \left[Q_{\mathbf{a}(v)} \, \delta_{\mathbf{a}(v)} \, \delta \left(\sum_{\substack{u \in \partial v \\ \text{ingoing}}} \mathbf{k}_{uv} - \sum_{\substack{u \in \partial v \\ \text{outgoing}}} \mathbf{k}_{uv} \right) \right]. \quad (19)$$

In the equation above $k_e := \|\mathbf{k}_e\|$ and

$$g_r(k) := \Omega_d \int_0^\infty z^{d-1} e^{-r\beta z^p} {}_0F_1 \left[\begin{matrix} - \\ \frac{d}{2} \end{matrix}; -\frac{k^2 z^2}{4} \right] dz, \quad (20)$$

where ${}_0F_1$ is a confluent hypergeometric function, defined as

$${}_0F_1 \left[\begin{matrix} - \\ b \end{matrix}; z \right] := \sum_{k=0}^\infty \frac{\Gamma(b)}{\Gamma(b+k)} \frac{z^k}{k!}, \quad (21)$$

and we have introduced the unit sphere's surface in $d-1$ dimensions

$$\Omega_d := \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \quad (22)$$

Note that a Dirac's delta function enforces the conservation of momentum on each vertex. As an additional prescription, one of the Dirac's delta has to be considered a Kronecker's delta in order to avoid an extra volume contribution. Feynman rules characterizing a generic diagram g for the construction of Eq. (19) are, at this point, given. An alternative and equivalent formulation for the momentum integration is given in Appendix D.

III. REPLICA SYMMETRIC ASSUMPTION AND SADDLE POINT APPROXIMATION

The results in the previous section allow us to express the replicated partition function in a form that is suitable for a saddle point computation, i.e.,

$$\overline{Z_w^n} \sim \left[\prod_{\mathbf{a}} \int_{-\infty}^{+\infty} dQ_{\mathbf{a}} \int_{-i\infty}^{+i\infty} \frac{d\hat{Q}_{\mathbf{a}}}{2\pi} \right] e^{-n\beta N S[\beta, Q, \hat{Q}]}, \quad (23)$$

where the product runs over the nonempty subsets of replica indexes $\mathbf{a} \subseteq [n]$. The action in the exponent in Eq. (23) has the structure

$$S[\beta, Q, \hat{Q}] := S_{\mathrm{mf}}[\beta, Q, \hat{Q}] + \sum_{\mathrm{g}: E_{\mathrm{g}} \geqslant 3}' S_{\mathrm{g}}[\beta, Q]. \quad (24)$$

The first contribution is the mean field term, corresponding to the biconnected graph with one edge only, plus other terms deriving from the constraints imposed for the introduction of

the order parameters $Q_{\mathbf{a}}$ and $\hat{Q}_{\mathbf{a}}$. It is given by

$$
\begin{aligned}
&-n\beta S_{\mathrm{mf}}\\
&= -\sum_{\mathbf{a}} Q_{\mathbf{a}}\,\hat{Q}_{\mathbf{a}} + \frac{1}{2}\sum_{\mathbf{a}} g_{|\mathbf{a}|}(0)\, Q_{\mathbf{a}}^2\\
&\quad + \ln\left[\prod_{a=1}^n \int_0^{2\pi} \frac{e^{i\lambda^a}\,d\lambda^a}{2\pi}\exp\left(\sum_{\mathbf{a}}\hat{Q}_{\mathbf{a}}\,e^{-i\sum_{l=1}^{|\mathbf{a}|}\lambda^{a_l}}\right)\right].
\end{aligned}
$$

$$(25)$$

Before performing the analytic continuation for small n, we assume a *replica symmetric ansatz*, i.e.,

$$
Q_{\mathbf{a}} \equiv Q_{|\mathbf{a}|} \quad\text{and}\quad \hat{Q}_{\mathbf{a}} \equiv \hat{Q}_{|\mathbf{a}|}. \tag{26}
$$

It is convenient, in order to take the $n\downarrow 0$ and $\beta\uparrow\infty$ limits, to introduce a functional representation for the order parameters, namely

$$
G(x) := \sum_{r=1}^{\infty} (-1)^{r-1}\frac{\hat{Q}_r\,e^{\beta r x}}{r!}. \tag{27}
$$

The saddle point condition with respect to $\{\hat{Q}_r\}_r$ in the $n\to 0$ limit immediately yields

$$
\frac{\delta S}{\delta\hat{Q}_r} = 0 \Rightarrow Q_r = \beta\int \frac{e^{\beta r x - G(x)}}{(r-1)!}\,dx. \tag{28}
$$

If we restrict ourselves to the mean field approximation, in the limit $n\downarrow 0$ we can express the saddle-point mean field action as function of G only (see Appendix A for a detailed computation):

$$
\begin{aligned}
S_{\mathrm{mf}} &= -\int (e^{-e^{\beta x}} - e^{-G(x)})\,dx + \int G(x)e^{-G(x)}\,dx\\
&\quad - \frac{1}{2}\int \rho(w)e^{-G(x)-G(y)}\frac{\partial J_0(2e^{\beta\frac{x+y-w}{2}})}{\partial x}\,dx\,dy\,dw,
\end{aligned}
$$

$$(29)$$

where $J_0(x)$ is a Bessel function of the first kind and

$$
\rho(w) = \frac{\Omega_d}{p}\,w^{\frac{d}{p}-1}\theta(w) \tag{30}
$$

is the distribution of the weight appearing in the graph with $E=1$. Taking the zero temperature limit of Eq. (29), we obtain the mean field cost

$$
\begin{aligned}
\mathcal{E}_{\mathrm{mf}} &= -\int [\theta(-x) - e^{-G(x)}]\,dx + \int G(x)e^{-G(x)}\,dx\\
&\quad + \frac{1}{2}\int \rho(x+y)\,e^{-G(x)-G(y)}\,dx\,dy.
\end{aligned}
$$

$$(31)$$

The saddle point condition for $G(u)$ in the mean field approximation is then

$$
\frac{\delta\mathcal{E}_{\mathrm{mf}}}{\delta G(u)} = 0 \Rightarrow G(u) = \int \rho(w)\,e^{-G(w-u)}\,dw, \tag{32}
$$

to be used in Eq. (31) to obtain the mean field approximation to the optimal cost. As anticipated, the mean field case was discussed in Ref. [9] in the study of the random link matching problem.

If we consider, instead, the complete action, each term S_{g} gives a correction to the mean field contribution that, in general, is of the same order of the mean field contribution itself, being the dependence of S_{g} from N already factorized out for large N, as in Eq. (23). However, it has been observed by Houdayer *et al.* [26] that the contribution of the different graphs is exponentially small in the dimension d of the Euclidean space. Defining the zero-temperature limits $\mathcal{E} := \lim_{\beta\uparrow\infty} S$ and $\mathcal{E}_{\mathrm{g}} := \lim_{\beta\uparrow\infty} S_{\mathrm{g}}$, which can be conveniently considered as functionals of $G(u)$, saddle point extremization gives

$$
\begin{aligned}
\frac{\delta\mathcal{E}}{\delta G(u)} = 0 &\Rightarrow G(u) = \int \rho(w)e^{-G(w-u)}\,dw\\
&\quad - e^{G(u)}\sum_{\mathrm{g}:E_{\mathrm{g}}\geqslant 3}{}' \frac{\delta\mathcal{E}_{\mathrm{g}}}{\delta G(u)}.
\end{aligned}
$$

$$(33)$$

The resulting order parameter G can then be used to evaluate the average optimal cost

$$
\hat{\mathcal{E}} = \mathcal{E}_{\mathrm{mf}} + \sum_{\mathrm{g}:E_{\mathrm{g}}\geqslant 3}{}' \mathcal{E}_{\mathrm{g}}. \tag{34}
$$

In the next section we take a first step beyond the mean field approximation, considering the terms in the series corresponding to graphs having a single loop, and ignoring the others.

IV. ONE-LOOP CONTRIBUTIONS

In this section we consider the one-loop terms appearing in the action in Eq. (24). We denote by p_E the one-loop graph having E vertexes and E edges and we will use the term *polygon* for such graphs. Polygons appear also as first finite size corrections in random link matching problem [20,43,44] and as first corrections in certain perturbative expansions around the Bethe approximation [45–47]. We shall denote by $S_E := S_{\mathsf{p}_E}$ the contribution of the polygon p_E to the action in Eq. (19). The symmetry factor of a polygon p_E is given by

$$
\sigma_{\mathsf{p}_E} = \frac{1}{2E}. \tag{35}
$$

Neglecting nonpolygonal contributions, we thus approximate the full replicated action in Eq. (24) by

$$
S_{\mathrm{poly}} := S_{\mathrm{mf}} + \sum_{E=3}^{\infty} S_E. \tag{36}
$$

To explicitly compute the terms S_E, we can proceed in analogy with the computation performed in Ref. [20] for the finite size corrections in the random link problem. We introduce the $(2^n - 1)\times(2^n - 1)$ matrix $\mathsf{T}(k)$, also called replicated transfer matrix, whose elements are given by

$$
T_{\mathbf{a}\mathbf{b}}(k) := \delta_{\mathbf{a}\cap\mathbf{b}=\emptyset}\, Q_{|\mathbf{a}|+|\mathbf{b}|}\sqrt{g_{|\mathbf{a}|}(k)g_{|\mathbf{b}|}(k)}. \tag{37}
$$

Here $\mathbf{a}$ and $\mathbf{b}$ are, as before, nonvoid elements of the power set of the replica indexes $[n]$, whose cardinality is expressed as $|\mathbf{a}|$ and $|\mathbf{b}|$, respectively, and $\delta_{\mathbf{a}\cap\mathbf{b}=\emptyset}$ is defined by

$$
\delta_{\mathbf{a}\cap\mathbf{b}=\emptyset} = \begin{cases} 1 & \text{if } \mathbf{a}\cap\mathbf{b}=\emptyset,\\ 0 & \text{otherwise.} \end{cases} \tag{38}
$$

Therefore the contribution of the polygon p_E, according to Eq. (19) and under the replica symmetric assumption, can be

written as

$$-n\beta S_E = \frac{1}{2E}\frac{\Omega_d}{(2\pi)^d}\int_0^\infty k^{d-1}\,\mathrm{tr}[\mathsf{T}^E(k)]\,dk. \quad (39)$$

To proceed further, we will diagonalize $\mathsf{T}(k)$ following the classical strategy of de Almeida and Thouless [48] and already adopted in Ref. [20]. In fact, the next steps of our calculation, reported in Appendix B, differ from the ones of Ref. [20] in the random link problem for the presence of the momentum variable k only.

The matrix $\mathsf{T}(k)$ is invariant under permutations of the replica indexes, therefore we block diagonalize it according to the irreducible representations of the permutation group. The subspaces that are invariant under the action of the symmetry group are classified according to the number q of distinguished replica indexes, in some appropriate basis spawning them (see Refs. [44,49] for an application of the same procedure to disordered Ising models). Particular care has to be taken in the limits $n \downarrow 0$ followed by $\beta \uparrow \infty$. We give here only the final result, whereas the required computation is presented in Appendix B. The polygon cost functional $\mathcal{E}_E$ is divided into two terms: the first one $\mathcal{E}^{(01)}$, accounting for the contribution of the subspaces $q = 0,1$, corresponds to the so-called longitudinal and anomalous sectors in spin glass literature; the second one $\mathcal{E}^{(2+)}$, accounts for all the other subspaces, $q \geqslant 2$, and it is nonzero for E odd only. The average optimal cost functional is thus given by

$$\mathcal{E}_{\mathrm{poly}} = \mathcal{E}_{\mathrm{mf}} + \sum_{E=3}^\infty \left(\mathcal{E}_E^{(01)} + \mathcal{E}_E^{(2+)}\right). \quad (40a)$$

The term $\mathcal{E}_{\mathrm{mf}}$ here is given by Eq. (31). The contributions $\mathcal{E}_E^{(01)}$ and $\mathcal{E}_E^{(2+)}$, with $\mathcal{E}_E = \mathcal{E}_E^{(01)} + \mathcal{E}_E^{(2+)}$ are given by

$$\mathcal{E}_E^{(01)} := \frac{(-1)^E\Omega_d}{2(2\pi)^d}\int_0^\infty k^{d-1}\,\mathrm{tr}[\mathsf{H}^{E-1}(0,k)\mathsf{K}(k)]\,dk, \quad (40b)$$

$$\mathcal{E}_E^{(2+)} := \begin{cases} \frac{\Omega_d}{E(2\pi)^d}\iint_0^\infty k^{d-1}\,\mathrm{tr}[\mathsf{H}^E(t,k)]\,dt\,dk & E\ \text{odd}, \\ 0 & E\ \text{even}. \end{cases} \quad (40c)$$

In the equations above we have introduced the operator

$$[\mathsf{H}(t,k)]_{uv} :$$

$$= \Omega_d\, e^{-\frac{G(u)+G(v)}{2}}\frac{x^{\frac{d}{p}-1}\,_0F_1\left[\frac{-}{\frac{d}{2}};-\frac{k^2x^{\frac{2}{p}}}{4}\right]\theta(x)}{p}\Bigg|_{x=u+v-2t}, \quad (41a)$$

and the operator

$$[\mathsf{K}(k)]_{uv} := \Omega_d\, e^{-\frac{G(u)+G(v)}{2}}\frac{x^{\frac{d}{p}}}{d}\,_0F_1$$

$$\times \left[\frac{-}{\frac{d}{2}+1};-\frac{k^2x^{\frac{2}{p}}}{4}\right]\theta(x)\Bigg|_{x=u+v}. \quad (41b)$$

As anticipated, the contribution $\mathcal{E}_E^{(2+)}$ has an expression that is analogous to the finite size corrections computed in Ref. [20]

for the random link matching problem, while the sectors with $q = 0,1$ produce a contribution $\mathcal{E}_E^{(01)}$ that has no equivalent in that computation.

The general saddle point equation for G is given by Eq. (33). However, keeping the polygonal contribution only, we can approximate Eq. (33) by

$$G(u) = \int \rho(w)e^{-G(w-u)}\,dw - e^{G(u)}\sum_{E=3}^\infty\left[\frac{\delta\mathcal{E}_E^{(01)}}{\delta G(u)} + \frac{\delta\mathcal{E}_E^{(2+)}}{\delta G(u)}\right]. \quad (42a)$$

The functional derivatives in Eq. (42a) are given by

$$\frac{\delta\mathcal{E}_E^{(01)}}{\delta G(u)} = \frac{(-1)^E\Omega_d}{2(2\pi)^d}$$

$$\times \int_0^\infty k^{d-1}\sum_{m=0}^{E-1}[\mathsf{H}^{E-1-m}(0,k)\mathsf{K}(k)\mathsf{H}^m(0,k)]_{uu}\,dk \quad (42b)$$

and similarly

$$\frac{\delta\mathcal{E}_E^{(2+)}}{\delta G(u)} = -\frac{\Omega_d}{(2\pi)^d}\iint_0^\infty k^{d-1}[\mathsf{H}^E(t,k)]_{uu}\,dk\,dt. \quad (42c)$$

The computation of the spectra of $\mathsf{H}(t,k)$ and $\mathsf{K}(k)$ allows us to evaluate the polygonal correction both to the average optimal cost and to the saddle point solution for $G(u)$. The results of this computation will be presented in the next section.

V. NUMERICAL RESULTS

In order to compute, for a given dimension d and cost exponent p, the polygonal approximation to the average optimal cost, we have to evaluate the cost functional $\mathcal{E}_{\mathrm{poly}}$, given in Eq. (40a), on the solution G_{poly} of the saddle point equation (42a). However, a naïve numerical computation of the terms of the series becomes rapidly infeasible as the number of edges E increases [e.g., the term $\mathcal{E}_E^{(2+)}$ in Eq. (40c) involves $E + 2$ integrations].

We adopted therefore a different strategy. We evaluated the spectrum of a discretized representation of the operator $\mathsf{H}(t,k)$. Typically, few of the largest eigenvalues, and the corresponding eigenvectors, are sufficient to approximate the operator within the required precision. The infinite sum over E in Eqs. (40a) and (42a) could also eventually be taken before the integrations in k and t. Proceeding in this way, we managed to compute efficiently $\mathcal{E}_{\mathrm{poly}}$ from Eq. (40a) for any given $G(u)$, using the expressions

$$\sum_{E\geqslant 3}\mathcal{E}_E^{(01)} = \frac{\Omega_d}{2(2\pi)^d}\int_0^\infty k^{d-1}\sum_\lambda\frac{-\lambda^2}{1+\lambda}\langle\lambda|\mathsf{K}(k)|\lambda\rangle\,dk, \quad (43)$$

$$\sum_{E\geqslant 3}\mathcal{E}_E^{(2+)} = \frac{\Omega_d}{(2\pi)^d}\iint_0^\infty k^{d-1}\sum_\lambda[\mathrm{atanh}(\lambda) - \lambda]\,dt\,dk, \quad (44)$$

where the sum runs over the eigenvalues of $\mathsf{H}(0,k)$ and $\mathsf{H}(t,k)$ and we omitted the dependence of λ from t and k.

(a) Eigen values of $\mathsf{H}(0,k)$.
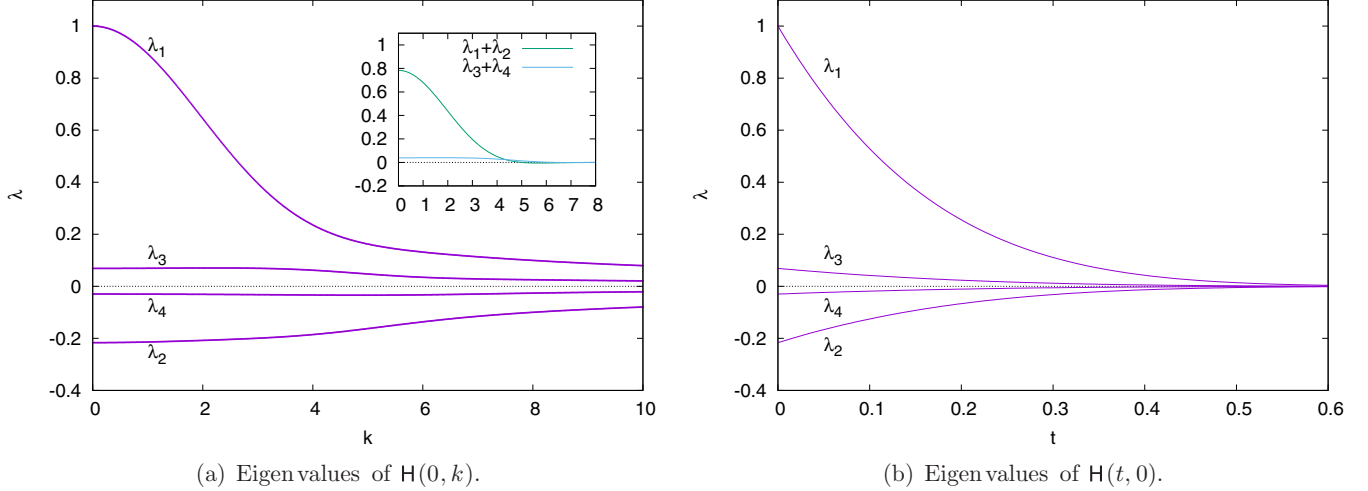

(b) Eigen values of $\mathsf{H}(t,0)$.

FIG. 4. The four largest magnitude eigenvalues of the operator $\mathsf{H}(t,k)$ as a function of the momentum k for $t=0$ (*left*) and as function of t for $k=0$ (*right*).

On the other hand, the computation of G_{poly} through iterations of Eq. (42a) proved to be much harder than in the mean field case, due to some numerical instabilities that prevented the iterative procedure to reach a fixed point, even truncating the expression to the $E=3$ term. We took the alternative approach of dealing with $\sum_E \mathcal{E}_E$ as a perturbation to $\mathcal{E}_{\text{mf}}$, evaluating $\mathcal{E}_{\text{poly}}$ in Eq. (40a) on G_{mf}, solution of Eq. (32). Observe that $G_{\text{mf}}(u)\,e^{-\sqrt{G_{\text{mf}}(u)}}$ is the leading eigenfunction of the operator $\mathsf{H}(0,0)$ with eigenvalue $\lambda_1(0,0)=1$, that is, we have a theory with a zero mass when $t=k=0$. The commutation of sum and integral leading to Eq. (44) is justified in spite of the singularity in the integrand for $t=k=0$, once one takes into account the behavior of the largest eigenvalue $\lambda_1 \sim e^{-at-bk^2}$, with $a,b>0$ and for small t and k, as we checked numerically (see also Fig. 4) and analytically (using perturbation theory).

We report the results of our estimates for the (rescaled) average optimal cost in Table I, in the case $p=1$ and using the first six eigenvalues of $\mathsf{H}(t,k)$. Our analytical predictions for $\hat{\mathcal{E}}$ are compared to the numerical values $\mathcal{E}_{\text{num}}$ obtained in Ref. [26], where the authors applied an exact algorithm to random instances of the EMP and averaged over many samples. Also, for comparison, we report the values obtained

for

$$\mathcal{E}_\triangle := \mathcal{E}_{\text{mf}} + \mathcal{E}_3, \tag{45}$$

with $\mathcal{E}_{\text{mf}}$ and $\mathcal{E}_3 = \mathcal{E}_3^{(01)} + \mathcal{E}_3^{(2+)}$ given by Eq. (31) and Eqs. (40), respectively. $\mathcal{E}_\triangle$ is therefore the cost comprehensive of the triangular correlations only, as considered in Refs. [12,26]. In Appendix C we show how our expression for $\mathcal{E}_3$ obtained through diagonalization in the invariant subspaces of the replica permutations group can be mapped into the expression given in Ref. [12].

We also defined $G_\triangle$ to be the saddle point solution for G corresponding to the triangular approximation $\mathcal{E}_\triangle$, and we computed it according to Eq. (34) of Ref. [12]. Note that a small mistake appears there in the final formulas [50]. Our results for $\mathcal{E}_\triangle$ computed on $G_\triangle$ are slightly different from the ones reported in Ref. [26] (β^{EC} in Table 5 of that paper). Since their numerical results were based on the analytical results in Ref. [12], we suspect that the discrepancy is due to the aforementioned error, that went unnoticed.

The results in Table I show that $\mathcal{E}_{\text{poly}}$, computed as a perturbation to $\mathcal{E}_{\text{mf}}$, is a consistent improvement over the mean-field result in any dimension. Comparison with $\mathcal{E}_\triangle$ in $G_\triangle$ is unfavorable in high dimension, though. Further investigation using the appropriate saddle point G_{poly} are due to assess the relevance of this particular diagrammatic class in the cluster expansion.

TABLE I. Comparison of the analytical predictions for the average optimal cost for many dimension and for $p=1$. The values for $\mathcal{E}_{\text{num}}$, corresponding to the average matching cost obtained by an actual matching procedure, are taken from Ref. [26].

d	$\mathcal{E}_{\text{num}}$	$\mathcal{E}_{\text{mf}}$ in G_{mf}	$\mathcal{E}_\triangle$ in G_{mf}	$\mathcal{E}_{\text{poly}}$ in G_{mf}	$\mathcal{E}_\triangle$ in $G_\triangle$
1	0.5	0.4112335	0.33624	–	0.33623
2	0.3104(2)	0.3225805	0.29699	0.31376	0.30291
3	0.3172(2)	0.3268392	0.31255	0.31998	0.31536
4	0.3365(3)	0.3432274	0.33399	0.33809	0.33554
5	0.3572(2)	0.3621749	0.35577	0.35825	0.35669
6	0.3777(1)	0.3814168	0.37678	0.37838	0.37735

VI. SUB-SUB-LEADING CORRECTION IN THE RANDOM LINK PROBLEM

We reconsider now the random link matching problem, that is the matching problem on K_N with costs independently and uniformly distributed in the interval $[0,1]$. The average optimal cost $\overline{\mathcal{E}_{\text{RL}}^*}$ has a finite asymptotic limit, computed in Ref. [9] through the replica method, as

$$\lim_{N\to\infty} \overline{\mathcal{E}_{\text{RL}}^*} \equiv \mathcal{E}_{\text{mf}} = \frac{\pi^2}{12}. \tag{46}$$

The $O(1/N)$ correction to the asymptotic cost has been obtained in Refs. [51–53]. In particular, in Ref. [52] it is shown that for large N,

$$\overline{\mathcal{E}^*_{\text{RL}}} = \mathcal{E}_{\text{mf}} + \frac{\Delta\mathcal{E}}{N} + o\left(\frac{1}{N}\right), \qquad (47a)$$

with

$$\Delta\mathcal{E} = \frac{\zeta(2)}{4} - \frac{\zeta(3)}{2} + \sum_{\substack{E \geqslant 3 \\ E \text{ odd}}} \frac{1}{2E} \int_0^\infty \text{tr}[\mathsf{H}^E(t,0)]\, dt, \quad (47b)$$

where $\zeta(z)$ is the Riemann zeta function. Here we recognize the same structure of the polygonal expansion in the Euclidean case. The main differences are the absence of the momentum integration and of the $\mathcal{E}^{(01)}$ term, which is equal to zero in this case [52]. The operator H is in fact the same we have defined in Eq. (41a) for our one-loop computation in the EMP, assuming $d = p = 1$.

We will show now how the particular form of Eqs. (47) allows us to predict the scaling with N of the next order finite size correction in the random link matching problem. Let us start observing that, for large E, the integral is dominated by the region around $t = 0$. We assume the behavior

$$\text{tr}[\mathsf{H}^E(t,0)] \sim \lambda^E(t) \sim \lambda^{-ctE}, \qquad (48)$$

for small t and large E, where the coefficient $c > 0$ can be explicitly computed using perturbation theory. Performing the t integration, we find that the coefficients of the series in Eq. (47b) decay as E^{-2}. We can extract the sub-sub-leading scaling with N of the optimal cost, which is due to counting correction in the number of loops at finite N, using a simple heuristic argument. A random path on the complete graph K_N of length ℓ has a probability of intersecting itself in the next step of order ℓ/N. Therefore, for a random path of length E the total probability of intersection is of order E^2/N and a crossover arises at the scale $E \sim \sqrt{N}$. As a consequence, at finite N, the sum in Eq. (47b) should be opportunely regularized. Choosing an appropriate regularizing function $f(x)$, with limits 1 and 0 for $x \downarrow 0$ and $x \uparrow \infty$, respectively, we have the relation

$$\sum_E \frac{1}{E^2} f\left(\frac{E}{\sqrt{N}}\right) \sim a + \frac{b}{\sqrt{N}}, \qquad (49)$$

as it can be easily showed approximating the sum with an integral. With these assumptions the first two finite size corrections to the asymptotic cost take the form

$$\overline{\mathcal{E}^*_{\text{RL}}} \sim \frac{\pi^2}{12} + \frac{e_1}{N} + \frac{e_{3/2}}{N^{3/2}}. \qquad (50)$$

The anomalous 3/2 exponent obtained using this simple argument is indeed perfectly consistent with the numerical simulations we performed using an exact optimization algorithm [54], see Fig. 5.

A refined computation of the terms appearing in the $O(1/N)$ corrections gives $e_1 = 0.0674(1)$ [52,55], in agreement with our numerical data. From numerical fit we then obtain the estimate $e_{3/2} = -1.24(4)$ for the coefficient of the $O(1/N^{3/2})$ correction.
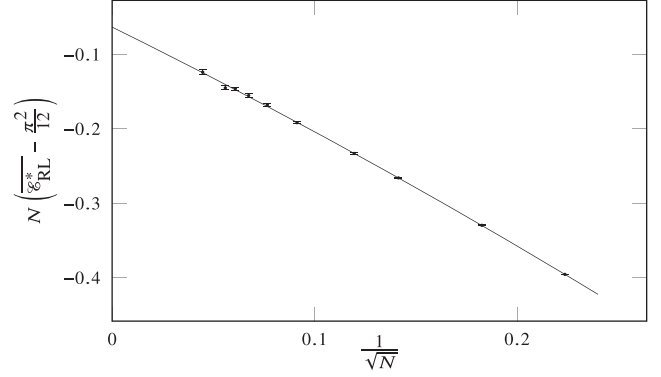

FIG. 5. Average optimal cost in the random link matching problem as a fuction of N^{-1}. Numerical data points are shown along a quadratic fit in $N^{-1/2}$. The fact that the data points are linearized by the chosen scaling of the axes implies a $O(N^{-3/2})$ finite size correction to the asymptotic average optimal cost.

The extension of these considerations to the polygonal contributions we computed in the Euclidean case to obtain a prediction for the exponent of the finite size correction remains an interesting open problem.

VII. CONCLUSIONS AND PERSPECTIVES

In the present work we have discussed the random EMP on the unit hypercube in the thermodynamic limit. We have adopted the classical replica approach. It is well known [12] that Euclidean correlations among weights can be considered as corrections to a mean field contribution corresponding to the purely random case. We have shown that the Euclidean corrections can be treated in a Mayer-type expansion of biconnected diagrams, each one of them representing a different order of correlation among weights. Subsequently we restricted our computation to the polygonal contribution in the replica symmetric hypothesis, showing that, in this case, the corrections can be properly evaluated using a transfer matrix approach. We have obtained an implicit expression for the average optimal cost in terms of the spectrum of two operators K and H. Finally, we have presented a numerical study of our results, comparing our predictions with the numerical simulations.

As specified above, in our calculation we did not evaluate nonpolygonal diagrams that should be included to obtain the leading contribution to the average optimal cost. These contributions correspond to the existence of additional inner loops. An analytic treatment of these contributions would greatly improve the final theoretical predictions. Moreover, other quantities of interest related to the problem, like correlation functions, were not considered here. A restatement of the previous results in a cavity method formalism is another interesting open problem.

APPENDIX A: MEAN FIELD ACTION

In the present Appendix we evaluate the mean field action presented in Sec. III. Let us start from the replicated action of Eq. (25) that under the replica symmetric hypothesis becomes

$$
n\beta S_{\mathrm{mf}}[\beta, Q, \hat{Q}] = \sum_{r \geqslant 1} \binom{n}{r} Q_r \, \hat{Q}_r - \frac{1}{2} \sum_{r \geqslant 1} \binom{n}{r} g_r(0) \, Q_r^2 - \ln \left[\prod_{a=1}^{n} \int_0^{2\pi} \frac{e^{i\lambda^a} \, d\lambda^a}{2\pi} \exp \left(\sum_{r \geqslant 1} \hat{Q}_r \sum_{a_1 < \cdots < a_r} e^{-i \sum_{l=1}^{r} \lambda^{a_l}} \right) \right]. \quad (A1)
$$

Let us now work out the $n \to 0$ limit. The result of this limit is presented already in the seminal work by Mézard and Parisi [9]. However, some intermediate nontrivial steps are missing in their exposition and therefore we present here a more detailed derivation. We start observing that

$$
\sum_{a_1 < \cdots < a_l} e^{-i \sum_{j=1}^{r} \lambda^{a_j}} = \frac{1}{r!} \left(\sum_a e^{-i\lambda^a} \right)^r. \quad (A2)
$$

It follows that

$$
\exp \left(\sum_{r \geqslant 1} \hat{Q}_r \sum_{a_1 < \cdots < a_r} e^{-i \sum_{l=1}^{r} \lambda^{a_l}} \right) = \exp \left[\sum_{r \geqslant 1} \frac{\hat{Q}_r}{r!} \left(\sum_a e^{-i\lambda^a} \right)^r \right] = \iint_{-\infty}^{+\infty} \exp \left[i\eta \left(x - \sum_a e^{-i\lambda^a} \right) + \sum_{r \geqslant 1} \frac{\hat{Q}_r x^r}{r!} \right] \frac{dx \, d\eta}{2\pi}. \quad (A3)
$$

The dependence on $\{\lambda^a\}_a$ factorizes and therefore we can calculate, for each value of a,

$$
\int_0^{2\pi} \frac{d\lambda^a}{2\pi} \exp(i\lambda^a - i\eta e^{-i\lambda^a}) = i \int_\gamma \frac{e^{-i\eta z}}{z^2} \frac{dz}{2\pi} = -i\eta, \quad (A4)
$$

where γ is the anticlockwise oriented unit circle in the complex plane. We have

$$
\left[\prod_{a=1}^{n} \int_0^{2\pi} \frac{e^{i\lambda^a} \, d\lambda^a}{2\pi} \right] \exp \left(\sum_{r \geqslant 1} \hat{Q}_r \sum_{a_1 < \cdots < a_r} e^{-i \sum_{l=1}^{r} \lambda^{a_l}} \right)
$$
$$
= \iint_{-\infty}^{+\infty} (-i\eta)^n \exp \left[i\eta x + \sum_{r \geqslant 1} \frac{\hat{Q}_r x^r}{r!} \right] \frac{dx \, d\eta}{2\pi} = \frac{d^n}{dx^n} \exp \left(\sum_{r=1}^{n} \frac{\hat{Q}_r x^r}{r!} \right) \bigg|_{x=0}. \quad (A5)
$$

In the $n \to 0$ limit,

$$
\iint_{-\infty}^{+\infty} (-i\eta)^n \exp \left[i\eta x + \sum_{r \geqslant 1} \frac{\hat{Q}_r x^r}{r!} \right] dx \, d\eta = 1 + n \iint_{-\infty}^{+\infty} \ln(-i\eta) \exp \left[i\eta x + \sum_{r \geqslant 1} \frac{\hat{Q}_r x^r}{r!} \right] dx \, d\eta + o(n). \quad (A6)
$$

Using now the integral representation for the logarithm

$$
\ln(x) = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt, \quad (A7)
$$

we observe that, for a generic function $f(x)$,

$$
\iint_{-\infty}^{+\infty} \ln(-i\eta) e^{i\eta x + f(x)} dx \, d\eta = \int_0^{+\infty} \frac{dt}{t} \left[\iint_{-\infty}^{+\infty} dx \, d\eta (e^{-t} - e^{i\eta t}) e^{i\eta x + f(x)} \right]
$$
$$
= \int_0^{+\infty} \frac{e^{f(0)-t} - e^{f(-t)}}{t} dt = \int_{-\infty}^\infty [e^{f(0)-e^y} - e^{f(-e^y)}] dy. \quad (A8)
$$

Therefore, using Eq. (27), we have

$$
\iint_{-\infty}^{+\infty} \ln(-i\eta) \exp \left[i\eta x + \sum_{r \geqslant 1} \frac{\hat{Q}_r x^r}{r!} \right] dx \, d\eta = \beta \int_{-\infty}^\infty [e^{-e^{\beta y}} - e^{-G(y)}] dy. \quad (A9)
$$

The other terms appearing in the mean field action can be evaluated on the saddle point using Eq. (28) and the fact that

$$\binom{n}{r} = \frac{(-1)^{r-1} n}{r} + o(n). \tag{A10}$$

In particular,

$$\sum_{r \geqslant 1} \frac{(-1)^{r-1}}{r} Q_r \hat{Q}_r = \beta \int e^{-G(x)} \sum_{r=1}^{\infty} \frac{\hat{Q}_r e^{\beta r x}}{r!} \, dx = \beta \int G(x) e^{-G(x)} \, dx \tag{A11}$$

and similarly

$$\sum_{r \geqslant 1} \frac{(-1)^{r-1} g_r(0)}{r} Q_r^2 = \beta^2 \iint e^{-G(x)-G(y)} \sum_{r \geqslant 1} \frac{(-1)^{r-1} g_r(0) e^{\beta r (x+y)}}{r!(r-1)!} \, dx \, dy$$

$$= -\beta \iiint e^{-G(x)-G(y)} \rho(w) \frac{\partial J_0(2 e^{\beta \frac{x+y-w}{2}})}{\partial x} \, dx \, dy \, dw. \tag{A12}$$

Collecting all contributions, we can finally write the mean field action at finite temperature,

$$S_{\mathrm{mf}}[\beta, Q, \hat{Q}] \equiv S_{\mathrm{mf}}[\beta, G] = \int G(x) e^{-G(x)} \, dx + \frac{1}{2} \iiint e^{-G(x)-G(y)} \rho(w) \frac{\partial J_0(2 e^{\beta \frac{x+y-w}{2}})}{\partial x} \, dx \, dy \, dw - \int_{-\infty}^{\infty} [e^{-e^{\beta y}} - e^{-G(y)}] \, dy, \tag{A13}$$

that has the structure of Eq. (29). The $\beta \to \infty$ limit of this quantity is immediately obtained using the fact that

$$J_0 \left[2 \exp \left(\frac{\beta x}{2} \right) \right] - 1 \xrightarrow{\beta \to \infty} -\theta(x), \tag{A14}$$

and therefore we have

$$\lim_{\beta \to \infty} S_{\mathrm{mf}}[\beta, Q, \hat{Q}] = \int G(x) e^{-G(x)} \, dx - \frac{1}{2} \iint e^{-G(x)-G(w-x)} \rho(w) \, dx \, dw - \int_{-\infty}^{\infty} [\theta(-x) - e^{-G(x)}] \, dx. \tag{A15}$$

The mean field approximation to the optimal cost is obtained substituting in the previous equation the mean field solution for $G(x)$, given by Eq. (32).

APPENDIX B: DERIVATION OF THE POLYGONAL CORRECTIONS

To derive the series of Eq. (40a), we proceed accordingly to the diagonalization strategy of De Almeida and Thouless [48]. The following computation is a straightforward generalization of the ones of Parisi and Mézard [51] and Parisi and Ratiéville [52], the main difference being the momentum dependence of some quantities, which is not present in previous mean field calculations.

Consider an eigenvector $\mathbf{c} = (c_{\mathbf{a}})_{\mathbf{a}}$ of the replicated transfer matrix T of Eq. (37). It must satisfy

$$\sum_{\mathbf{b}} T_{\mathbf{ab}} c_{\mathbf{b}} = \sum_{\mathbf{b}: \mathbf{a} \cap \mathbf{b} = \emptyset} Q_{|\mathbf{a}|+|\mathbf{b}|} \sqrt{g_{|\mathbf{a}|}(k) g_{|\mathbf{b}|}(k)} c_{\mathbf{b}} = \lambda c_{\mathbf{a}}. \tag{B1}$$

We will look for eigenvectors $\mathbf{c}^q$ with q distinguished replicas, in the form

$$c_{\mathbf{a}}^q = \begin{cases} 0 & \text{if } |\mathbf{a}| < q, \\ d_{|\mathbf{a}|}^i & \text{if } \mathbf{a} \text{ contains } q-i+1 \text{ different} \\ & \text{indexes}, i = 1, \dots, q+1. \end{cases} \tag{B2}$$

For $q \geqslant 2$, if we consider $q-1$ distinguished replicas, it can be proved [51] that the following orthogonality condition holds:

$$\sum_{k=0}^{q-j} \binom{k}{q-j} \binom{|\mathbf{a}| - (k+j)}{n-q} d_{|\mathbf{a}|}^{q+1-(k+j)} = 0. \tag{B3}$$

The orthogonality condition provides a relation between all the different values $d_{|\mathbf{a}|}^i$, showing that we can keep one value only, say $d_{|\mathbf{a}|}^1$, as independent. Using this assumption, the eigenvalues of the original $\mathsf{T}(k)$ matrix can be evaluated diagonalizing the infinite dimensional matrices $\mathsf{N}^{(q)}(k)$ [20] whose elements, in the $n \to 0$ limit, are given by

$$N_{ab}^{(q)}(k) = (-1)^b \frac{\Gamma(a+b)\Gamma(b) Q_{a+b} \sqrt{g_a(k) g_b(k)}}{\Gamma(a)\Gamma(b-q+1)\Gamma(b+q)}. \tag{B4}$$

In particular, for $q = 0$ a direct computation gives

$$N_{ab}^{(0)}(k) = \binom{n-a}{b} Q_{a+b} g_b(k)$$

$$\xrightarrow{n \to 0} (-1)^b \frac{\Gamma(a+b)}{\Gamma(a) b!} Q_{a+b} \sqrt{g_a(k) g_b(k)}, \tag{B5}$$

whereas for $q = 1$ we obtain

$$N_{ab}^{(1)}(k) = \binom{n-a}{b} \frac{b}{b-n} Q_{a+b} \sqrt{g_a(k) g_b(k)}$$

$$\xrightarrow{n \to 0} N_{ab}^{(0)}(k) + \frac{n}{b} N_{ab}^{(0)}(k) + o(n). \tag{B6}$$

Summarizing, we can write

$$\mathrm{tr}[\mathsf{T}^E(k)] = \sum_{q=0}^{\infty} \left[\binom{n}{q} - \binom{n}{q-1} \right] \mathrm{tr}[(\mathsf{N}^{(q)}(k))^E]. \tag{B7}$$

We distinguish now the sectors $q \geqslant 2$ from the sectors $q = 0,1$, due to the fact that the two sets requires a different analytic treatment.

1. Sectors $q \geqslant 2$

Computing the spectrum of the matrix $\mathsf{N}^{(q)}$ for $q \geqslant 2$ is equivalent to the computation of the spectrum of $\mathsf{M}^{(q)}(k)$ that has elements

$$M_{ab}^{(q)}(k) := (-1)^{a+b} \sqrt{\frac{g_{b+q}(k)}{g_{a+q}(k)}} \frac{\Gamma(a+1)\Gamma(b+q)}{\Gamma(b+1)\Gamma(a+q)} N_{b+q\ a+q}^{(q)}(k) = (-1)^{a+q} \frac{\Gamma(a+b+2q)}{\Gamma(a+2q)b!} Q_{a+b+2q} g_{b+q}(k). \tag{B8}$$

The eigenvalue equation for $\mathsf{M}^{(q)}(k)$ has the form

$$\lambda c_a^{(q)} = \sum_{b=1}^{\infty} M_{ab}^{(q)}(k) c_b^{(q)} = \beta(-1)^q \int \frac{(-1)^a e^{(a+q)\beta u}}{\Gamma(a+2q)} \phi^{(q)}(u;k)\, du, \tag{B9}$$

where we have introduced

$$\phi^{(q)}(u;k) := \sum_{b=1}^{\infty} \frac{e^{(b+q)\beta u - \frac{G(u)}{2}}}{b!} c_b^{(q)} g_{b+q}(k). \tag{B10}$$

Equation (B9) can be written as

$$\lambda \phi^{(q)}(u;k) = (-1)^q \int [\mathsf{A}^{(q)}(k)]_{uv} \phi^{(q)}(v;k)\, dv, \tag{B11}$$

where $\mathsf{A}^{(q)}(k)$ is the operator

$$[\mathsf{A}^{(q)}(k)]_{uv} := \beta e^{-\frac{G(u)+G(v)}{2}+q\beta(u+v)} \sum_{a=1}^{\infty} \frac{(-1)^a e^{a\beta(u+v)}}{\Gamma(a+2q)a!} g_{a+q}(k). \tag{B12}$$

In the $n \to 0$ limit, from Eq. (B7) we have therefore

$$\sum_{q=2}^{\infty} \left[\binom{n}{q} - \binom{n}{q-1} \right] \mathrm{tr}[(\mathsf{N}^{(q)}(k))^E]$$

$$= \sum_{q=2}^{\infty} (-1)^{qE} \left[\binom{n}{q} - \binom{n}{q-1} \right] \mathrm{tr}[(\mathsf{A}^{(q)}(k))^E] \xrightarrow{n\to 0} n \sum_{q=2}^{\infty} (-1)^{q(E+1)} \frac{2q-1}{q(1-q)} \mathrm{tr}[(\mathsf{A}^{(q)}(k))^E]$$

$$= n \sum_{q=1}^{\infty} \frac{4q-1}{2q(1-2q)} \mathrm{tr}[(\mathsf{A}^{(2q)}(k))^E] + (-1)^E n \sum_{q=1}^{\infty} \frac{4q+1}{2q(2q+1)} \mathrm{tr}[(\mathsf{A}^{(2q+1)}(k))^E]. \tag{B13}$$

Here we have used the fact that

$$\binom{n}{q} - \binom{n}{q-1} \xrightarrow{n\to 0} \begin{cases} 1 & \text{if } q = 0, \\ -1 + n & \text{if } q = 1, \\ n(-1)^q \frac{2q-1}{q(1-q)} & \text{if } q > 1. \end{cases} \tag{B14}$$

2. Sectors $q = 0$ and $q = 1$

Let us now evaluate the contributions of the sectors $q = 0$ and $q = 1$. We have

$$\sum_{q=0}^{1} \left[\binom{n}{q} - \binom{n}{q-1} \right] \mathrm{tr}[(\mathsf{N}^{(q)}(k))^E] = \mathrm{tr}[(\mathsf{N}^{(0)}(k))^E] + (n-1)\, \mathrm{tr}[(\mathsf{N}^{(1)}(k))^E] + o(n). \tag{B15}$$

To evaluate the traces appearing in the previous expression, we define the operator $\mathsf{M}^{(0)}(k)$,

$$M_{ab}^{(0)}(k) := (-1)^{a+b} \sqrt{\frac{g_b(k)}{g_a(k)}} \frac{a}{b} N_{ba}^{(0)}(k) = (-1)^a \frac{\Gamma(a+b)}{\Gamma(a)b!} Q_{a+b} g_b(k), \tag{B16a}$$

and the operator $\tilde{\mathsf{M}}^{(1)}(k)$,

$$\tilde{M}_{ab}^{(1)}(k) := (-1)^{a+b} \sqrt{\frac{g_b(k)}{g_a(k)}} \frac{a}{b} N_{ba}^{(1)} = (-1)^a \frac{\Gamma(a+b)}{\Gamma(a)b!} Q_{a+b} g_b(k) + n(-1)^a \frac{\Gamma(a+b)}{\Gamma(a+1)b!} Q_{a+b} g_b(k). \tag{B16b}$$

Repeating the considerations presented for the $q \geqslant 2$ case, we can introduce the operator $\mathsf{A}^{(0)}(k)$ as follows:

$$[A^{(0)}(k)]_{uv} = \beta e^{-\frac{G(u)+G(v)}{2}} \sum_{a=0}^{\infty} \frac{(-1)^a e^{a\beta(u+v)} g_a(k)}{\Gamma(a)a!} = \Omega_d e^{-\frac{G(u)+G(v)}{2}} \int_0^{\infty} z^{d-1} {}_0F_1 \left[\begin{array}{c} - \\ \frac{d}{2} \end{array} ; -\frac{k^2 z^2}{4} \right] \frac{\partial J_0(2e^{\beta \frac{y}{2}})}{\partial y} \bigg|_{y=u+v-z^p} dz, \tag{B17}$$

having the same eigenvalues of $\mathsf{M}^{(0)}(k)$, in such a way that

$$\mathrm{tr}[(\mathsf{N}^{(0)}(k))^E] = \mathrm{tr}[(\mathsf{M}^{(0)}(k))^E] = \mathrm{tr}[(\mathsf{A}^{(0)}(k))^E]. \tag{B18}$$

Similarly, we have that the eigenvalues of $\tilde{\mathsf{M}}^{(1)}(k)$ are obtained from

$$\lambda \tilde{c}_a = \sum_b \tilde{M}_{ab}^{(1)}(k) \tilde{c}_b = \sum_{r'} (-1)^a \frac{\Gamma(a+b)Q_{a+b}g_b(k)}{\Gamma(a)b!} \left(1+\frac{n}{a}\right) \tilde{c}_b = \int \frac{e^{au\beta - \frac{G(u)}{2}}}{\Gamma(a)} \left(1+\frac{n}{a}\right) \tilde{\phi}(u;k)\, du, \tag{B19}$$

where $\tilde{\phi}(u;k)$ is given by

$$\tilde{\phi}(u;k) := \sum_{b=1}^{\infty} \frac{e^{b\beta u - \frac{G(u)}{2}}}{b!} \tilde{c}_b g_b(k). \tag{B20}$$

It is natural, therefore, to introduce the operator $\tilde{\mathsf{A}}^{(1)}(k)$ defined as follows:

$$[\tilde{A}^{(1)}(k)]_{uv} := \beta e^{-\frac{G(u)+G(v)}{2}} \sum_{a=1}^{\infty} \frac{(-1)^a e^{a\beta(u+v)}}{\Gamma(a)a!} g_a(k) \left(1+\frac{n}{a}\right) = [A^{(0)}(k)]_{uv} + n[B(k)]_{uv}. \tag{B21}$$

The operator $\mathsf{B}(k)$ introduced above is

$$[B(k)]_{uv} := \beta e^{-\frac{G(u)+G(v)}{2}} \sum_{a=1}^{\infty} \frac{(-1)^a e^{a\beta(u+v)}}{\Gamma(a+1)a!} g_a(k) = \Omega_d \beta e^{-\frac{G(u)+G(v)}{2}} \int_0^{\infty} z^{d-1} {}_0F_1 \left[\begin{array}{c} - \\ \frac{d}{2} \end{array} ; -\frac{k^2 z^2}{4} \right] [J_0(2e^{\beta \frac{u+v-z^p}{2}}) - 1]\, dz. \tag{B22}$$

We have then, up to higher orders in n,

$$\mathrm{tr}[(\mathsf{N}^{(0)}(k))^E] + (n-1)\,\mathrm{tr}[(\mathsf{N}^{(1)}(k))^E] = \mathrm{tr}[(\mathsf{A}^{(0)}(k))^E] + (n-1)\,\mathrm{tr}[(\mathsf{A}^{(0)}(k)+n\mathsf{B}(k))^E]$$

$$= n\,\mathrm{tr}[(\mathsf{A}^{(0)}(k))^E] + nE\,\mathrm{tr}[(\mathsf{A}^{(0)}(k))^{E-1}\mathsf{B}(k)]. \tag{B23}$$

3. Zero-temperature limit

For each one of the quantities above, we need to calculate the $\beta \to \infty$ limit, being interested in the optimal cost. Let us consider the $q \geqslant 2$ contribution. First, we introduce the identity

$$\sum_{r=1}^{\infty} \frac{(-x)^r}{\Gamma(r+2q)r!} = \frac{i}{2\pi} \oint_{\gamma_\epsilon} e^{-\zeta - 2q\ln(-\zeta) + \frac{x}{\zeta}}\, d\zeta. \tag{B24}$$

The path γ_ϵ, in the complex plane, is the Hankel path, represented in Fig. 6. This identity can be proved starting from the Hankel representation for the reciprocal gamma function [56]

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \oint_{\gamma_\epsilon} e^{-\zeta - z\ln(-\zeta)}\, d\zeta. \tag{B25}$$

Using Eq. (B24), we can rewrite Eq. (B12) for $q \geqslant 2$ as

$$[A^{(q)}(k)]_{uv} = \frac{i\beta\Omega_d}{2\pi} e^{-\frac{G(u)+G(v)}{2}}$$

$$\times \int_0^{+\infty} dw \oint_{\gamma_\epsilon} d\zeta \frac{w^{\frac{d}{p}-1}}{p} {}_0F_1 \left[\begin{array}{c} - \\ \frac{d}{2} \end{array} ; -\frac{k^2 w^{\frac{2}{p}}}{4} \right]$$

$$\times \exp\left(\beta q(u+v-w) - w - 2q\ln(-\zeta) \right.$$

$$\left. + \frac{e^{\beta(u+v-w)}}{\zeta} \right). \tag{B26}$$

To compute the $\beta \to \infty$ limit, we perform a saddle point approximation, obtaining

$$\zeta_{\mathrm{sp}} = -q, \quad w_{\mathrm{sp}} = u+v - \frac{2\ln q}{\beta}. \tag{B27}$$
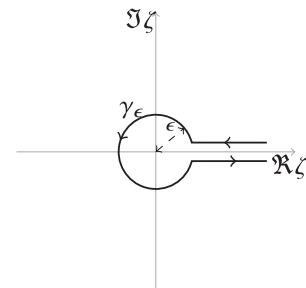

FIG. 6. Hankel path in the complex plane.

The saddle point has fixed position assuming that $\ln q = t\beta$ for some t. Taking instead q fixed and $\beta \to \infty$, it is easily seen from Eq. (B12) that

$$\lim_{\beta \to \infty} [\mathsf{A}^{(q)}(k)]_{uv} = \begin{cases} \infty & \text{for } u + v > 0, \\ 0 & \text{for } u + v < 0. \end{cases} \quad \text{(B28)}$$

Indeed, only for $u + v - 2t > 0$ the saddle point is inside the range of integration. For this reason, we take $\frac{\ln q}{\beta} = t$ fixed, obtaining the limit operator $\mathsf{H}(t,k)$,

$$[\mathsf{H}(t,k)]_{uv} := \lim_{\substack{\beta \to \infty, \, q \to \infty \\ \beta^{-1}\ln q = t}} [\mathsf{A}^{(q)}(k)]_{uv}$$

$$\approx \frac{\Omega_d}{p} e^{-\frac{G(u)+G(v)}{2}} x^{\frac{d}{p}-1} {}_0F_1$$

$$\times \left[\begin{matrix} - \\ \frac{d}{2} \end{matrix} ; -\frac{k^2 x^{\frac{2}{p}}}{4} \right] \theta(x) \Bigg|_{x=u+v-2t}. \quad \text{(B29)}$$

Observing that $\sum_{q=2}^{\infty} \frac{1}{\beta q} \to \int_0^{+\infty} dt$ the contribution to the (rescaled) average optimal cost from the $q \geqslant 2$ sectors is

$$\mathcal{E}_E^{(2+)} := \begin{cases} \frac{\Omega_d}{E(2\pi)^d} \iint_0^{\infty} k^{d-1} \, \mathrm{tr}[\mathsf{H}^E(t,k)] \, dt \, dk & E \text{ odd}, \\ 0 & E \text{ even}. \end{cases}$$
$$\text{(B30)}$$

For the sectors $q = 0$ and $q = 1$ the $\beta \to \infty$ limit can be performed quite straightforwardly. In particular, using Eq. (A14), we obtain the limit operators $\mathsf{H}(0,k)$,

$$[\mathsf{A}^{(0)}(k)]_{uv} \xrightarrow{\beta \to \infty} -[\mathsf{H}(0,k)]_{uv}$$

$$\equiv -\Omega_d e^{-\frac{G(u)+G(v)}{2}} \frac{x^{\frac{d}{p}-1}}{p} {}_0F_1 \left[\begin{matrix} - \\ \frac{d}{2} \end{matrix} ; -\frac{k^2 x^{\frac{2}{p}}}{4} \right] \theta(x) \Bigg|_{x=u+v},$$
$$\text{(B31a)}$$

and the operator $\mathsf{K}(k)$,

$$[\mathsf{B}(k)]_{uv} \xrightarrow{\beta \to \infty} -\beta[\mathsf{K}(k)]_{uv}:$$

$$= -\Omega_d \beta e^{-\frac{G(u)+G(v)}{2}} \frac{x^{\frac{d}{p}}}{d} {}_0F_1 \left[\begin{matrix} - \\ \frac{d}{2}+1 \end{matrix} ; -\frac{k^2 x^{\frac{2}{p}}}{4} \right] \theta(x) \Bigg|_{x=u+v}.$$
$$\text{(B31b)}$$

The contribution to the (rescaled) average optimal cost from the sectors $q = 0$ and $q = 1$ is

$$\mathcal{E}_E^{(01)} := (-1)^E \frac{\Omega_d}{2(2\pi)^d} \int_0^{\infty} k^{d-1} \, \mathrm{tr}[\mathsf{H}^{E-1}(0,k)\mathsf{K}(k)] \, dk.$$
$$\text{(B32)}$$

Collecting the results above, Eq. (40a) is immediately obtained.

APPENDIX C: THE TRIANGULAR CONTRIBUTION

As stressed in the main text, in Ref. [12] only the contribution triangular contribution $E = 3$ was considered and discussed. Moreover, their computation formally differs from our triangular term, since Parisi and Mézard computed the joint distribution of edge lengths, which is feasible in the triangular case, therefore they did not need to step into momentum space. Also the t integration along the sectors of the transfer matrix decomposition is not present in their result. For the sake of completeness, here we report their calculation and bridge it with our formalism, specifying all details of the computation. We will show that the expression in Eq. (40a) for $E = 3$ is equivalent to their classical result.

We proceed in the replica symmetric hypothesis. We observe that in this case Eq. (15) becomes

$$-n\beta S_3[\beta, Q] = \sum_{r_1, r_2, r_3} \frac{n! \overline{\prod_{e=1}^{3} e^{-\beta r_e w_e}}^{\mathsf{K}_3}}{6(n - r_1 - r_2 - r_3)!} \prod_{i=1}^{3} \frac{Q_{r_i + r_{i+1}}}{r_i!}$$

$$= \frac{n}{6} \sum_{r_1, r_2, r_3} \overline{\prod_{e=1}^{3} e^{-\beta r_e w_e}}^{\mathsf{K}_3} (r_1 + r_2 + r_3 - 1)!$$

$$\times \prod_{i=1}^{3} \frac{(-1)^{r_i - 1} Q_{r_i + r_{i+1}}}{r_i!} + o(n). \quad \text{(C1)}$$

We have used the fact that $\sigma_{\mathsf{K}_3} = \frac{1}{6}$ and, moreover, there are

$$\frac{n!}{(n - r_1 - r_2 - r_3)! r_1! r_2! r_3!} \quad \text{(C2)}$$

ways to organize $r_1 + r_2 + r_3$ different replica indexes in three groups of cardinality r_1, r_2, r_3, respectively. Using Eq. (28) we can write the previous expression as

$$-\beta S_3[\beta, Q] = \sum_{r_1, r_2, r_3} \frac{n! \overline{\prod_{e=1}^{3} e^{-\beta r_e w_e}}^{\mathsf{K}_3}}{6(n - r_1 - r_2 - r_3)!} \prod_{i=1}^{3} \frac{Q_{r_i + r_{i+1}}}{r_i!}$$

$$= \frac{1}{6} \left[\prod_{i=1}^{3} \iint dx_i \, dw_i \, G'(x_i) e^{-G(x_i)} \right]$$

$$\times \rho_{\mathsf{K}_3}(\{w_i\}) K[\{\beta(x_i + x_{i+1} - w_i)\}_i], \quad \text{(C3)}$$

where we have introduced the function

$$K(\{x_i\}_i) = \sum_{r_1, r_2, r_3} (r_1 + r_2 + r_3 - 1)! \prod_{i=1}^{3} \frac{(-1)^{r_i - 1} e^{-r_i x_i}}{(r_i + r_{i+1})! r_i!}$$
$$\text{(C4)}$$

and ρ_{K_3} is given by Eq. (D2) in the form

$$\rho_{\mathsf{K}_3}(\{w_e\}) = \left[\prod_{e=1}^{3} \int_{\mathbb{R}^d} d^d z_e \, \delta(w_e - \|\mathbf{z}_e\|^p) \right] \delta\left(\sum_{e=1}^{3} \mathbf{z}_e \right). \quad \text{(C5)}$$

We refer to Ref. [12] for the explicit result of the integration above. Using the expression in Eq. (A13), we can write the action, in the triangular approximation, at finite

temperature,

$$
S_{\mathrm{mf}}[\beta, G] + S_3[\beta, G] = \int G(x) e^{-G(x)} \, dx + \frac{1}{2} \iiint e^{-G(x)-G(y)} \rho(w) \frac{\partial J_0(2e^{\beta \frac{x+y-w}{2}})}{\partial x} \, dx \, dy \, dw
$$
$$
- \int_{-\infty}^{\infty} [e^{-e^{\beta y}} - e^{-G(y)}] dy - \frac{1}{6\beta} \left[\prod_{i=1}^{3} \iint dx_i \, dw_i \, G'(x_i) e^{-G(x_i)} \right]
$$
$$
\times \rho_{\mathrm{K}_3}(\{w_i\}) K[\{\beta(x_i + x_{i+1} - w_i)\}_i]. \tag{C6}
$$

We have to evaluate the $\beta \to \infty$ limit. Using the identity

$$
\frac{1}{\Gamma(p_1 + p_2 + 1)} = \frac{1}{\Gamma\left(p_1 + \frac{1}{2}\right)\Gamma\left(p_2 + \frac{1}{2}\right)} \int_0^1 x^{p_1 - \frac{1}{2}} (1-x)^{p_2 - \frac{1}{2}} \, dx, \tag{C7}
$$

we can write

$$
\frac{K(\{\beta x_i\}_i)}{\beta} = - \int_{-\infty}^{+\infty} e^{-e^{-\beta w}} \left[\prod_{i=1}^{3} \int_0^1 \frac{du_i}{\sqrt{u_i(1-u_i)}} \right] \left[\sum_{p=1}^{\infty} \frac{(-u_i u_{i-1} e^{\beta(x_i - w)})^p}{p! \Gamma^2\left(p + \frac{1}{2}\right)} \right] dw \xrightarrow{\beta \to \infty} \frac{1}{\pi^3} \left[\int_0^1 \frac{du}{\sqrt{u(1-u)}} \right]^3
$$
$$
\times \int_0^{+\infty} \prod_{i=1}^{3} \theta(x_i) \theta(x_i - w) \, dw = \min_i(\{x_i\}) \theta(x_1) \theta(x_2) \theta(x_3). \tag{C8}
$$

To perform the last limit, we have used the fact that

$$
\lim_{\beta \to \infty} \sum_{p=1}^{\infty} \frac{(-z e^{\beta x})^p}{p! \Gamma^2\left(p + \frac{1}{2}\right)} = - \frac{\theta(x)}{\pi}. \tag{C9}
$$

This property can be obtained applying the following

Proposition. Let $f(p)$ be an holomorphic function in the semiplane $\mathrm{Re}(p) > -\epsilon$, for some $\epsilon \in (0,1)$. Moreover assume that $|f(p)| e^{-\pi|p|} \leqslant M|p|^{-k}$ with $k > 1$ as $\mathrm{Im}(p) \to \pm\infty$. Then the following identity holds:

$$
\lim_{x \to +\infty} \sum_{p=0}^{\infty} f(p)(-x)^p = 0. \tag{C10}
$$

Proof. The series in Eq. (C10) admits a representation as an integral over the Hankel path γ_ϵ in the complex plane, see Fig. 6, with $\epsilon \in (0,1)$:

$$
\sum_{p=0}^{\infty} f(p)(-x)^p = \frac{1}{2i} \int_{\gamma_\epsilon} \frac{f(\zeta) x^\zeta}{\sin(\pi\zeta)} \, d\zeta = \frac{x^{-\epsilon}}{2i} \int_{-\infty}^{+\infty} \frac{f(-\epsilon + iy) x^{iy}}{\sin\left[\pi(iy - \epsilon)\right]} \, dy, \tag{C11}
$$

where in the second equality we have deformed the path to the vertical line $\mathrm{Re}(\zeta) = -\epsilon$. It follows that

$$
\left| \sum_{p=0}^{\infty} f(p)(-x)^p \right| \leqslant x^{-\epsilon} \int_{-\infty}^{+\infty} \left| \frac{f(-\epsilon + iy)}{2 \sin\left[\pi(iy - \epsilon)\right]} \right| dy, \tag{C12}
$$

Given the assumptions on $f(p)$, the last integral is convergent, and the thesis follows taking the limit $x \to +\infty$. ∎

Noticeably, Eq. (C10) implies

$$
\lim_{x \to +\infty} \sum_{p=1}^{\infty} f(p)(-x)^p = -f(0) \tag{C13}
$$

from which Eq. (C9) follows immediately.

Combining the results above with the expression for the mean field action in Eq. (A15), we obtain the saddle point action in the triangular approximation and in the zero temperature limit, $\mathcal{E}_\triangle := \mathcal{E}_{\mathrm{mf}} + \mathcal{E}_3$, where

$$
\mathcal{E}_3 = - \frac{1}{6} \left[\prod_{i=1}^{3} \iint dw_i \, dx_i \, G'(x_i) e^{-G(x_i)} \right] \rho_{\mathrm{K}_3}(\{w_e\}_e) \min_i(\{x_i + x_{i+1} - w_i\}_i) \prod_{i=1}^{3} \theta(x_i + x_{i+1} - w_i). \tag{C14}
$$

The value of the average optimal cost can be obtained using for $G(x)$ the solution of the saddle point equation Eq. (32), or the solution of the saddle point equation obtained from the action $\mathcal{E}_\triangle$, as showed in Ref. [12].

Equation (C14) can be written in a different form. Indeed, expanding again the expression for $\mathcal{E}_3$ using Eq. (D2) and the relation

$$\min(x_1,x_2,x_3)\theta(x_1)\theta(x_2)\theta(x_3) = \int_0^\infty \theta(x_1 - t)\theta(x_2 - t)\theta(x_3 - t)\, dt, \tag{C15}$$

we can verify that the triangular contribution can be written in terms of the operators $\mathsf{H}(t,k)$ and $\mathsf{K}(k)$ introduced in Appendix B as

$$\mathcal{E}_3 = -\frac{1}{6(2\pi)^d}\int_0^{+\infty} dt \int_{\mathbb{R}^d} d^dk \left[\prod_{i=1}^3 \iint d^dz_i\, dx_i\, G'(x_i)e^{-G(x_i)+i\mathbf{k}\cdot\mathbf{z}_i}\theta(x_i + x_{i+1} - \|\mathbf{z}_i\|^p - t)\right]$$

$$= -\frac{2\Omega_d}{3(2\pi)^d}\iint_0^{+\infty} k^{d-1}\,\mathrm{tr}[\mathsf{H}^3(t,k)]\,dt\,dk + \frac{\Omega_d}{(2\pi)^d}\iint_0^{+\infty} k^{d-1}\,\mathrm{tr}[\mathsf{H}^3(t,k)]\,dt\,dk - \frac{\Omega_d}{2(2\pi)^d}\int_{\mathbb{R}^d} k^{d-1}\,\mathrm{tr}[\mathsf{H}^2(0,k)\mathsf{K}(k)]\,d^dk$$

$$= \frac{\Omega_d}{3(2\pi)^d}\iint_0^{+\infty} k^{d-1}\,\mathrm{tr}[\mathsf{H}^3(t,k)]\,dt\,dk - \frac{\Omega_d}{2(2\pi)^d}\int_{\mathbb{R}^d} k^{d-1}\,\mathrm{tr}[\mathsf{H}^2(0,k)\mathsf{K}(k)]\,d^dk. \tag{C16}$$

The second and the third contributions in the second line derive from the fact that, given a set of three numbers $\{a_1,a_2,a_3\}$, the simple identity

$$\int_{-\infty}^\infty \theta(t)\sum_{k=0}^2 \delta'(a_{1+k}-t)\delta(a_{2+k}-t)\theta(a_{3+k}-t)\,dt + \int_{-\infty}^\infty \theta(t)\sum_{k=0}^2 \delta(a_{1+k}-t)\delta'(a_{2+k}-t)\theta(a_{3+k}-t)\,dt$$

$$= -\int_{-\infty}^\infty \theta(t)\sum_{k=0}^2 \delta(a_{1+k}-t)\delta(a_{2+k}-t)\delta(a_{3+k}-t)\,dt + \sum_{k=0}^2 \delta(a_{1+k})\delta(a_{2+k})\theta(a_{3+k}) \tag{C17}$$

holds. We finally have that Eq. (C16) is exactly the contribution appearing in Eq. (40a) for $E = 3$.

APPENDIX D: DIAGRAMMATIC RULES FOR S_{g}

The contribution $S_{\mathrm{g}}[\beta,Q]$ for a generic biconnected graph to the action in Eq. (24) can be written in a quite general form in relation to the topological structure of the graph g itself. Let us first observe that, for a given graph g with V vertices and E edges, we can define a *cycle basis* as follows [57]. Every cycle in the graph can be represented in the space $\mathcal{C} \subseteq \{0,1\}^E$ by a vector $\mathsf{L} = (\ell_e)_e$ such that $\ell_e = 1$ if the edge e belongs to L, $\ell_e = 0$ otherwise. Remember that in a cycle, each vertex has even degree by definition, and a cycle is called *circuit* if all vertexes have degree equal to two, i.e., a circuit corresponds to a "loop" in the nomenclature adopted in the body of the paper [58]. In the introduced representation we can sum two cycles $\mathsf{L}_1 = (\ell_e^{(1)})_e$ and $\mathsf{L}_2 = (\ell_e^{(2)})_e$, in such a way that $\mathsf{L}_1 \oplus \mathsf{L}_2 = (\ell_e^{(1)} + \ell_e^{(2)} \mod 2)_e \in \mathcal{C}$. We say that $\mathcal{L}_{\mathrm{g}}$ is a cycle basis for g if it is a set of circuits such that every cycle in g can be expressed as sum of circuits in $\mathcal{L}_{\mathrm{g}}$, and, moreover, its cardinality $L := |\mathcal{L}_{\mathrm{g}}|$ is minimal. The number L is called circuit rank and, for a connected graph, it satisfies the fundamental property [57]

$$L = E - V + 1. \tag{D1}$$

In a planar graph a basis $\mathcal{L}_{\mathrm{g}}$ can be always easily identified considering, as basis circuits, the faces of the graph. With these definitions in mind, the distribution $\rho_{\mathrm{g}}(\{w_e\})$ in Eq. (18) can be written in terms of a cycle basis of the graph g as

$$\rho_{\mathrm{g}}(\{w_e\}) = \left[\prod_{e=1}^E \int_{\mathbb{R}^d} d^dz_e\,\delta(w_e - \|\mathbf{z}_e\|^p)\right]\prod_{\mathsf{L}\in\mathcal{L}_{\mathrm{g}}}\delta\left(\sum_{e\in\mathsf{L}}\mathbf{z}_e\right), \tag{D2}$$

and therefore, denoting by $r_e := |\mathbf{a}^e|$,

$$\overline{\prod_{e\in\mathrm{g}} e^{-\beta r_e w_e}}^{\mathrm{g}}$$

$$= \left[\prod_{e=1}^E \int_{\mathbb{R}^d} d^dz_e\, e^{-\beta r_e\|\mathbf{z}_e\|^p}\right]\prod_{\mathsf{L}\in\mathcal{L}}\delta\left(\sum_{e\in\mathsf{L}}\mathbf{z}_e\right)$$

$$= \prod_{\mathsf{L}}\int_{\mathbb{R}^d}\frac{d^dk_{\mathsf{L}}}{(2\pi)^d}\prod_{e=1}^E\left[\int_{\mathbb{R}_d} d^d\kappa_e\, g_{r_e}(\kappa_e)\delta\left(\kappa_e - \sum_{\mathsf{L}:e\in\mathsf{L}}\mathbf{k}_{\mathsf{L}}\right)\right]. \tag{D3}$$

In the equation above we have introduced the function

$$g_r(\kappa) := \int_{\mathbb{R}^d} e^{i\kappa\cdot\mathbf{z}-\beta r\|z\|^p}\,d^dz$$

$$= \Omega_d \int_0^\infty z^{d-1}e^{-\beta r z^p}{}_0F_1\left[\frac{-}{\frac{d}{2}}; -\frac{\kappa^2 z^2}{4}\right]dz. \tag{D4}$$

Equation (D3) can be pictorially interpreted as follows. We associate to each circuit L of our basis a "momentum" $\mathbf{k}_{\mathsf{L}}$ and to
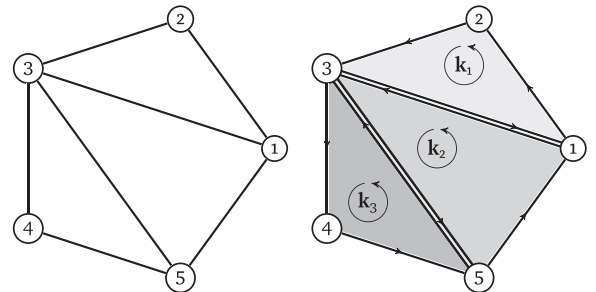

FIG. 7. A planar graph and its decomposition in basis circuits.

each edge of the graph the quantity $g_{r_e}(\kappa_e)$, with the additional constraint that κ_e is the algebraic sum of the momenta flowing in the basis circuits to which the edge e belongs (see Fig. 7). Inserting Eq. (D3) in Eq. (15) we obtain a new expression depending explicitly on the topology of the graph. We can list a set of diagrammatic rules for the evaluation of S_g at finite temperature. In particular, a momentum $\mathbf{k}_L$ must be associated with each basis circuit L; we must associate a set

of replica indexes $\mathbf{a}^e$ and a quantity $g_{r_e}(\kappa_e)\delta(\kappa_e - \sum_{L:e\in L} \mathbf{k}_L)$ to each edge e, and a quantity $Q_{\mathbf{a}(v)}\delta_{\mathbf{a}(v)}$ to each vertex v. We must finally sum on all $\{\mathbf{a}^e\}_e$ and integrating on all momenta. Observe that the case of polygons is particularly simple, being in this case $L = 1$, and therefore Eq. (D3) becomes

$$\overline{\prod_{e\in \mathrm{p}_E} e^{-\beta r_e w_e}}^{\mathrm{p}_E} = \frac{\Omega_d}{(2\pi)^d} \int dk\, k^{d-1} \prod_{e=1}^{E} g_{r_e}(k). \quad (D5)$$

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