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Exact probability distribution functions for Parrondo's games

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We study the discrete time dynamics of Brownian ratchet models and Parrondo's games. Using the Fourier transform, we calculate the exact probability distribution functions for both the capital dependent and history dependent Parrondo's games. In certain cases we find strong oscillations near the maximum of the probability distribution with two limiting distributions for odd and even number of rounds of the game. Indications of such oscillations first appeared in the analysis of real financial data, but now we have found this phenomenon in model systems and a theoretical understanding of the phenomenon. The method of our work can be applied to Brownian ratchets, molecular motors, and portfolio optimization.

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Parrondo's games [1–8] are related to Brownian ratchets [9–15] and exhibit interesting phenomena at the intersection of game theory, econophysics, and statistical physics (see [8] for interdisciplinary applications). In the case of Brownian ratchets, a particle moves in a potential, which randomly changes between two versions. For each there is a detailed balance condition. However, for random switches between the two potentials, there is on average a directed motion. This phenomenon is fundamentally related to portfolio optimization [16,17], and corresponds to the "volatility pumping" strategy in portfolio optimization. For a two-asset portfolio one half of the capital is kept in the first asset, and the other half in the second asset with high volatility [18].

Parrondo invented a game-theoretic model of a Brownian flashing ratchet, thus producing a discrete-time model of the ratchet effect [1]. An agent tosses biased coins using one of two strategies (games), and both strategies are losing. In some cases a random combination of the losing games is a winning game. Of course, the opposite situation is also possible; a random combination of two winning games can give a losing game.

The state of the system is characterized by the current value of the capital X, and the choice of the strategy. X is defined on a one-dimensional axis with discrete points, a "chain." In analogy to ratchets, there may be a periodicity M in the rules, how capital X can increase or decrease. This version corresponds to a particle moving on a ladder geometry with several rungs [19]. Originally $M = 3$ games were considered, then $M = 2$ versions of Parrondo's games were constructed [7,20]. For the history dependent versions of the game the current rules of the game depend on the past, whether there was growth of capital in the previous rounds or not.

Later many modifications of the games were invented, i.e., games with different integers M for both games [21], the Allison mixture [22] where random mixing of two random sequences creates autocorrelation [23], and two envelope game problems [24]. Especially intriguing is an anti-Parrondo effect or *Verschlimmbesserung* where reduced confidence in a measurement results from an increase in the number of observations that are in agreement [25].

As the variances of distributions (volatilities) are important in economics, we calculated the variance for the history independent case with specific parameters in our recent work [26]. And vice versa, for obtaining the unknown parameters of a model that describes empirical data, the complete distributions have to be available. This is the problem we address here by applying a Fourier transform technique to solve exactly the probability distribution function. This approach allows for an efficient calculation of the long time asymptotics from saddle-point contributions. We discover that under certain conditions subleading saddle points become degenerate in absolute value with the leading saddle point. Still, the degenerate saddle points differ in phases which leads to strong fluctuations.

We apply this method to random walks on chains and ladders corresponding to capital dependent Parrondo's models. We calculate the entire probability distribution for the capital, then solve the same problem for history dependent games.

A biased discrete space and time random walk. As an illustration let us consider the discrete time random walk on a chain, where the probability of right and left jumps are p and q, respectively. We can write the master equation for the probability $P(n,t)$ at position n after t steps:

$$P(n,t+1) = pP(n-1,t) + qP(n+1,t) + (1-p-q)P(n,t). \tag{1}$$

The initial distribution is $P(n,0) = \delta_{n,0}$.

For the motion on the infinite axis we can always write a Fourier transform like

$$P(n,t) = \int_{-\pi}^{\pi} dk\, e^{ikn} \bar{P}(k,t),$$
$$\bar{P}(k,t) = \frac{1}{2\pi} \sum_n P(n,t)e^{-ikn}. \tag{2}$$

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For the initial distribution the Fourier transform is $\bar{P}(k,0) = 1/2\pi$.

Equation (1) transforms into

$$\bar{P}(k,t+1) = [pe^{-ik} + qe^{ik} + (1-p-q)]\bar{P}(k,t). \quad (3)$$

We obtain the solution

$$\bar{P}(k,t) = \lambda^t(k)\cdot \bar{P}(k,0),$$
$$\lambda(k) = [pe^{-ik} + qe^{ik} + (1-p-q)]. \quad (4)$$

As λ is a linear polynomial in e^{ik} and e^{-ik}, λ^t is a polynomial with monomials ranging from e^{-tik} to e^{tik}.

The Fourier transform from momentum to spatial representation yields

$$P(n,t) = \int_{-\pi}^{\pi} dk\, e^{tV(ik)+ikn}\bar{P}(k,0),$$
$$V(\kappa) := \ln[pe^{-\kappa} + qe^{\kappa} + (1-p-q)], \quad (5)$$

where we defined the function $V(\kappa)$. Using that λ^t has a finite expansion in powers of e^{ik} we obtain with $\bar{P}(k,0) = 1/2\pi$

$$P(n,t) = \frac{1}{2t}\sum_{m=1}^{2t} e^{tV(im\pi/t)+imn\pi/t}. \quad (6)$$

Also for the study of more involved models, we will use both representations (5) and (6).

Note that Eqs. (5) and (6) are exact for any t and n. By use of the saddle-point approximation we derive the large t and $n = xt$ asymptotics. We are allowed to move the integration contour because of the analytic dependence of the integrand on k resulting in ($\kappa = ik$)

$$P(n,t) = \frac{\exp[tu(x)]}{\sqrt{2\pi t V''(\kappa)}},$$
$$x = -V'(\kappa), \quad u(x) = V(\kappa) + \kappa x. \quad (7)$$

As $u(x)$ is the Legendre transform of $-V(\kappa)$, we also have $V''(k) = -1/u''(x)$ and hence

$$P(n,t) = \frac{1}{\sqrt{2\pi t}}\exp[tu(x) + 1/2\ln|u''(x)|].$$

Let us assume an expansion for $V(\kappa)$,

$$V(\kappa) = r\kappa + K\kappa^2/2. \quad (8)$$

It then follows that

$$\langle n\rangle = rt, \quad \langle (n-\langle n\rangle)^2\rangle = Kt. \quad (9)$$

Random walks with periodicity. Consider the case of a random walk on an axis, using rules with period M. We divide the entire x axis in intervals of length M, $[(n-1)M,nM[$, and label points by (n,l) where $l = \text{Mod}(X,M)$. We represent the sets of p_X (the discrete probability distribution of the capital value) by $P_l(n,t), 0 \leqslant l < M$. The integer t represents time. This bookkeeping allows us to consider the geometry with periodicity M as a multirung ladder or as a chain of unit cells containing M points.

We study the following master equation [5]:

$$P_l(n,t+1) = p_{l_-}P_{l_-}(\hat{n},t) + q_{l_+}P_{l_+}(\bar{n},t)$$
$$+ (1-p_l-q_l)P_l(n,t), \quad (10)$$

where $l_- = \text{Mod}(l-1,M)$, $l_+ = \text{Mod}(l+1,M)$, $\hat{n} = n$ for $l-1 \geqslant 0$, and $\hat{n} = n-1$ for $l-1 < 0$; $\bar{n} = n$ for $l+1 < M$ and $\bar{n} = n+1$ for $l+1 > M$. Thus the model is characterized by the parameters p_l,q_l, where p_l and q_l are the probabilities to win and lose for capital X with $l = \text{Mod}(X,M)$.

Again we consider the Fourier transform

$$P_l(n,t) = \int_{-\pi}^{\pi} dk\, e^{ikn}\bar{P}_l(k,t), \quad (11)$$

and obtain

$$\bar{P}_l(k,t+1) = p_{l_-}e^{ik(\hat{n}-n)}\bar{P}_{l_-}(k,t) + q_{l_+}e^{ik(\bar{n}-n)}\bar{P}_{l_+}(k,t)$$
$$+ [1-(p_l+q_l)]\bar{P}_l(k,t)$$
$$\equiv \sum_{m=0}^{M-1}\hat{Q}_{lm}(ik)\bar{P}_m(k,t). \quad (12)$$

Using the eigenvalues and eigenvectors $\lambda_m(\kappa), v_{ml}(\kappa)$ ($m = 0,\ldots,M-1$), of the matrix $\hat{Q}(\kappa)$, we find

$$\bar{P}_l(n,t) = \int_{-\pi}^{\pi} dk\, e^{ikn}\sum_m c_m \exp[tV_m]v_{ml}, \quad (13)$$

where $V_m(\kappa) := \ln[\lambda_m(\kappa)]$. The factors $c_m(\kappa)$ are determined by the initial distribution. For using Eq. (7), we choose as $V(\kappa)$ the eigenvalue function $V_m(\kappa)$ with the largest saddle point. For generic parameters and close to the maximum of the distribution $u(x)$, the choice is unique. Saddle points with smaller value do not contribute to the large time asymptotics. However, we will encounter the possibility of several eigenvalues degenerate in absolute value. To compare our results for the gain-loss rate with the formulas in [7] we have to multiply the rate r in Eq. (9) for the case of the multirung ladder with a factor M, as one step in n in our approach equals M ordinary steps.

The eigenvalues ± 1. Next we investigate more closely the case of zero probability for holding the capital at the current value, i.e., $p_l + q_l = 1$ for all l in (12).

Consider first the case of odd M: $\hat{Q}(0)$ has one eigenvalue $+1$ with left eigenstate $(1,1,1,1\ldots)$, and $\hat{Q}(\pi i)$ has one eigenvalue -1 with left eigenstate $(1,-1,1,-1\ldots)$. Hence, in Fourier representation the two "momenta" $\kappa = 0$ and $\kappa = \pi i$ contribute to the asymptotic behavior.

Let us now consider the matrices $\hat{Q}(\kappa)$ and $\hat{Q}(\kappa+\pi i)$ for arbitrary κ. It is easy to see that the spectra are simply related. Let $(x_0,x_1,x_2,\ldots)^T$ be a right eigenvector of $\hat{Q}(\kappa)$ with eigenvalue $\lambda(\kappa)$, then $(x_0,-x_1,x_2,\ldots)^T$ is a right eigenvector of $\hat{Q}(\kappa+\pi i)$ with eigenvalue $-\lambda(\kappa)$.

Let us assume an expansion like Eq. (8) for the leading $V(\kappa)$ near $\kappa = 0$, then

$$V(\pi+\kappa) = \pi i + r\kappa + K\kappa^2/2. \quad (14)$$

Let v^+ and v^- be the right eigenstates of $\hat{Q}(0)$ and $\hat{Q}(\pi i)$ with eigenvalues $+1$ and -1. There are constants α and β such that

$$P_l(n,t) = \frac{\alpha v_l^+ + (-1)^{n+t}\beta v_l^-}{\sqrt{2\pi Kt}}e^{-(n-rt)^2/2Kt}. \quad (15)$$

We see oscillations caused by the rapid sign change of the second term. The coefficients α and β are determined by the initial probability distribution. If this was peaked at $n = l = 0$, then $\alpha = \beta$ (with v^+ and v^- related as pointed out above). In this case $P_l(n,t)$ is nonzero (zero) for even (odd) $l + n + t$.

Consider now the case of even M. Here, $\hat{Q}(0)$ has one eigenvalue 1 with left eigenstate $(1,1,1,1\ldots)$ and one eigenvalue -1 with left eigenstate $(1,-1,1,-1\ldots)$. Hence, only the "momentum" $\kappa = 0$ contributes in the Fourier representation to the asymptotic behavior, but with two eigenvalues. Let $(x_0,x_1,x_2,\ldots)^T$ be a right eigenvector of $\hat{Q}(\kappa)$ with eigenvalue $\lambda(\kappa)$, then $(x_0,-x_1,x_2,\ldots)^T$ is also a right eigenvector of $\hat{Q}(\kappa)$, but with eigenvalue $-\lambda(\kappa)$. Now we find similar to the case above

$$P_l(n,t) = \frac{\alpha v_l^+ + (-1)^t \beta v_l^-}{\sqrt{2\pi K t}} e^{-(n-rt)^2/2Kt}. \tag{16}$$

Note that the oscillating factor does not depend on n. For a probability initially peaked at $n = l = 0$ we find $P_l(n,t)$ is nonzero (zero) for even (odd) $l + t$.

The findings in both cases, odd M and even M, can however be summarized: $P_l(n,t)$ is nonzero (zero) for even (odd) $l + nM + t$.

It is quite interesting to consider the quantity

$$\hat{P}(n,t) = \sum_{l=0}^{M-1} P_l(n,t). \tag{17}$$

It shows nonzero oscillations in dependence on n and t for odd M. Such oscillations do not exist for even M. The reason for this is easily understood: the term entering $\hat{P}(n,t)$ with a $(-1)^t$ factor is $\sum_l v_l^-$. This is the scalar product of $(1,1,1,1\ldots)$ with $(x_0,-x_1,x_2,-x_3,\ldots)^T$ which are the left and right eigenvectors of $\hat{Q}(0)$ with different eigenvalues $+1$ and -1, and hence this product must be zero.

Explicit expressions for the capital growth rates. Consider the capital depending Parrondo's game with $p_1,\ldots,p_M$ for the winning probabilities and periodicity M. We find the corresponding $\hat{Q}$ matrix

$$\hat{Q}(\kappa) = \begin{pmatrix} 0 & q_2 & \cdots & p_M e^{-\kappa} \\ p_1 & 0 & \cdot & \cdot \\ \cdot & p_2 & \cdot & q_M \\ q_1 e^{\kappa} & 0 & \cdots & 0 \end{pmatrix}. \tag{18}$$

Applying the method of [7] gives

$$r = \frac{\sum_i (p_i - q_i) x_i}{\sum_i x_i}. \tag{19}$$

We prove that Eqs. (8) and (9) give the same result. Let us denote by $\lambda(\kappa)$ the largest eigenvalue of $\hat{Q}(\kappa)$ with left and right eigenstates $\langle y(\kappa)|$ and $|x(\kappa)\rangle$. For $\kappa = 0$ we have $\lambda(0) = 1$ and $\langle y(0)| = (1,1,\ldots,1)$. The growth rate r is the first derivative of $\ln \lambda(\kappa)$ at $\kappa = 0$. As $\lambda(0) = 1$ we have $r = \lambda'(0)$, and hence

$$r = \frac{\partial}{\partial \kappa} \frac{\langle y(\kappa)|\hat{Q}(\kappa)|x(\kappa)\rangle}{\langle y(\kappa)|x(\kappa)\rangle} = \frac{\langle y(0)|\hat{Q}'(0)|x(0)\rangle}{\langle y(0)|x(0)\rangle}, \tag{20}$$

where the last equality follows from the Hellmann-Feynman theorem. Using the explicit form of the matrix $\hat{Q}(\kappa)$, $\langle y(0)| = (1,1,\ldots,1)$, and $|x(0)\rangle = (x_1,x_2,\ldots,x_M)^T$ we find

$$r = \frac{p_M x_M - q_1 x_1}{\sum_i x_i}. \tag{21}$$

Now we prove the equivalence of Eqs. (19) and (21). The eigenvalue equation for the right eigenstate $(x_1,x_2,\ldots,x_M)^T$ of $\hat{Q}(0)$ for eigenvalue 1 is

$$p_{i-1} x_{i-1} + q_{i+1} x_{i+1} = x_i, \tag{22}$$

for all i. From this we derive

$$p_{i-1}x_{i-1} - q_i x_i = x_i - q_{i+1}x_{i+1} - q_i x_i = p_i x_i - q_{i+1} x_{i+1},$$

where the first equality is simply (22) and the second equality is due to $q_{i+1} = 1 - p_{i+1}$. Hence $p_{i-1}x_{i-1} - q_i x_i$ is independent of i and the sum over this term for all i is simply M times the first term for $i = 1$. The sum over all terms can be written as

$$\sum_i (p_i - q_i) x_i = M(p_0 x_0 - q_1 x_1) = M(p_M x_M - q_1 x_1), \tag{23}$$

where we used the cyclic "boundary condition" $x_0 = x_M$. This completes the proof.

The $M = 3$ Parrondo's games. Let us apply the theory of the previous section to the concrete case of the $M = 3$ Parrondo's game. We have two elementary games. The first game is a random walk on the one-dimensional axis with probability h for the right jumps and probability $(1-h)$ for the left jumps. For the second game the jump parameters depend on the capital value. The probability for the right jumps is h_1 for $\mathrm{mod}(X,3) \neq 0$ and h_0 for the case $\mathrm{mod}(X,3) = 0$. We randomly choose the game every round. For this we have an effective $M = 3$ Parrondo's game with probability for right jumps $(h + h_1)/2$ for $\mathrm{mod}(X,3) \neq 0$ and $(h + h_0)/2$ for the case $\mathrm{mod}(X,3) = 0$.

We solve the master equation (10) for calculating the probability distribution after t rounds. The results of iterative numerics are given in Figs. 1 and 2. We see that after $t = 50$ there is an oscillation near the maximum, then as time passes the number of oscillations grows.

For the analytic solution with Eqs. (12) and (13) we obtain the matrix $\hat{Q}(\kappa)$,

$$\begin{pmatrix} 0 & (1-p_1) & p_2 e^{-\kappa} \\ p_1 & 0 & (1-p_2) \\ (1-p_1)e^{\kappa} & p_1 & 0 \end{pmatrix}, \tag{24}$$

where $p_1 = (h + h_0)/2,\ p_2 = (h + h_1)/2$.

The $M = 3$ game with zero probability for holding the capital at the current value, has peculiar properties: the probability distribution is nonzero for odd differences in the capital after an odd number of time steps, and for even differences after even time steps. We checked that there are smooth limiting distributions for even and odd n's (see Fig. 2).

We have seen above that strong oscillations exist in the case of zero probability for holding the capital at the current value, $p_l + q_l = 1$. It is interesting and important to understand if this—namely, the existence of degenerate saddle points—may also appear under other conditions. We have to leave the answer to this question to future work.

Consider the $M = 2$ case. Again, the first game is a random walk on the one-dimensional axis with probability p for right jumps and probability q for left jumps. For the second game we have the right jump probabilities p_1,p_2 and left jump probabilities q_1,q_2. For the random combination of the games

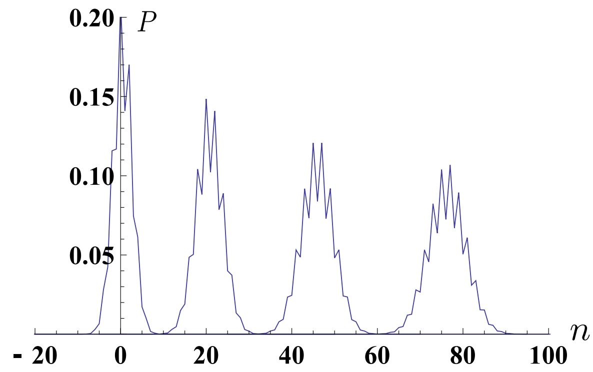

FIG. 1. The probability distribution for the capital growth $\hat{P}(n,t)$, see Eq. (17), for $t = 50,100,150,200$ for the $M = 3$ Parrondo's model with $p = 0.5 - \epsilon$, $p_1 = 0.75 - \epsilon$, $p_2 = 0.1 - \epsilon$, $\epsilon = 0.005$ [cf. Eq. (24)]. For a proper illustration of the distributions we moved the graphs horizontally. Without this shift, the maximum of the distribution after t rounds is located at the point $n = 0.0052t$. Our analytical results by Eq. (6) are identical to the results of the numerics.

we have the matrix $\hat{Q}(\kappa)$,

$$\begin{pmatrix} 1-\frac{p_1+q_1+p+q}{2} & \frac{q+q_2}{2}+\frac{p+p_2}{2}e^{-\kappa} \\ \frac{p+p_1}{2}+\frac{q+q_1}{2}e^{\kappa} & 1-\frac{p_2+q_2+p+q}{2} \end{pmatrix}. \tag{25}$$

Games with state dependence on history. Consider the case of random walks with memory. We define the current state by (X,α_1,α_2) where X is the current value of the capital, α_1 is $+$ $(-)$ if the last change of the capital was gain (loss), and α_2 is $+$ $(-)$ if the second last change of the capital was gain (loss). The parameters of the motion are allowed to depend on α_1,α_2 and we get

$$P(X,+,\alpha,t+1) = \sum_\beta P(X-1,\alpha,\beta,t)p_{\alpha,\beta},$$
$$P(X,-,\alpha,t+1) = \sum_\beta P(X+1,\alpha,\beta,t)(1-p_{\alpha,\beta}). \tag{26}$$

Let us introduce $w(X,t), y(X,t), z(X,t), h(X,t)$ for the cases $(-,-),(-,+),(+,-),(+,+)$, with corresponding probabilities of the right jumps p_1,p_2,p_3,p_4. Then we have the master equations

$$w(X,t+1) = w(X+1,t)(1-p_1) + z(X+1,t)(1-p_3),$$
$$y(X,t+1) = w(X-1,t)p_1 + z(X-1,t)p_3,$$
$$z(X,t+1) = y(X+1,t)(1-p_2) + h(X+1,t)(1-p_4),$$
$$h(X,t+1) = y(X-1,t)p_2 + h(X-1,t)p_4. \tag{27}$$

Performing the Fourier transform and subsequent analysis as above we get

$$w(X,t) = v_1 \exp[tu(X/t)], \quad y(X,t) = v_2 \exp[tu(X/t)],$$
$$z(X,t) = v_3 \exp[tu(X/t)], \quad h(X,t) = v_4 \exp[tu(X/t)], \tag{28}$$

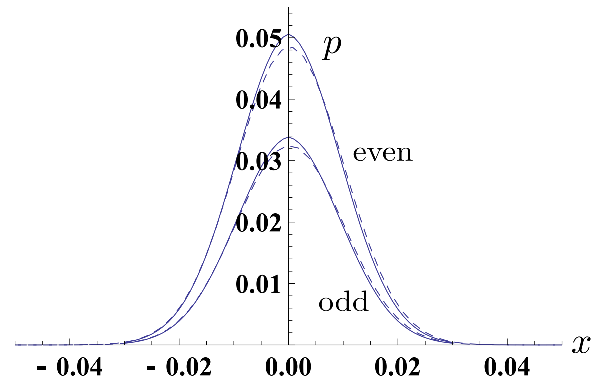

FIG. 2. Illustration of $p(x) = \hat{P}(n,t)$, $x = n/t - r$ with $t = 1000$ for even n (upper two lines) and for odd n (lower two lines) of the $M = 3$ Parrondo's game with the same parameters as for Fig. 1. The smooth lines are derived according to our asymptotic formulas and Eq. (15); the dashed lines correspond to the numerics.

where $u(x)$ is obtained from the largest eigenvalue λ of the system of equations

$$\lambda v_1 = (1-p_1)e^{\kappa}v_1 + (1-p_3)e^{\kappa}v_3,$$
$$\lambda v_2 = p_1 e^{-\kappa}v_1 + p_3 e^{-\kappa}v_3,$$
$$\lambda v_3 = (1-p_2)e^{\kappa}v_2 + (1-p_4)e^{\kappa}v_4,$$
$$\lambda v_4 = p_2 e^{-\kappa}v_2 + p_4 e^{-\kappa}v_4. \tag{29}$$

In conclusion, we considered general versions of Parrondo's games. For applications it is most important to find the capital growth rate and the variance of the distribution. We calculated not only these characteristics of the models, but also found an exact distribution function. Furthermore, we calculated analytically the asymptotics of the distribution $u(x)$ for large t. The function $u(x)$ satisfies a highly nonlinear differential equation, but has an explicit expression as the Legendre transform of a computable function, where for $M > 1$ we have to carry out an eigenvalue analysis. Before our work, the simple matrix $\hat{Q}(0)$ has been used to analyze Parrondo's games [6]. The average growth of the capital is determined by the eigenstate with the maximum eigenvalue 1. Here, we found that the matrix $\hat{Q}(\pi)$ and its eigenvalue -1 lead to fundamental changes of the characteristics of the distribution function. The existence of this eigenvalue creates oscillations in the probability distribution of the capital and results in the existence of two limiting distributions. This is a typical situation with real data of stock fluctuations in financial markets, and it is interesting that our simple model describes this phenomenon. How realistic is the Parrondo's phenomenon? As we underlined, it assumes the possibility to use a simple switch to realize different degrees of mixing between several strategies, either strengthening the system or attenuating it. As we discussed in [27], the latter situation is typical for sufficiently complex living systems.

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[1] J. M. R. Parrondo, How to cheat a bad mathematician, in EEC HC&M Network on Complexity and Chaos (# ERBCHRX-CT940546), 1996 (unpublished).
[2] G. P. Harmer and D. Abbott, Stat. Sci. **14**, 206 (1999).
[3] G. P. Harmer and D. Abbott, Nature (London) **402**, 864 (1999).
[4] J. M. R. Parrondo, G. P. Harmer, and D. Abbott, Phys. Rev. Lett. **85**, 5226 (2000).
[5] H. Moraal, J. Phys. A: Math. Gen. **33**, L203 (2000).
[6] G. P. Harmer, D. Abbott, P. G. Taylor, and J. M. R. Parrondo, J. Chaos **11**, 705 (2001).
[7] G. P. Harmer and D. Abbott, Fluct. Noise Lett. **02**, R71 (2002).
[8] D. Abbott, Fluct. Noise Lett. **09**, 129 (2010).
[9] A. Ajdari and J. Prost, C. R. Acad. Sci. Paris **315**, 1635 (1992).
[10] R. D. Astumian and M. Bier, Phys. Rev. Lett. **72**, 1766 (1994).
[11] R. D. Astumian, Science **276**, 917 (1997).
[12] H. Qian, Phys. Rev. Lett. **81**, 3063 (1998).
[13] A. Gnoli, A. Petri, F. Dalton, G. Pontuale, G. Gradenigo, A. Sarracino, and A. Puglisi, Phys. Rev. Lett. **110**, 120601 (2013).
[14] P. Amengual, A. Allison, R. Toral, and D. Abbott, Proc. R. Soc. London, Ser. A **460**, 2269 (2004).
[15] Y. Chen and W. Just, Phys. Rev. E **90**, 042102 (2014)
[16] W.-X. Zhou, G.-H. Mu, W. Chen, and D. Sornette, PLoS ONE **6**, e24391 (2011).
[17] Z. Forró, R. Woodard, and D. Sornette, Phys. Rev. E **91**, 042803 (2015).
[18] D. G. Luenberger, *Investment Science* (Oxford University Press, New York, 1997).
[19] V. Galstyan and D. B. Saakian, Phys. Rev. E **86**, 011125 (2012).
[20] R. D. Astumian, Am. J. Phys. **73**, 178 (2005).
[21] D. Wu and K. Y. Szeto, Phys. Rev. E **89**, 022142 (2014).
[22] L. J. Gunn, A. Allison, and D. Abbott, Int. J. Mod. Phys.: Conf. Ser. **33**, 1460360 (2014).
[23] D. Abbott, in *Applications of Nonlinear Dynamics*, edited by V. P. Longhini and A. Palacios (Springer-Verlag, Berlin, 2009), p. 307.
[24] D. Abbott, B. R. Davis, and J. M. R. Parrondo, Fluct. Noise Lett. **09**, 1 (2010).
[25] L. J. Gunn, F. Chapeau-Blondeau, M. D. McDonnell, B. R. Davis, A. Allison, and D. Abbott, Proc. R. Soc. London, Ser. A **472**, 20150748 (2016).
[26] D. B. Saakian, J. Stat. Mech. (2016) 043213.
[27] D. B. Saakian, Phys. Rev. E **71**, 016126 (2005).