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Zero-temperature directed polymer in random potential in 4 + 1 dimensions

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Zero-temperature directed polymer in random potential in $4+1$ dimensions is described. The fluctuation $\Delta E(t)$ of the lowest energy of the polymer varies as t^{β} with $\beta = 0.159 \pm 0.007$ for polymer length t and ΔE follows $\Delta E(L) \sim L^{\alpha}$ at saturation with $\alpha = 0.275 \pm 0.009$, where L is the system size. The dynamic exponent $z \approx 1.73$ is obtained from $z = \alpha/\beta$. The estimated values of the exponents satisfy the scaling relation $\alpha + z = 2$ very well. We also monitor the end to end distance of the polymer and obtain z independently. Our results show that the upper critical dimension of the Kardar-Parisi-Zhang equation is higher than $d = 4 + 1$ dimensions.

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I. INTRODUCTION

Over the past 30 years there have been considerable studies on the problem of directed polymers (DPs) in random potentials [1–7]. Through various mappings, it is related to many other physical problems such as the Eden model [8], the restricted solid-on-solid (RSOS) model [9], ballistic aggregation [10], domain walls in the two-dimensional random bond Ising model [1], Burgers' equation [11], and the Kardar-Parisi-Zhang (KPZ) equation [12]. The KPZ equation describes the evolution dynamics of the interface

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\vec{r},t), \tag{1}$$

where $h(x,t)$ is the local height at time t and the random white noise η satisfies

$$\langle \eta(\vec{r},t)\eta(\vec{r}\,',t')\rangle = 2D\delta(\vec{r}-\vec{r}\,')\delta(t-t'). \tag{2}$$

The directed polymer in the random potential problem [3,4,6,7] is one of the simplest models described by the KPZ equation. Because the polymers are directed, it allows only paths in the forward direction with no reverse step. A simple directed polymer model in random potential in lower dimensions was studied before [4]. For completeness, we briefly describe the model here. Consider a directed polymer on a discrete hypercubic structure. The random potentials $\mu(\mathbf{x},t)$ are assigned to each site where $\mathbf{x}$ and t are the $(d-1)$-dimensional transverse vector and the longitudinal length of the polymer, respectively. The energy of a polymer

$$E = \int dt\left[\gamma\left(\frac{d\mathbf{x}}{dt}\right)^2 + \mu(\mathbf{x},t)\right] \tag{3}$$

is given by following the path. There are two competing terms: One is a bending energy forcing the polymer straight and the other is the random potential $\mu(\mathbf{x},t)$ making the polymer deformed through lower potential sites. A polymer starts from the substrate at $t = 0$ and its path is restricted by $|\mathbf{x}(t) - \mathbf{x}(t+1)| = 0$ or 1. There is a bending energy γ, which represents the stretching energy of the polymer against a transverse jump $|\mathbf{x}(t) - \mathbf{x}(t+1)| = 1$. The initial condition is given as $E(\mathbf{x},0) = 0$. At zero temperature, the minimum energy $E(\mathbf{x},t)$ among all the paths arriving at $(\mathbf{x},t)$ can be obtained recursively, for example, in $d = 1+1$, [4]

$$\begin{aligned} E(\mathbf{x},t) = \min[&E(\mathbf{x}-1,t-1) + \mu(\mathbf{x}-1,t-1) \\ &+ \gamma, E(\mathbf{x},t-1) + \mu(\mathbf{x},t-1), E(\mathbf{x}+1,t-1) \\ &+ \mu(\mathbf{x}+1,t-1) + \gamma], \end{aligned} \tag{4}$$

where $\min[A,B,C]$ takes the minimum value among A, B, and C.

One of the interesting quantities of the directed polymer problem is the standard deviation $\Delta E(t)$ of the minimum energy $E(\mathbf{x},t)$,

$$\Delta E(t) = \langle (E-\bar{E})^2\rangle^{1/2}. \tag{5}$$

Here $\bar{E}(t)$ is the spatial average of the minimum energy at t, $\langle\cdots\rangle$ denotes the average over various samples, and $\Delta E(t)$ is an order parameter similar to the surface width characterizing the roughness of the interface in the surface problems [2,13]. Starting from a flat substrate in a finite system of lateral size L, the mean-square energy fluctuation $\Delta E(t)$ scales as [13]

$$\Delta E(t) = L^{\alpha} f\left(\frac{t}{L^z}\right) \sim \begin{cases} t^{\beta}, & t \ll L^z \\ L^{\alpha}, & t \gg L^z, \end{cases} \tag{6}$$

where α and z are the roughness and the dynamic exponents, respectively. The scaling function $f(x)$ is x^{β} for $x \ll 1$ and constant for $x \gg 1$. The dynamic exponent z describing the polymer dynamics follows the relation $z = \alpha/\beta$.

There has been much theoretical effort to study the universal properties of the KPZ equation. Due to the invariance of the KPZ equation under an infinitesimal tilt of the interface, there exists the scaling relation $\alpha + z = 2$ [12]. Most of the recent works have been devoted to determining these critical exponents. Despite the simple mathematical representation of the directed polymer problem, it is not completely understood except in two dimensions. In $d = 1+1$ dimensions, where there is one transverse and one longitudinal direction, the exponents are known [12] to be $\beta = 1/3$ and $z = 3/2$ and extensive numerical work has confirmed them [2]. Also, recent studies of the largest eigenvalue of a random matrix have shown the exact probability distribution for the height of the KPZ equation [14]. There is some variation among the values of the exponents quoted by various authors in higher dimensions [2]. Based on the numerical simulation results of the RSOS growth

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model, we suggested $\beta = \frac{1}{d+1}$ [9]; however, this seems to be consistent with neither the large-scale numerical simulation results [4,6,15–20] nor the analytic calculations [21–23], which actually disagree with each other. So far, there is no clear understanding of the values of the exponents in the higher dimension. In $d = 3+1$ dimensions it is generally believed that a phase transition exists between a strong-coupling fixed point and an Edwards-Wilkinson-type trivial point. This transition was tested numerically and the critical coupling constant at the transition point was discussed [4,17,24–27]. Also, there is a controversy about the existence of the upper critical dimension of the KPZ equation [28–32]. Above the critical dimension, the exponents take the values $z = 2$ and $\beta = 0$. Most of the analytic studies including the mode-coupling-type approaches of the KPZ equation suggest that the upper critical dimension is less than or equal to $4+1$ [21,22,28–30,32]. However, the numerical studies of the RSOS growth model and the directed polymer problem, which belong to the KPZ universality class, show the existence of a strong-coupling regime in $4+1$ dimensions [15,16,18,20]. The existence of an upper critical dimension is still under debate [15,16,18,20–22,28–32].

Recently, we studied the roughening properties of the discrete growth model [15,16] with the RSOS condition

$$\nabla h = |h(r) - h(r')| \leqslant N, \tag{7}$$

where N is a restriction parameter confining the height difference between the nearest neighbors. In general, the RSOS model shows good scaling behavior of the KPZ universality class [9]. Through numerical simulations, we have shown that the model with the restriction parameter $N = 1$ in higher dimensions suffers from an artifact induced by the discrete nature of the model. This is due to the small width in comparison with the unit of the discrete height [15,16]. To avoid this artifact, we consider a directed polymer problem where the energy parameter can take a continuum value.

Here we present a more detailed analysis of the directed polymer problem in random potentials in particular in $d = 4+1$ dimensions. We employ a transfer-matrix method for the model and obtain $\beta \approx 0.159$. The other exponents α and z are also estimated independently.

II. NUMERICAL SIMULATION OF THE DP PROBLEM AT ZERO TEMPERATURE AND THE SCALING EXPONENTS

Starting from a four-dimensional hypercubic substrate of linear dimension L with periodic boundary conditions in the transverse directions, the random site potential $\mu(\mathbf{x},t)$ is drawn uniformly from the interval (0,1) and the bending energy $\gamma = 1$ is chosen. The time t is defined as the longitudinal length of the polymer. At zero temperature, the minimum energy $E(\mathbf{x},t)$ among all paths arriving at $(\mathbf{x},t)$ is obtained recursively following Eq. (4).

To determine the exponent β governing the growth rate of the energy fluctuation, $\Delta E(t)$ is monitored as a function of t for system size $L = 170$ with $\gamma = 1.0$ averaged over 320 different realizations of the random potentials. At zero temperature, $\Delta E^2(t)$ grows as $t^{2\beta}$ at the beginning, as shown in Fig. 1. From the least-squares fit of the data we obtain

$$\beta = 0.159 \pm 0.07. \tag{8}$$

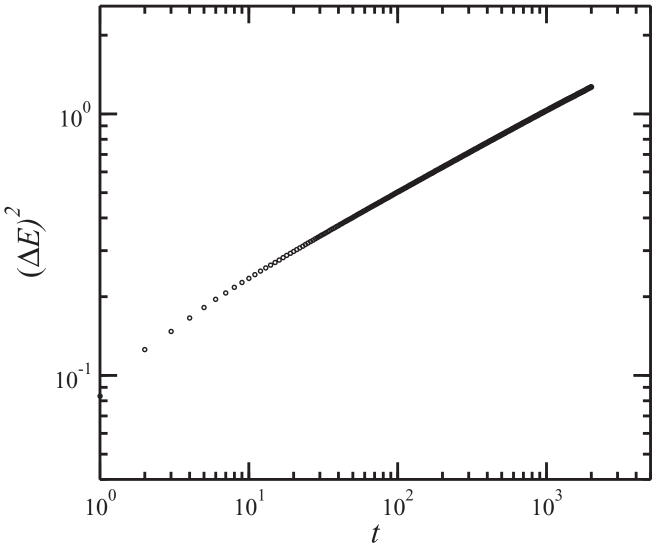

FIG. 1. Mean-square deviation $(\Delta E)^2$ as a function of t plotted on a log-log scale for the system size $L = 170$ in $4+1$ dimensions.

To measure the roughness exponent α describing the energy fluctuations in the late-time region, we monitor the energy fluctuations for the various system sizes L in the saturated region as shown in the inset of Fig. 2. By fitting the data to the relation $(\Delta E)^2(L) \sim L^{2\alpha}$,

$$\alpha = 0.275 \pm 0.009 \tag{9}$$

is estimated. We obtain the dynamic exponent $z \approx 1.73$ from $z = \alpha/\beta$ with the values of β and α. Our results show $\alpha + z \approx 2.01$, which is consistent with the scaling relation $\alpha + z = 2$. The scaling plot of $(\Delta E)^2/L^{2\alpha}$ as a function of t/L^z with the measured exponents $\alpha = 0.275$ and

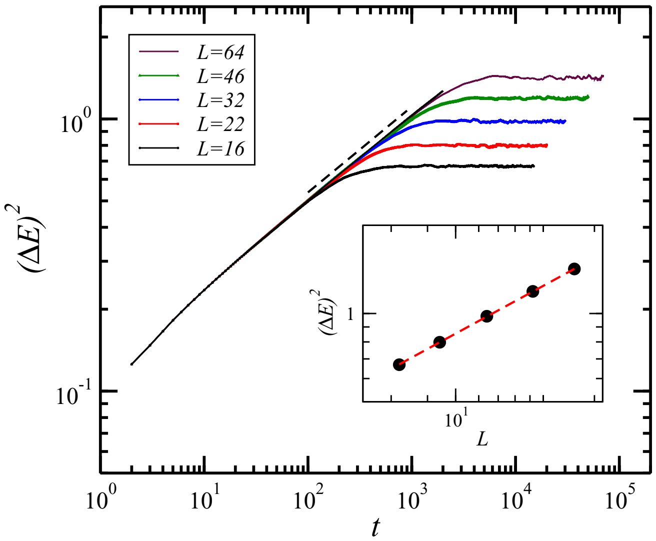

FIG. 2. Plot of $(\Delta E)^2(L,t)$ as a function of t on a log-log scale for the various system sizes $L = 16$, 22, 32, 46, and 64 from the bottom to the top. In the inset, the variation of polymer energy $(\Delta E)^2(L)$ at saturation is plotted as a function of L in $4+1$ dimensions.

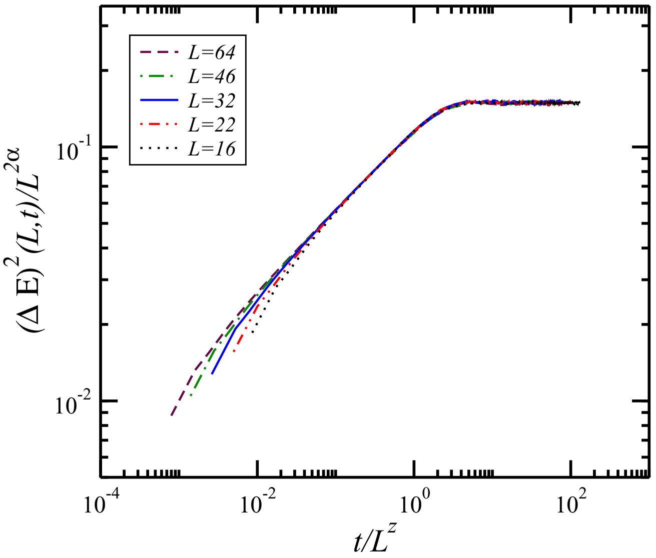

FIG. 3. Scaling plot of $(\Delta E)^2(L,t)$ using $\alpha = 0.275$ and $z = 1.73$ for various system sizes $L = 16$, 22, 32, 46, and 64.

$z = 1.73$ is shown in Fig. 3. A good data collapse is obtained, indicating that $(\Delta E)^2(L,t)$ follows the scaling formula very well. Our obtained values $\beta \approx 0.159$ and $\alpha \approx 0.275$ are in good agreement with the recent results for the RSOS growth model in 4 + 1 dimensions [15].

Another interesting quantity is the end-to-end transverse distance $\Delta X(t)$ of the minimum energy path, which is defined as

$$\Delta X(t) \equiv \langle [\mathbf{x}(t) - \mathbf{x}(0)]^2 \rangle^{1/2}. \quad (10)$$

The angular brackets denote the average over many different realization of the random site energies. The advantage of our study in the DP problem is that not only the exponents β and α but also the dynamic exponent z can be obtained directly. To determine z independently, we also monitor $\Delta X(t)$ averaged over 320 different realization of the random site energies with system size $L = 170$. Here $\Delta X(t)$ is expected to increase as $t^{1/z}$ with t for $t \ll L^z$. As shown in Fig. 4, the plot of $\log[\Delta X(t)]^2$ against $\log(t)$ is a straight line. Using the relation $\Delta X(t) \sim t^{1/z}$, we obtain $z = 1.75 \pm 0.02$. It is slightly larger than but consistent with $z \approx 1.73$ (with an error of less than 2%) indirectly estimated from β and α.

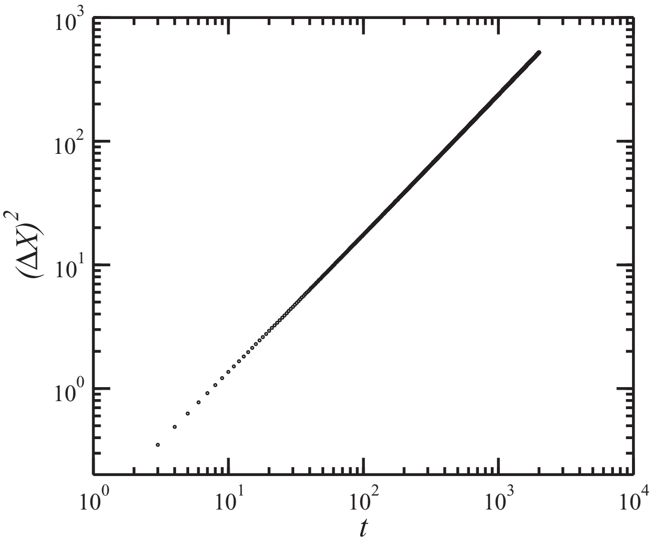

FIG. 4. Mean-square end-to-end distance $(\Delta X)^2(t)$ as a function of t plotted on a log-log scale for the system size $L = 170$ in 4 + 1 dimensions.

III. CONCLUSION

We have presented a numerical analysis of directed polymers in 4 + 1 dimensions. It is known that the directed polymer problem in random potential belongs to the KPZ universality class. We obtained $\beta \approx 0.159$, $\alpha \approx 0.275$, and $z \approx 1.73$, which are in good agreement with the recent results of $\beta \approx 0.158$ and $\alpha \approx 0.273$ from the RSOS model in 4 + 1 dimensions. The estimated β is slightly less than but close to our conjectured value of 1/6. Considering the finite-size effect, our numerical results do not exclude 1/6. We also estimated $z = 1.75 \pm 0.02$ independently by measuring the end-to-end distance of the polymer. Because the obtained z is less than 2, the transverse walk of the polymer becomes superdiffusive, implying that the upper critical dimension is higher than $d = 4 + 1$. Further analytical and numerical studies are required to define precisely the critical values of the exponents.

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