Collective dynamics of time-delay-coupled phase oscillators in a frustrated geometry

Bhumika Thakur,[1] Devendra Sharma,[1] Abhijit Sen,[1] and George L. Johnston[2]
[1]*Institute for Plasma Research, HBNI, Bhat, Gandhinagar 382428, India*
[2]*EduTron Corporation, 5 Cox Road, Winchester, Massachusetts 01890, USA*
(Received 14 July 2016; revised manuscript received 30 August 2016; published 6 January 2017)

We study the effect of time delay on the dynamics of a system of repulsively coupled nonlinear oscillators that are configured as a geometrically frustrated network. In the absence of time delay, frustrated systems are known to possess a high degree of multistability among a large number of coexisting collective states except for the fully synchronized state that is normally obtained for attractively coupled systems. Time delay in the coupling is found to remove this constraint and to lead to such a synchronized ground state over a range of parameter values. A quantitative study of the variation of frustration in a system with the amount of time delay has been made and a universal scaling behavior is found. The variation in frustration as a function of the product of time delay and the collective frequency of the system is seen to lie on a characteristic curve that is common for all natural frequencies of the identical oscillators and coupling strengths. Thus time delay can be used as a tuning parameter to control the amount of frustration in a system and thereby influence its collective behavior. Our results can be of potential use in a host of practical applications in physical and biological systems in which frustrated configurations and time delay are known to coexist.

DOI: 10.1103/PhysRevE.95.012204

I. INTRODUCTION

Geometrical frustration is a condition that occurs when topological constraints prevent the simultaneous minimization of the energy of all the interacting pairs of subunits of a system. A well-known example is that of three Ising spins that are antiferromagnetically coupled to each other and placed on the corners of an equilateral triangle. It is impossible to arrive at a configuration where each pair of spins is antiparallel, and as a consequence the system continually flips between different states in trying to find a minimum energy state and thereby displays a multistable behavior. A large lattice model consisting of such triangular units can therefore lead to high ground-state degeneracy and multiple phase transitions with increasing temperature [1]. In complex systems, frustration can arise from a combination of geometry and the nature of interaction among the subsystems, and it can give rise to a rich variety of collective behavior. Frustration plays an important role in the dynamics of many complex magnetic systems such as spin liquids, spin glasses, and a host of magnetic alloys [1–3]. More recently, geometrical frustration arising in neuronal networks has also been recognized as a pivotal factor influencing the cortical dynamics of the brain, and it is believed to be responsible for introducing metastability and variability in the brain's collective states [4,5]. The existence of metastability (or multiple operating regimes) is an essential and crucial feature for biological systems since it provides them with functional flexibility. Frustration and its dynamical consequences are therefore receiving a great deal of theoretical and experimental attention in a wide variety of physical [6–8] and biological systems [9,10].

One of the important factors that can influence the collective dynamics of a complex network is the presence of time delay in the coupling between the network nodes or between individual functional elements of the network. Time delay is ubiquitous in most natural systems due to the finiteness of the speed of propagation of electrical signals or reaction times in chemical interactions or the finite conductance of neuronal connections. It is well known from past studies [11–14] that time delay can significantly impact synchronization and other collective behavior in nonfrustrated systems, and therefore it is of interest to investigate its influence on the dynamics of frustrated systems. Our present work is devoted to such a study and is motivated by its potential utility in diverse practical applications.

Our model investigation is carried out on a system of repulsively coupled phase oscillators that are configured in a frustrated geometry. The intimate relation between frustration and network topology has been well established through many past studies. Some of the common topologies that lead to frustrated networks are triangular lattices [15], hexagonal lattices, and more exotic configurations such as the Kagome lattice [16]. Systems of repulsively coupled oscillators were studied earlier in the absence of time delay by Daido [17], who was one of the earliest to discuss the concept of frustration in a system of oscillators. More recently, Kaluza [18] showed that frustration in an ensemble of repulsively coupled Kuramoto phase oscillators can lead to a considerable increase in the number of stationary states and to multistability. Our particular focus is to examine in a quantitative manner the variations induced in the amount of frustration in a given system as a function of time delay in the coupling. For our study, we have considered three geometrical configurations that are representative of frustrated networks, namely a simple triangle, a triangular lattice with six nodes, and a hexagonal 4×4 lattice. Each node in these configurations is populated by a Kuramoto phase oscillator, and the links representing the interaction between the oscillators are repulsive in nature with an intrinsic time delay τ. Our principal findings are that time delay can significantly influence the amount of frustration in a system and thereby control the number and nature of the equilibrium states of the system. For an amount of delay beyond a critical value, the system can transit to an in-phase collective state—an equilibrium condition that is normally not possible for frustrated configurations. The transition to such a synchronous state is found to occur with a characteristic

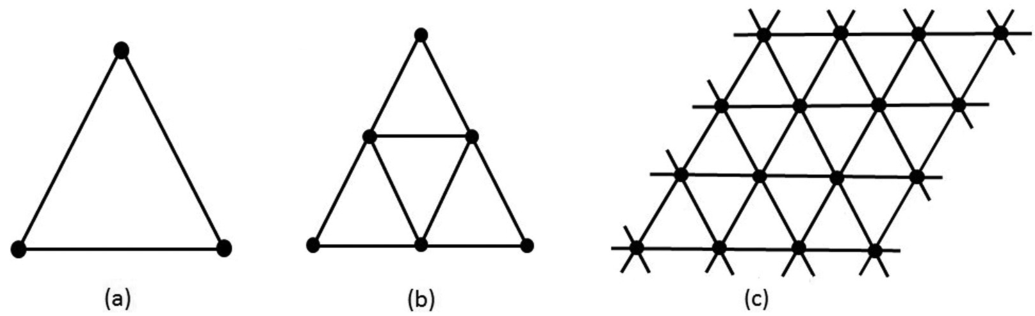

FIG. 1. Frustrated systems of oscillators. Circles on the vertices represent phase oscillators, and the edges represent repulsive coupling links between the oscillators.

behavior akin to first-order phase transitions. We also find that the variation of frustration as a function of the product of time delay and the collective frequency of the system follows a similar pattern for all three systems studied, which is suggestive of a universal scaling behavior.

The paper is organized as follows. In Sec. II, we present our model equations. Then we quantify frustration by introducing a frustration parameter that is a function of the time-delayed phase differences between the oscillators. In Sec. III, we study the variation of this frustration parameter as a function of coupling delay in a system of three coupled oscillators repulsively linked to each other in a triangular configuration. We next study, in Sec. IV, a system of six oscillators repulsively linked in a configuration that encloses a triangle within a triangle. The study is further extended to a triangular 4×4 lattice of oscillators in Sec. V. We discuss our findings and conclude in Sec. VI.

II. THE MODEL

We consider a system of N delay coupled phase oscillators with the dynamics of their phases ϕ_i governed by the equations

$$\frac{d\phi_i}{dt} = \omega_i + \frac{\kappa}{\nu_i} \sum_j A_{ij} \sin[\phi_j(t-\tau) - \phi_i(t)]. \qquad (1)$$

Here ω_i is the natural frequency of the ith oscillator, κ is the coupling strength, τ is the time delay in the interactions between the oscillators, and ν_i denotes the number of neighbors to which the ith unit is connected. A_{ij} is the adjacency matrix with $A_{ii} = 0$, $A_{ij} = 1$ if $i \neq j$ and units i and j are connected, otherwise $A_{ij} = 0$. We consider the oscillators to be identical, therefore we set $\omega_i = \omega$ and the coupling between the oscillators is nearest neighbor only. Since we are primarily interested in the dynamics of repulsively coupled oscillators, we take $\kappa = -|\kappa|$ unless otherwise stated.

We further define the frustration in the above system by a parameter F given as

$$F = 1 - \frac{1}{\sum_i \nu_i} \frac{\kappa}{|\kappa|} \sum_{i,j=1}^{N} A_{ij} \cos[\phi_j(t-\tau) - \phi_i(t)], \qquad (2)$$

where we have added the quantity 1 on the right-hand side so that $F = 0$ defines the no frustration state. From (2) it is evident that $0 \leqslant F \leqslant 2$, where $F = 0$ corresponds to nonfrustrated systems and $F = 2$ corresponds to maximally frustrated systems. In the absence of delay ($\tau = 0$), the definition of our frustration parameter reduces to the global frustration defined in [19,20]. Next, we explore the dynamics of some frustrated networks and investigate the role of time delay in influencing that dynamics.

III. THREE OSCILLATORS

We first consider a system of three repulsively coupled oscillators linked in a triangular configuration [Fig. 1(a)]. The dynamical equations (1) then reduce to

$$\dot{\phi}_{i=1,2,3} = \omega + \frac{\kappa}{2} \sum_{j=1, j\neq i}^{3} \sin[\phi_j(t-\tau) - \phi_i(t)]. \qquad (3)$$

A. Equilibrium states and their stability

In the absence of time delay, when the coupling between the oscillators is attractive, i.e., $\kappa > 0$, the oscillators can synchronize in-phase, whereas repulsively coupled ($\kappa < 0$) oscillators try to maximize the phase difference between them and synchronize to a state where there is a finite phase difference between them. In the simplest case of two repulsively coupled oscillators, they always synchronize in an antiphase state with a phase difference of π between them. For three repulsively coupled oscillators [Fig. 1(a)], all the pairs of oscillators cannot simultaneously attain a phase difference of π. The equilibrium state of this system is one in which all the oscillators are frequency-synchronized with a phase difference of $\Delta\Phi = 2\pi/3$ along each link. The presence of time delay in the coupling can significantly change this simple picture. It can make the in-phase synchronous state as well as the out-of-phase synchronous state stable for both attractively coupled as well as repulsively coupled oscillators, depending upon the choice of time delay and other parameters. A generalized stability criterion for the in-phase synchronous state in networks of identical phase oscillators with delayed sinusoidal coupling has been derived by Earl and Strogatz [21] to be

$$\kappa \cos(\Omega_{\text{in}}\tau) > 0, \qquad (4)$$

where the collective frequency Ω_{in} of the in-phase synchronous state is given by

$$\Omega_{\text{in}} = \omega - \kappa \sin(\Omega_{\text{in}}\tau). \qquad (5)$$

This stability condition holds for any network in which all the oscillators have the same number of connections (i.e., $\nu_i = \nu$), independent of all other details of its topology. Using this criterion, we can deduce that when the coupling between the oscillators is repulsive (i.e., $\kappa = -|\kappa|$), the in-phase state is stable for $(2n + \frac{1}{2})\pi < \Omega_{\rm in}\tau < (2n + \frac{3}{2})\pi$ such that $n = 0,1,2,3,\ldots$, and when the coupling between the oscillators is attractive (i.e., $\kappa = |\kappa|$), the in-phase state is stable for $2n\pi \leqslant \Omega_{\rm in}\tau < (2n + \frac{1}{2})\pi$ and $(2n + \frac{3}{2})\pi < \Omega_{\rm in}\tau \leqslant (n+1)2\pi$.

For a state with a finite phase difference between the oscillators, a generalized stability analysis has been carried out by D'Huys *et al.* [22], who considered the phase-locked solutions of a ring of bidirectionally coupled oscillators. The phase-locked state can be characterized as

$$\phi_m(t) = \Omega t + m\Delta\Phi, \tag{6}$$

where $m = 1,\ldots,N$ and $\Delta\Phi = 2j\pi/N$, $j \leqslant N/2$. The collective frequency Ω of this state is obtained from the expression

$$\Omega = \omega - \kappa \sin(\Omega\tau)\cos(\Delta\Phi). \tag{7}$$

To determine the stability of these phase-locked solutions, D'Huys *et al.* [22] performed a linear stability analysis and obtained the following equation for the eigenvalue λ:

$$\lambda = \kappa \cos\Delta\Phi \cos\Omega\tau\left[-1 + e^{-\lambda\tau}\left(\cos\frac{2m\pi}{N} + i\tan\Omega\tau\tan\Delta\Phi\sin\frac{2m\pi}{N}\right)\right]. \tag{8}$$

The above equation can be conveniently used to determine the stability of the phase-locked states for the simple systems of two coupled oscillators as well as our model of three coupled oscillators. The antiphase synchronous state of two coupled oscillators corresponds to $\Delta\Phi = \pi$ and will only be stable when $\kappa\cos(\Omega\tau) < 0$, as discussed in [22]. For three coupled oscillators, depending upon the choice of parameters and initial conditions, the stable equilibrium state of the system is either an in-phase synchronous state for which the phase difference is $\Delta\Phi = 0$ or an out-of phase state with $\Delta\Phi = 2\pi/3$. The collective frequency and the stability of the in-phase synchronous state is given by Eqs. (5) and (4), respectively. The collective frequency $\Omega_{\rm out}$ of the out-of-phase synchronous state is obtained by substituting $\Delta\Phi = 2\pi/3$ in Eq. (7), and it reads

$$\Omega_{\rm out} = \omega + \frac{\kappa}{2}\sin(\Omega_{\rm out}\tau) \tag{9}$$

and is stable when none of the eigenvalues obtained from Eq. (8) have a positive real part. Using these criteria, we have plotted in Fig. 2 the variation of the stable in-phase collective frequencies $\Omega_{\rm in}$ (black solid lines) and stable out-of-phase collective frequencies $\Omega_{\rm out}$ (red dashed lines) as functions of the delay parameter τ for a coupling strength $\kappa = -2$ and an intrinsic frequency $\omega = 6$. We note that for $\tau = 0$ one can only have an out-of-phase state, but beyond a certain value of τ an in-phase state can also become stable. There is also a small overlap region where both states are stable and the initial conditions would dictate the choice of the equilibrium state. The collective frequencies of both states decrease as a function of τ—the well-known phenomenon of frequency suppression that has been noted before for attractively coupled oscillators [23,24].

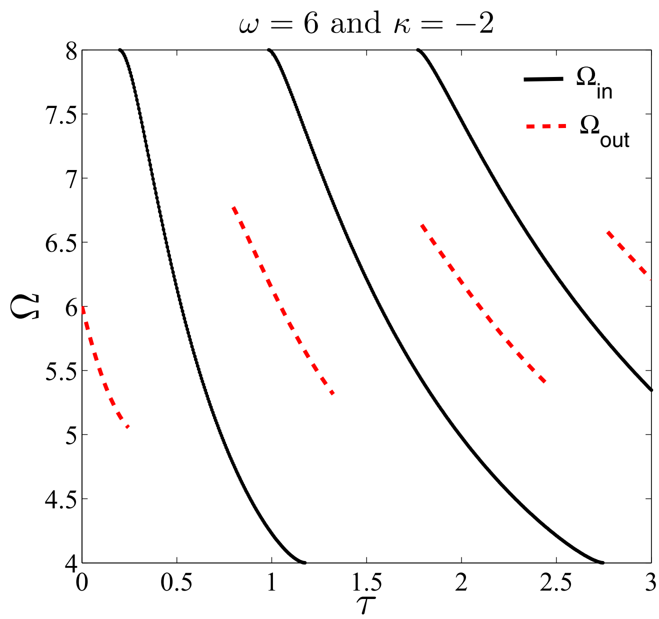

FIG. 2. Plot of the stable collective frequencies $\Omega_{\rm in}$ and $\Omega_{\rm out}$ of the oscillators in the in-phase state (black solid line) and the out-of-phase state (red dashed line) as functions of the delay parameter τ.

B. Frustration analysis

As discussed above, the equilibrium state in the absence of time delay for the present case of three repulsively coupled Kuramoto oscillators is one in which the phase difference along each link is $2\pi/3$. Setting $\tau = 0$, $N = 3$, $\kappa = -2$, $\nu_i = 2$, and the phase differences to be $2\pi/3$ in (2), the frustration value for such a state is seen to be $F = 0.5$. In the presence of time delay, (2) shows that frustration can vary with τ. This variation of F with τ is plotted in Fig. 3(a) for various values of the intrinsic frequencies ω of the individual oscillators and for a fixed value of κ. In Fig. 3(b), we have plotted the variation of F with τ for various values of κ and a fixed value of ω. We see that in all cases F increases at first as a function of τ, but beyond a certain value of delay, which changes for different values of ω and κ, there is a sudden drop in the value of F and thereafter it starts to decrease and eventually goes to zero such that the three oscillators become synchronized in-phase. The precipitous decrease in F beyond a critical value of τ is suggestive of a first-order phase-transition phenomenon marking the evolution of the system from a finite phase-locked state to an in-phase synchronized state. Before the critical delay, the oscillators are in a phase-locked state with $\Delta\Phi = 2\pi/3$, and beyond the critical delay the oscillators are synchronized in-phase. This abrupt change in the relative phase between the oscillators at a critical delay is known as a phase-flip bifurcation [25] and is a general feature of time-delay coupled systems. This generic behavior is seen for all the values of ω plotted in Fig. 3(a), with the curves shifted from one another as a function of ω at a constant κ. Likewise in Fig. 3(b), the curves are shifted from one another as a function of κ at a fixed ω. However, the data of the shifted curves can be made to lie on a single universal curve if we plot F as a function of $\Omega\tau$ instead of τ as shown in Fig. 3(c). Note that data points representing the value of F for a particular set of ω and κ values that disappear

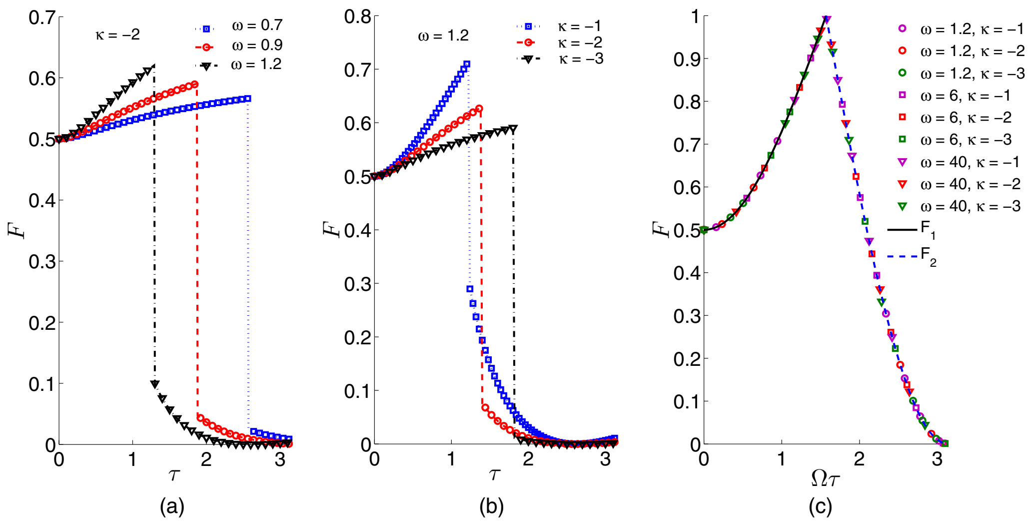

FIG. 3. Plots showing the variation of the frustration parameter F as a function of the delay parameter. F is calculated using Eq. (2) from numerical solutions of Eq. (1) for the same set of initial phases for (a) three values of intrinsic frequency ω at fixed $\kappa = -2$, and (b) three values of coupling strength κ at fixed $\omega = 1.2$. (c) In this figure, F is plotted against the product of collective frequency Ω and delay parameter τ for various values of ω and κ and a given set of initial phases. Frustration values for different intrinsic frequencies as well as coupling strengths are seen to lie on a common curve given by F_1 and F_2, which are obtained from Eqs. (10) and (11), respectively.

beyond a value of $\Omega\tau$ on the left side of the curve continue their progress on the lower half of the right side curve, reflecting the trends seen in Figs. 3(a) and 3(b). This consolidated curve, which is common for all ω and κ values, provides a universal scaling behavior for the variation of the frustration parameter in a time delay coupled system of three Kuramoto oscillators configured in a frustrated triangular configuration. As we will shortly show, this scaling behavior continues to hold for even larger systems and different geometrical configurations and is thus of a universal nature. The physical nature of this curve can be easily understood for the three-oscillator case from the expression for F given by (2). In the presence of time delay, the phase differences between the oscillators have two contributing factors: one is the phase shift of $2\pi/3$ because of the repulsive coupling, and other is the phase shift equivalent to $\Omega\tau$. It is the interplay between these two factors that governs the amount of frustration in the system and also determines the critical transition point. Analytic expressions for the universal curve can be easily obtained from the general expression for F given in (2) for the variation of frustration with $\Omega\tau$. As mentioned before, at $\tau = 0$, the oscillators have a phase difference of $[\phi_j(t) - \phi_i(t) = \pm 2\pi/3]$ between them and hence $F = 0.5$. When we introduce delay, the time-delayed phase differences between the oscillators have another contributing factor of $\Omega\tau$, i.e., $[\phi_j(t-\tau) - \phi_i(t) = -\Omega\tau \pm 2\pi/3]$. Substituting these time-delayed phase differences in the expression of frustration [Eq. (2)] gives one branch of the analytical curve that corresponds to the out-of-phase state [shown by the solid black line in Fig. 3(c)],

$$F_1 = 1 + \frac{1}{2}\left[\cos\left(\frac{2\pi}{3} - \Omega\tau\right) + \cos\left(\frac{-2\pi}{3} - \Omega\tau\right)\right]$$
$$= 1 - \frac{1}{2}\cos(\Omega\tau). \tag{10}$$

At $\Omega\tau = \pi/2$, the frustration value reaches a maximum value of unity. Beyond $\Omega\tau = \pi/2$, the stability condition for the in-phase synchronous state, namely $\cos(\Omega\tau) < 0$, is satisfied and all the oscillators are now synchronized in-phase. The phase shift introduced by the repulsive coupling vanishes and F_1 is no longer a valid expression for determining the frustration values. Since the in-phase synchronous state corresponds to $\phi_j = \Omega t$, phase differences $[\phi_j(t-\tau) - \phi_i(t) = -\Omega\tau]$ and we get the second half of the curve corresponding to the in-phase state [shown by the dashed blue line in Fig. 3(c)],

$$F_2 = 1 + \cos(\Omega\tau). \tag{11}$$

F_2 decreases with an increase in $\Omega\tau$ and frustration becomes 0 at $\Omega\tau = \pi$. Therefore, the system is most frustrated at $\Omega\tau = \pi/2$ with $F = 1$ and is least frustrated at $\Omega\tau = \pi$ with $F = 0$. Beyond $\Omega\tau = \pi$, F increases monotonically until $\Omega\tau = 3\pi/2$, beyond which it starts decreasing until $\Omega\tau = 2\pi$. At $\Omega\tau = 3\pi/2$, the system transits from an in-phase state to an out-of-phase state. In other words, the behavior of F in the region $\pi < \Omega\tau < 2\pi$ is a mirror image of its behavior in the range $0 < \Omega\tau < \pi$. The behavior of frustration parameter F is 2π-periodic with respect to the product $\Omega\tau$. This is also apparent from the analytic expressions for the frustration parameter and the stability conditions for the in-phase and out-of-phase states, where $\Omega\tau$ always comes in the argument of sine or cosine, which are 2π-periodic.

To return to the analogy of a phase transition occurring at a critical value of $\tau = \tau_c$ where the precipitous drop in F is seen, it is instructive to look at the time evolution of F for various values of τ and for a fixed set of values of ω and κ. We find that as τ increases, the frustration parameter takes a longer and longer time to settle down to a constant value corresponding to the final equilibrium state. We denote the time after which the system settles down to an equilibrium state by t_{sat}. As τ approaches τ_c, this time increases in a resonant fashion and a graphical depiction of the saturation time as a function of τ is shown in Fig. 4. The functional behavior of t_{sat} can be closely represented by the following expression obtained from

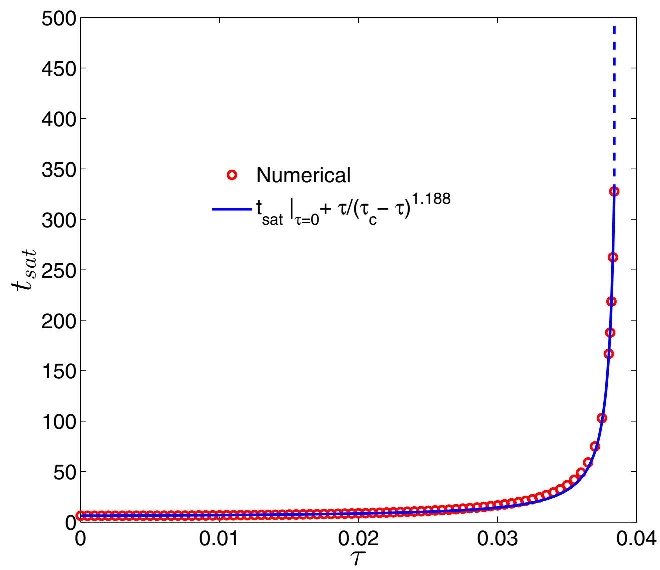

FIG. 4. Numerical data points and analytical fit of the variation of the saturation time t_{sat} are plotted with time delay τ for a system of three repulsive oscillators for $\omega = 40$ and $\kappa = -2$, for which the critical time delay $\tau_c = 0.0389$.

an analytical fit (solid line curve) to the numerical data points (red circles):

$$t_{\mathrm{sat}} = t_{\mathrm{sat}}\,|_{\tau=0} + \frac{\tau}{(\tau_c - \tau)^{\alpha}}. \tag{12}$$

In Fig. 4 we have $\omega = 40$, for which $\tau_c = 0.0389$ and α in the analytical fit curve is 1.188. The dashed line shows that $t_{\mathrm{sat}} \to \infty$ as $\tau \to \tau_c$. As we will see, this algebraic behavior of the saturation time close to the critical value of the delay time is found in other frustrated configurations as well with a "critical index" that is greater than unity.

IV. SIX OSCILLATORS

We next study the collective states of a network comprised of six phase-repulsive oscillators configured as shown in Fig. 1(b). Here oscillators are distinguishable in terms of the number of their nearest neighbors. This configuration is one among a host of repulsive networks that can exhibit multiple final dynamical states characterized by different values of frustration, as discussed in [19]. In the absence of time delay, while the network of three oscillators studied in the previous section shows only one equilibrium state, the present configuration of six oscillators has two equilibrium states characterized by different values of the frustration parameter. In Ref. [19], the frustration values F for this network were computed for 10^4 different initial phases in the absence of time-delayed interactions. The author has reported that 42% of the initial phases resulted in the equilibrium state with $F = 0.5$, and the remaining 58% settled to an equilibrium state with $F = 0.5505$. In an equilibrium state, the oscillators are frequency-synchronized with phase-locked motion. The oscillators sharing the same phases are considered to form a cluster. The oscillators in different clusters have the same frequency but different instantaneous values of phases. The

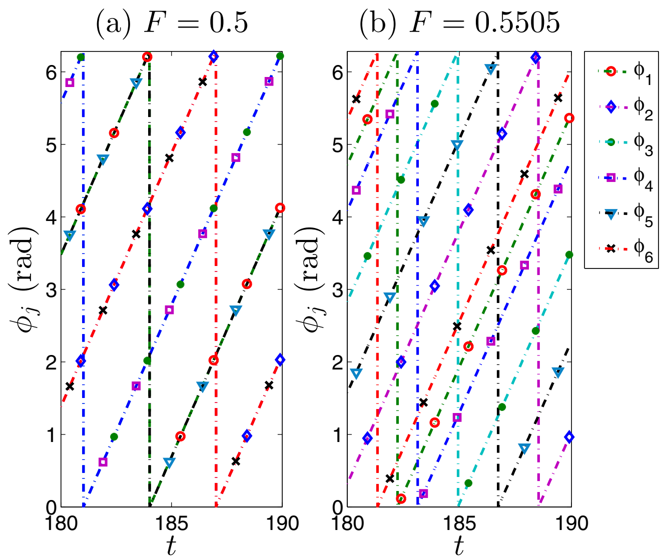

FIG. 5. The phase patterns corresponding to the two equilibria of the six-oscillator network shown in Fig. 1(b) in the absence of time delay. In this figure, the phases of all six oscillators are plotted together. Part (a) is obtained from an initial condition that settles to the equilibrium state corresponding to $F = 0.5$. This equilibrium state corresponds to a three-cluster pattern such that there are two oscillators in each cluster, and the oscillators in the same cluster share the same phase. Part (b) is obtained from an initial condition whose equilibrium state corresponds to $F = 0.5505$. This equilibrium state has a six-cluster pattern, i.e., the instantaneous values of individual phases of all six oscillators are different. We have taken $\omega = 0.7$ and $\kappa = -2$ to get this figure.

equilibrium state with $F = 0.5$ corresponds to the three-cluster pattern with two oscillators in each cluster. This is shown in Fig. 5(a), where the time evolutions of the phases of all six oscillators are plotted together. The equilibrium state corresponding to $F = 0.5505$ exhibits the six-cluster pattern where the instantaneous values of individual phases of all six oscillators are different, and it is plotted in Fig. 5(b). The existence of multiple equilibrium states makes this system an interesting candidate to study how frustration evolves with time delay for different initial conditions for such systems. We find that this six-oscillator network shows a similar tendency for the F versus τ and F versus $\Omega\tau$ variation as seen for the simpler three-oscillator system. In Fig. 6(a), we have plotted F against $\Omega\tau$. For some values of $\Omega\tau$, frustration F has two values. The value of F switches between these two values depending upon the choice of the initial phases. In the left half of Fig. 6(a), the lower curve corresponds to the three-cluster equilibrium state and the upper curve corresponds to the six-cluster equilibrium state. The universal scaling behavior where the variation in frustration with $\Omega\tau$ is seen to lie on a common curve for all ω and κ values holds for this six-oscillator configuration also. Even though the frustration for some values of $\Omega\tau$ depends on the choice of initial phases, for a given initial condition the frustration values for different intrinsic frequencies lie on the same curve. Just like the case with three coupled oscillators, the system is most frustrated at $\Omega\tau = \pi/2$ with $F = 1$ and nonfrustrated at

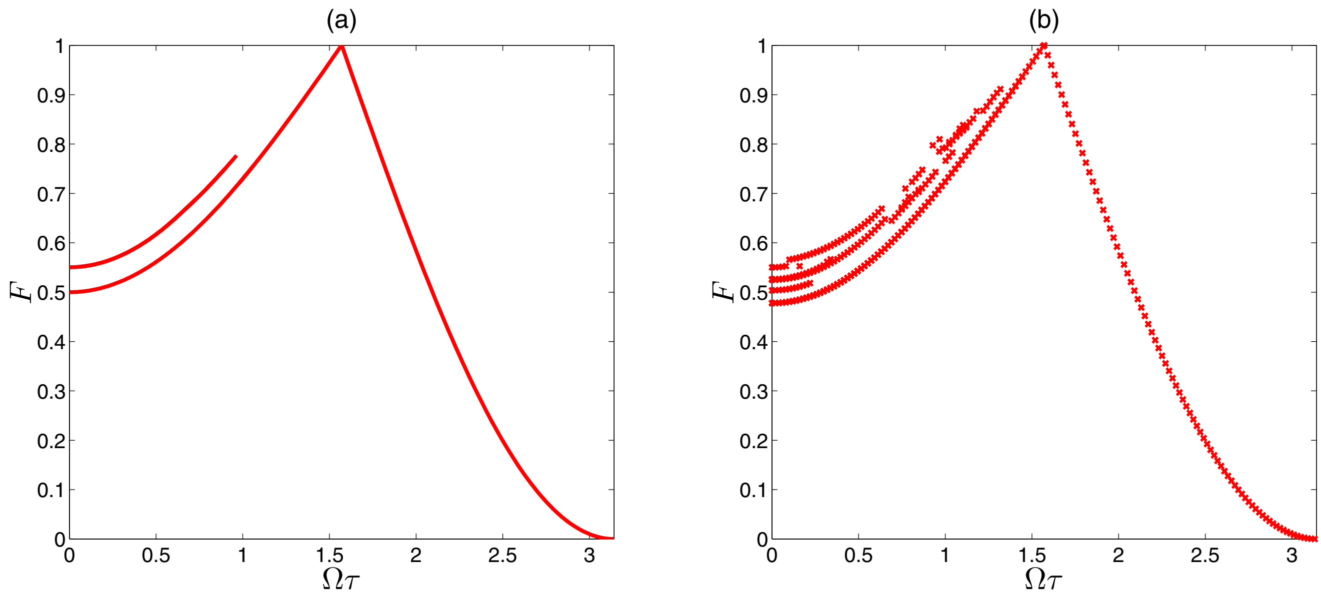

FIG. 6. In this figure, F is plotted against the product of collective frequency Ω and delay parameter τ for (a) the system of six oscillators and (b) the 4×4 triangular lattice with free boundaries. These systems show the existence of multiple equilibrium states characterized by different values of the frustration parameter for some values of delay, and therefore frustration F can have multiple values at some values of $\Omega\tau$. Depending on the choice of initial phases, the system can settle down to any of these equilibrium states. We have taken $\kappa = -2$ to obtain all the curves.

$\Omega\tau = \pi$. The functional behavior of t_{sat} for the lower branch of F variation with τ is similar to the three-oscillator system with $\alpha = 1.325$ and $\tau_c = 0.0384$ in Eq. (12). In Fig. 7, we have shown the variation in the sizes of the basins of different equilibrium states of the six-oscillator system with time delay. The results have been obtained by evolving the system for 10^4 different initial phases. We find that in the absence of delay ($\tau = 0$), similar to the results obtained in [19], about 42% of the initial conditions settle to the equilibrium state with $F = 0.5$, and the remaining 58% settle down to the final state with $F = 0.5505$. These percentages change with an increase in delay, and after a certain value of delay, for some initial conditions the system settles down to the in-phase state instead. The basin of attraction of the in-phase state increases with a further increase in delay.

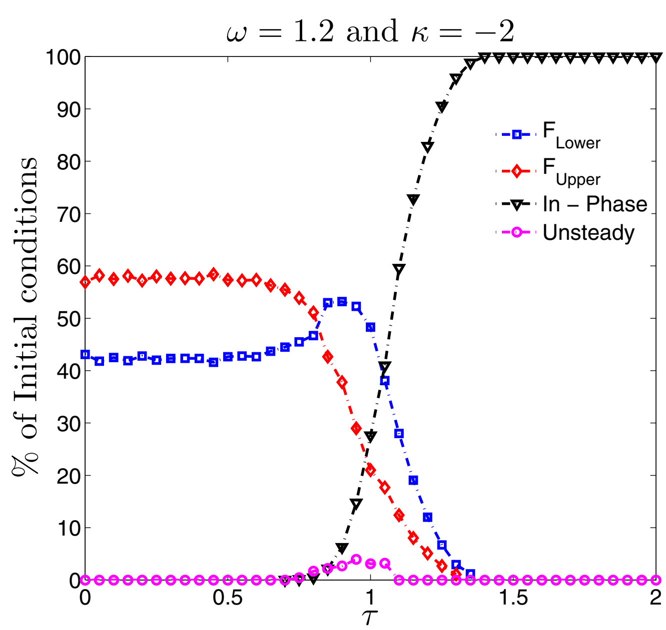

FIG. 7. The blue dashed line with squares shows the percentage of initial phases that lead to an equilibrium state with a lower value of F denoted by F_{lower}. The red dashed line with diamonds shows the percentage of initial phases that lead to the equilibrium states with a higher value of F denoted by F_{upper}. The black dashed line with triangles shows the percentage of initial phases that lead to the in-phase equilibrium state labeled by In-Phase, and the magenta dashed line with circles shows the percentage of initial conditions for which the system has not settled to any stable equilibrium state even for a very large time span ($t_{\text{span}} = 2000$ in simulations), and it is labeled as Unsteady.

V. TRIANGULAR LATTICE

We now investigate an extended network consisting of a triangular lattice formed by joining triangles along the edges thereby forming a multiunit system whose basic unit cell is the system of three oscillators which we have studied in detail in Sec. III. We consider a 4×4 triangular lattice with a hexagonal coupling configuration as shown in Fig. 1(c). Repulsive coupling in combination with the hexagonal coupling pattern turns all bonds into frustrated ones. Since frustration leads to the growth of the number of coexisting attractors, this system is highly multistable [15] even in the absence of time delay. Boundary conditions also play an important role. When the boundary conditions are periodic, then each oscillator has the same number of nearest neighbors [$\nu_i = 6$ in Eq. (1)]. When the interactions between the oscillators are instantaneous ($\tau = 0$), the system exhibits a variety of cluster patterns [15] but never goes to a single cluster state where all the oscillators

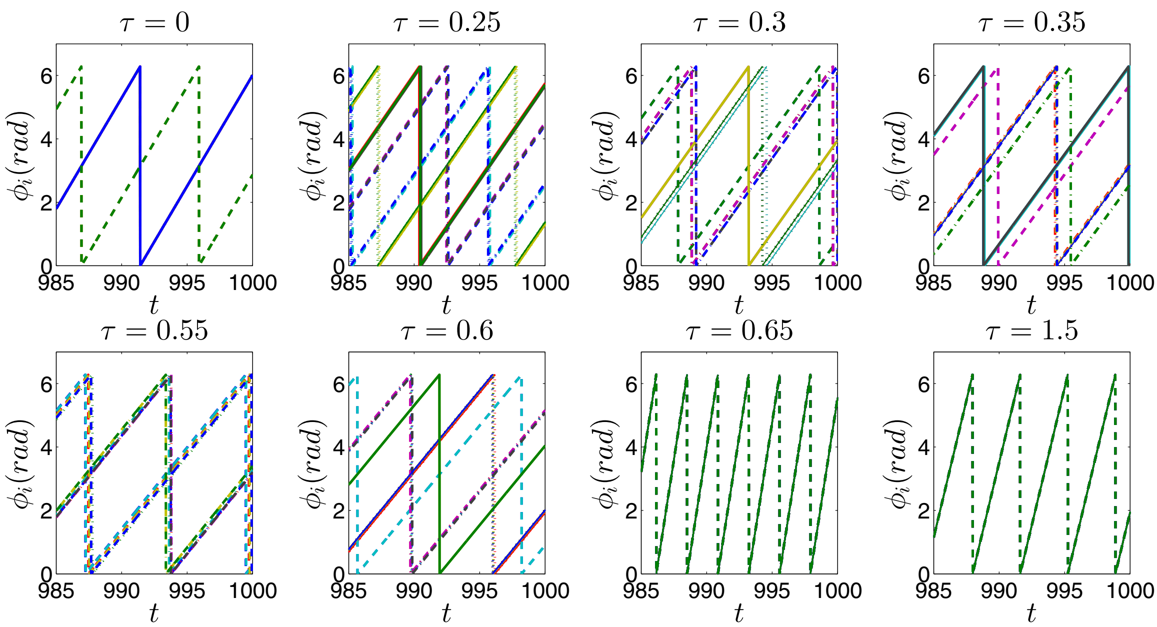

FIG. 8. The variation in the cluster pattern exhibited by the 4×4 lattice of repulsively coupled oscillators with a change in the time-delay parameter τ for a given set of initial phases. At a certain value of delay, the system starts exhibiting the single cluster in-phase state. The collective frequency of the in-phase state decreases with a further increase in delay. We have taken $\omega = 0.7$ and $\kappa = -2$.

are synchronized in-phase. Upon introducing time delay in the coupling between the oscillators, they are able to synchronize in-phase when time delay exceeds a critical value. Starting from the same set of initial phases if we keep on varying the value of the delay parameter τ, the cluster pattern exhibited by the system also varies as shown in Fig. 8. Beyond a certain value of delay, all the oscillators are synchronized in-phase into a single cluster. Since this system is highly multistable even in the absence of delay, the system can go to different cluster states at a given value of delay for different choices of initial conditions. Hence the critical value of delay at which the system first goes to the single cluster state can also be different depending upon the choice of initial conditions. The collective frequency of the in-phase state and its stability condition obey the same expressions [Eqs. (5) and (4), respectively] as the system of three coupled oscillators. The phase-locked equilibrium states exhibiting different cluster patterns at a given value of $\Omega\tau$ correspond to the same value of frustration parameter F. In the absence of time delay, the value of frustration for this system is $F = 0.6667$. The variation of F with τ and $\Omega\tau$ is similar to the systems of three repulsively coupled oscillators. The universal scaling behavior continues to hold for the 16-oscillator triangular lattice, too.

However, if the boundaries of the triangular lattice are free, then the oscillators are distinguishable in terms of the number of their nearest neighbors, unlike the lattice with periodic boundaries where each oscillator has six neighbors. Hence this system exhibits distinct equilibrium states corresponding to different values of frustration parameter F and cluster patterns. In the absence of delay ($\tau = 0$), the system exhibits five equilibrium states—S_1, S_2, S_3, S_4, and S_5—with corresponding frustration values $F(S_1) = 0.4779$, $F(S_2) = 0.5037$, $F(S3) = 0.5249$, $F(S4) = 0.5263$, and $F(S5) = 0.5502$, respectively. In the presence of time delay, similar to the six-oscillator network presented in the previous section, this system also exhibits multiple equilibrium states characterized by different values of frustration at some values of τ or $\Omega\tau$, as shown in Fig. 6(b). The universal scaling behavior continues to hold. The system can switch between these equilibria corresponding to the different F values depending upon the choice of initial phases. For each value of time delay, we have obtained the values of frustration parameter by evolving the system for 500 different initial conditions. In addition to the equilibrium states shown in Fig. 6(b), there might be other equilibrium states also but with the basins of attraction so small that it becomes extremely difficult to observe them numerically.

VI. SUMMARY AND DISCUSSION

To summarize, in this paper we have studied the effect of time-delayed coupling on the collective dynamics of various frustrated systems of repulsively coupled Kuramoto phase oscillators. For our study, we have chosen three typical frustrated configurations, namely (i) a set of three oscillators in a triangular configuration that represents the simplest possible two-dimensional frustrated geometry, (ii) a set of six oscillators configured as a triangle within a triangle that presents a slightly more complex geometry, and (iii) a set of 16 oscillators in a hexagonal lattice geometry representing a generalization of the basic triangular configuration of three oscillators. The three configurations also have distinctly different characteristics in the absence of time delay, e.g., (i) has a single equilibrium state with a unique value of the frustration parameter, (ii) has two equilibrium states (and hence two different frustration values) that the system can go to depending upon the initial conditions, and (iii) has multiple equilibrium states when open-boundary conditions are applied to the lattice, and hence it displays multistable behavior that is dependent on the initial conditions.

We study the dynamical behavior of these prototypical systems by a quantitative investigation of a suitably defined generalized frustration parameter that is a function of the time delay. Our numerical investigations reveal a generic behavior in all the systems where we observe that the frustration parameter initially increases with delay and then precipitously falls after a critical value of delay to a much lower value and then decays to zero, thereby transitioning to an in-phase synchronous state. Thus the presence of a time delay in the coupling that is larger than a critical value removes a basic constraint of frustrated systems and permits them to attain a synchronous state. This happens due to the interplay between the phase-difference contributions arising from the geometry and the time delay. We also find that this behavior can be characterized by a single universal curve representing the variation of F with the product of the collective frequency Ω of the synchronous state and the time delay parameter τ. The curve is common for all values of natural frequencies ω of the individual oscillators as well as the coupling strength κ between the oscillators. An analytic description of this curve is given for three coupled oscillators. The nature of the transition is seen to have the characteristics of a first-order phase transition whose behavior near the transition point can be expressed in terms of an algebraic relation that has a "critical exponent" that is larger than (but close to) unity.

The universal scaling behavior of F with $\Omega\tau$ that holds for all three systems is the most significant finding of our investigation and underscores the important role that time delay can play in the dynamics of frustrated systems. In particular, it shows that time delay can serve as a tuning parameter to steer the system towards different values of the frustration parameter and hence to different collective multistable states. This property can be exploited in physical systems in which time delay can be varied by changing the media characteristics to change the speed of signal propagation. In biological systems, e.g., in neuronal networks where frustration and time delay are coexistent, our results may prove useful in gaining a better understanding of their underlying dynamical behavior.

[1] *Frustrated Spin Systems*, edited by H. T. Diep (World Scientific, Singapore, 2004).

[2] *Introduction to Frustrated Magnetism: Materials, Experiments, Theory*, edited by C. Lacroix, P. Mendels, and F. Mila (Springer-Verlag, Berlin, Heidelberg, 2011).

[3] I. Gilbert, C. Nisoli, and P. Schiffer, Phys. Today **69**(7), 54 (2016).

[4] L. L. Gollo and M. Breakspear, Philos. Trans. R. Soc. London, Ser. B **369**, 20130532 (2014).

[5] L. L. Gollo, A. Zalesky, R. M. Hutchison, M. van den Heuvel, and M. Breakspear, Philos. Trans. R. Soc. London, Ser. B **370**, 20140165 (2015).

[6] Y. Han, Y. Shokef, A. M. Alsayed, P. Yunker, T. C. Lubensky, and A. G. Yodh, Nature (London) **456**, 898 (2008).

[7] N. Choudhury, L. Walizer, S. Lisenkov, and L. Bellaiche, Nature (London) **470**, 513 (2011).

[8] A. Ortiz-Ambriz and P. Tiernoa, Nat. Commun. **7**, 10575 (2016).

[9] M. H. Jensen, S. Krishna, and S. Pigolotti, Phys. Rev. Lett. **103**, 118101 (2009).

[10] S. Krishna, S. Semsey, and M. H. Jensen, Phys. Biol. **6**, 036009 (2009).

[11] H. G. Schuster and P. Wagner, Prog. Theor. Phys. **81**, 939 (1989).

[12] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Phys. Rev. Lett. **80**, 5109 (1998).

[13] R. Dodla, A. Sen, and G. L. Johnston, Phys. Rev. E **69**, 056217 (2004).

[14] *Complex Time-delay Systems*, edited by F. M. Atay (Springer, New York, 2010).

[15] F. Ionita, D. Labavić, M. Zaks, and H. Meyer-Ortmanns, Eur. Phys. J. B **86**, 511 (2013).

[16] M. Nixon, E. Ronen, A. A. Friesem, and N. Davidson, Phys. Rev. Lett. **110**, 184102 (2013).

[17] H. Daido, Phys. Rev. Lett. **68**, 1073 (1992).

[18] P. Kaluza and H. Meyer-Ortmanns, Chaos **20**, 043111 (2010).

[19] Z. Levnajić, Phys. Rev. E **84**, 016231 (2011).

[20] D. H. Zanette, Europhys. Lett. **72**, 190 (2005).

[21] M. G. Earl and S. H. Strogatz, Phys. Rev. E **67**, 036204 (2003).

[22] O. D'Huys, R. Vicente, T. Erneux, J. Danckaert, and I. Fischer, Chaos **18**, 037116 (2008).

[23] S. Kim, S. H. Park, and C. S. Ryu, Phys. Rev. Lett. **79**, 2911 (1997).

[24] E. Niebur, H. G. Schuster, and D. M. Kammen, Phys. Rev. Lett. **67**, 2753 (1991).

[25] A. Prasad, J. Kurths, S. K. Dana, and R. Ramaswamy, Phys. Rev. E **74**, 035204(R) (2006).