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Finite-size scaling analysis of a nonequilibrium phase transition in the naming game model

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We realize an extensive numerical study of the naming game model with a noise term which accounts for perturbations. This model displays a nonequilibrium phase transition between an absorbing ordered consensus state, which occurs for small noise, and a disordered phase with fragmented clusters characterized by heterogeneous memories, which emerges at strong noise levels. The nature of the phase transition is studied by means of a finite-size scaling analysis of the moments. We observe a scaling behavior typical of a discontinuous transition and we are able to estimate the thermodynamic limit. The scaling behavior of the clusters size seems also compatible with this kind of transition.

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I. INTRODUCTION

The contributions of statistical physicists to the understanding of the spread of cultural traits, opinions, or conventions of different natures are nowadays well established [1]. Several works focus on the general mechanisms responsible for the ordering dynamics that generates global consensus as an emergent phenomenon [2]. The introduction of minimal models is a standard practice of the area, which is intended to uncover the possible existence of some universal character shared by different real systems. Such an idea of universal properties is in fact supported by the classical theory of phase transitions, which is expected to give at least a partial legacy for these nonequilibrium systems.

In this work we focus our attention to a model of conventions spreading successfully used for describing linguistic dynamics [3]. The model, usually called naming game [4], is characterized by a collective dynamics which implements a memory-based negotiation strategy, where a sequence of trials shapes and reshapes the system memories, allowing for intermediate individual states and feedback effects. These rules appear to be more realistic than simple imitation or local majority mechanisms, commonly implemented by the use of Ising-like dynamics. The introduction of a noise term, which can account for external or internal perturbations or agents' irresolute attitude, generates global consensus for small noise levels [4]. On the other hand, strong noise conducts the system to a stationary state characterized by several coexisting conventions. On the basis of some analytical considerations, supported by numerical simulations, it was suggested that the onset of the consensus state can be described as a nonequilibrium phase transition giving rise to order [5].

The purpose of this work is to study and clearly characterize the nature of this transition throughout an accurate finite-size scaling analysis. For this reason, we consider the implementation of the model on a two-dimensional (2D) square lattice, where our Monte Carlo simulations are performed.

In Sec. II we will describe the details of the model and we will introduce a comparison with prior results in the literature, and in Sec. III we will report the numerical analysis for the characterization of the phase transition and we will discuss our results.

II. THE MODEL

The dynamics rules defining the model are quite simple. Each player is characterized by an inventory which can contain an infinite number of conventions. In fact, it is structured by an array of potentially infinite cells where each cell is set on one of an infinite number of possible numerable states. In the initial state players start with an empty inventory. At each time step, a pair of agents is randomly selected. The first agent selects one of its conventions or, if its inventory is empty, it creates a new one. After that, the convention is transmitted to the second agent. If this last agent possesses the transmitted convention, with a probability β, the two agents update their inventories to keep only the convention involved in the interaction. Conversely, with probability $1-\beta$, no actions are performed by the couple of agents. Otherwise, if the second agent does not possess the transmitted convention, the interaction is a failure and it adds the new convention in its inventory.

The game is simulated on a regular 2D square lattice with $L \times L$ sites and periodic boundary conditions. This implementation defines a short-range interaction system, where agents communicate only with their four nearest neighbors. The special case where $\beta = 1$ corresponds to the original naming game embedded on a low-dimensional lattice. This model was extensively studied and it has shown different convergence behaviors from the mean-field case. In fact, consensus is reached by means of a coarsening process which needs less agents' memory effort but longer convergence times than the mean-field model [6].

The model with general values of β, which effectively generates the described transition, was studied only in one previous work [5]. That paper studied a special case of the mean-field model by means of an analytical approximation where agents can store a maximum of only two different conventions. In this specific approximation of the original model, it was shown that a shift from a consensus state to a polarized one exists at $\beta_c = 1/3$. Simulations of the

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original model, i.e., with an unlimited number of conventions, suggested that the transition happen at the same value of β obtained for the analytical approximation [5]. This fact was inferred looking at the divergence of the convergence time near those values. In detail, the convergence time required by the system to reach the consensus state (t_{conv}) presents two different behaviors [5]: one for the simplified case with just two conventions, where $t_{\mathrm{conv}} \propto (\beta - \beta_c)^{-1}$, and one for the case of the original model, with an unlimited number of conventions, where $t_{\mathrm{conv}} \propto (\beta - \beta_c)^{-0.3}$. Finally, using similar arguments, the authors showed that homogeneous random networks and heterogeneous topologies with power-law degree distributions present the same β value for the transition if the pair selection criterion consists in randomly choosing a link [5].

III. RESULTS AND DISCUSSION

In the following, we develop a finite-size scaling analysis to clearly characterize the phase transition of the model implemented on a 2D square lattice. The system presents very slow relaxation time close to the transition. For this reason, we analyze the final state reached after running 4×10^{10} Monte Carlo steps. Such very long simulations force us to adopt L values limited between 20 and 60.

The phase transition is marked by the passage from an active stationary state of disordered and fragmented clusters to an absorbing state of a single cluster represented by the same word. In fact, for small β, convergence is not attained and small clusters, with one or more different words, characterize the system. In particular, for β values slightly smaller than the one which generates a single cluster (β_c), the fragmented state is characterized by the presence of just two conventions, in accordance with the mean-field behavior of the model [5] (see Fig. 1).

For these reasons the relative size of the largest cluster present in the system is an ideal parameter to characterize the transition [7,8]. This is defined as the size of the largest cluster composed by agents sharing the same unique convention normalized over the system size: $s_{\max}/L^2$. In Fig. 2 we can observe its behavior for different values of L. In accordance with typical phase transitions, we can observe a clear scaling looking at the rise of the critical value of $\beta_c(L)$ and at the characteristic steeper transitions for larger system sizes.

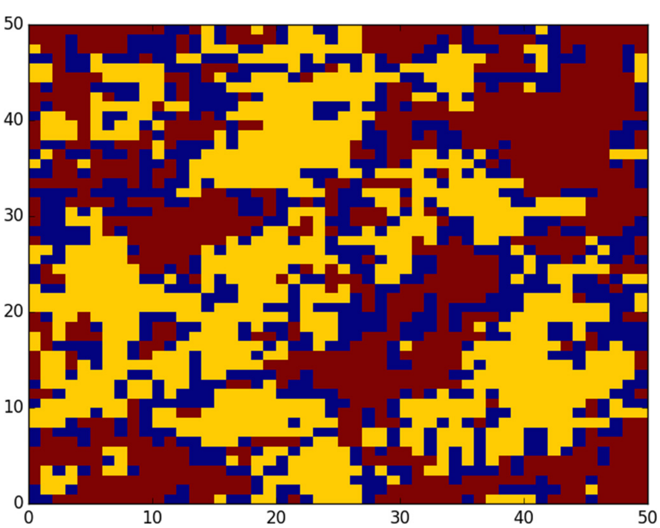

FIG. 1. An active and fragmented steady state ($\beta = 0.32$ and $L = 50$). Two conventions are exchanged: In the red and yellow sites only one convention is present, and in the blue sites the two conventions coexist.

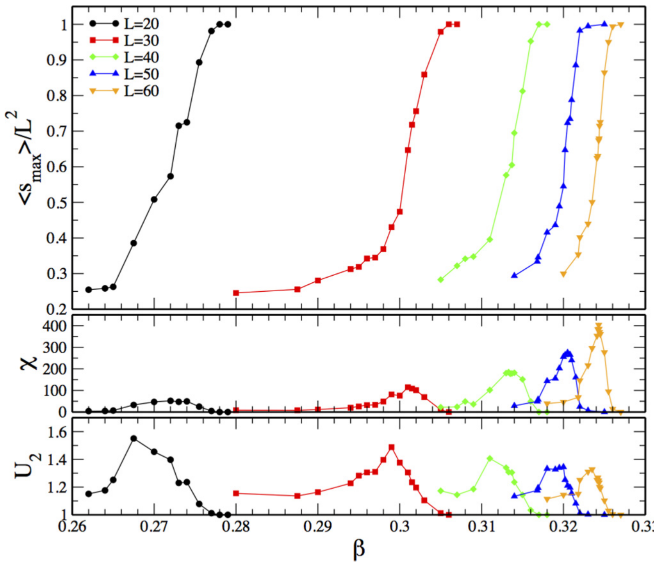

FIG. 2. From top to bottom: Mean of the normalized largest cluster size, its variance χ, and the moment ratio U_2 as a function of β for different system sizes. Each point is averaged over 100 simulations.

For a clear characterization of the type of the transition (continuous or discontinuous) and a quantitative estimation of β_c it is useful to evaluate the fluctuations of the size of the largest cluster:

$$\chi = L^2\left(\langle s_{\max}^2\rangle - \langle s_{\max}\rangle^2\right)$$

and its moment ratio (reduced cumulant) [9]:

$$U_2 = \frac{\langle s_{\max}^2\rangle}{\langle s_{\max}\rangle^2};$$

where $\langle\ \rangle$ stands for averages over different simulations.

As can be seen in Fig. 2, these quantities peak around $\beta_c(L)$. In particular, the maxima of the fluctuations are characterized by higher values for increasing values of L. In fact, the use of the maxima of these quantities has recently been demonstrated to be a very robust approach for performing a finite-size scaling analysis of a discontinuous phase transition into absorbing states [10]. In this case, the asymptotic transition point (asymptotic coexistence point) can be obtained looking at the convergence of the finite-size transition points $\beta_c(L)$ as estimated by the localization of the maxima of the fluctuations or the maxima of the moment ratio. In both cases, the convergence is expected to follow an algebraic behavior: $\beta_c(L) = \beta_c + aL^{-2}$, which is the usual equilibrium scaling [10,11].

These scaling laws are well verified by our data. Looking at Fig. 3, we can see that the maxima positions for χ and U_2 effectively decrease as $1/L^2$. An extrapolation for

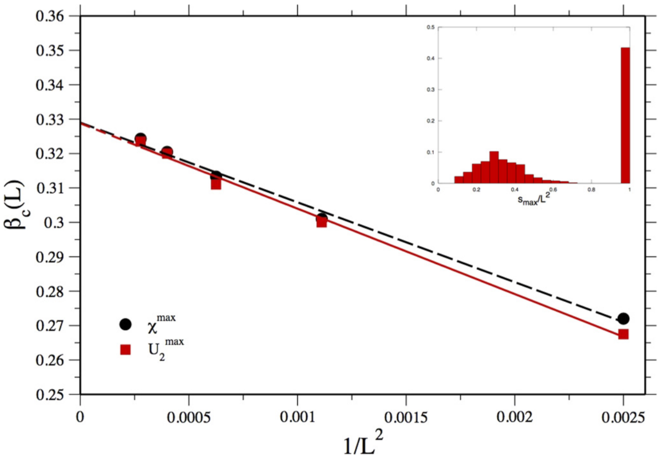

FIG. 3. Convergence to the asymptotic transition point β_c of the finite-size transition points $\beta_c(L)$ measured from the variance and the U_2 maxima. In the inset, a typical example of the shape of the probability distribution function of the normalized largest cluster size near $\beta_c(L)$ (in this case $\beta = 0.301, L = 30$).

$L \to \infty$ yields $\beta_c = 0.329 \pm 0.001$ for the two cases, an excellent agreement between them.

An alternative approach [12] for the estimation of the asymptotic transition point uses the location of the observed discontinuity in the normalized size of the largest cluster (the first value of β for which $\langle s_{\max}\rangle/L^2$ is less than 1). Using this method, the convergence is supposed to be exponential and the extrapolation for $L \to \infty$ gives $\beta_c = 0.328 \pm 0.002$ (see Fig. 4). In the same figure we can observe how the difference $\beta_c - \beta_c(L)$ clearly presents the expected exponential behavior.

Additional consistency checks of the above results can be performed verifying if the measured quantities present the typical scaling of a discontinuous transition near the transition point, a standard procedure for equilibrium finite-size scaling analysis [13]. The scaling plot of $\langle s_{\max}\rangle/L^2$ should be obtained by simply considering the rescaled control parameter $\beta^* = (\beta - \beta_c)L^d$, where d is the system dimension. In a similar fashion, the scaling plot of the fluctuations should be obtained considering the rescaled fluctuation χL^{-d} and the rescaled parameter β^*. As shown in detail in Fig. 4, it is possible to obtain a reasonable collapse which satisfies these relations. In fact, data roughly collapse to a single curve, strongly suggesting the validity of the finite-size scaling ansatz expected for a discontinuous transition.

We conclude our study analyzing the behavior of the size distribution of the clusters present in the fragmented phase near the transition. Obviously we examine only the simulations which do not converge to a unique cluster and, among them, the domains characterized by agents sharing the same unique convention (the regions with memory containing more than one convention are not considered). The probability distribution of the clusters size s is presented in Fig. 5. In the range of the system size we study, the distribution decays as a power law.

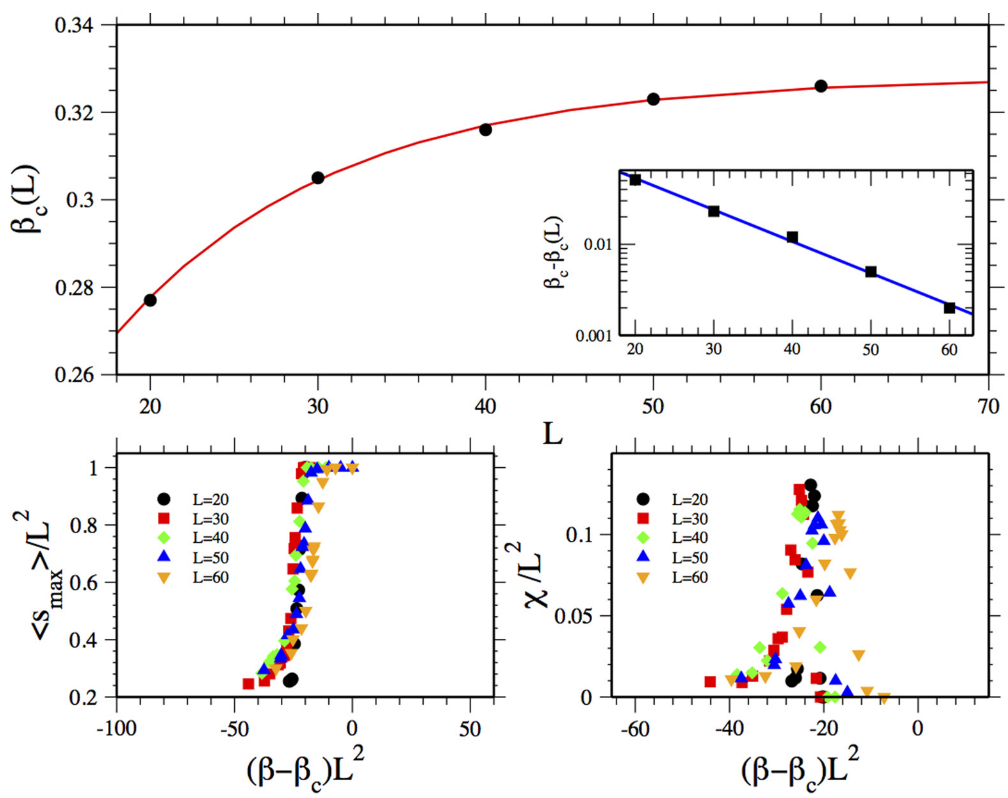

FIG. 4. Top: Convergence to the asymptotic transition point β_c of the finite-size transition points $\beta_c(L)$ measured from the jump location in the normalized size of the largest cluster. The continuous line represents the best fitting function: $\beta_c(L) = 0.328 - 0.23\exp(-0.076L)$. In the inset, the semilogaritmic plot shows in details the expected exponential behavior of the difference $\beta_c - \beta_c(L)$. The continuous line represents an exponential fitting. Bottom: Rescaled plot for $\langle s_{\max}\rangle/L^2$ (on the left) and its fluctuations (on the right).

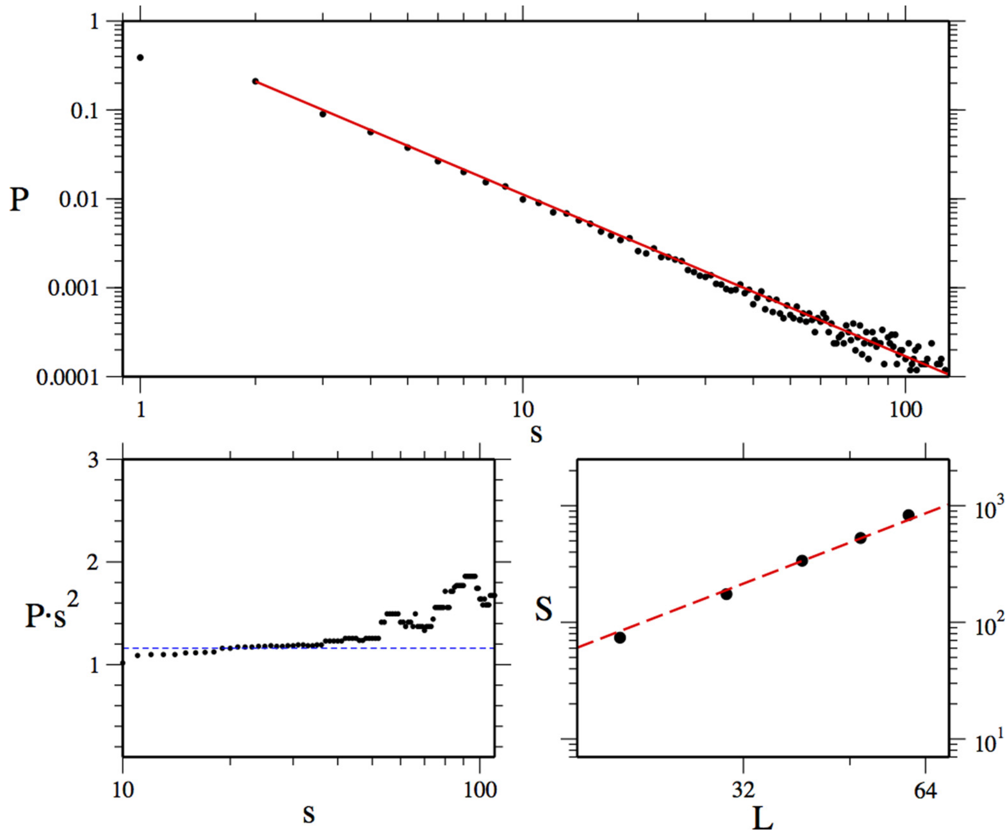

FIG. 5. Top: Probability distribution of the size of clusters $P(s)$ for $\beta = 0.3235$ and $L = 60$. The continuous line represents the power-law fitting of the data. Bottom: On the left, $P(s)s^2$ as a function of s. The probability distribution is binned and it shows an exponent very close to 2 at the intermediate region of s, just before the development of the finite-size effects. On the right, the average cluster size S is plotted as a function of L at $\beta_c(L)$. The dashed line has slope 2. Results have been averaged over 100 samples.

An accurate estimation of the exponent is difficult for the system size we are using. A data fitting all over the region of considered s gives an exponent near 1.8. However, the exponent is very close to 2 in the intermediate region of s, just before the finite-size effects which generate an accumulation on the higher bins responsible for the exponent decrease (see Fig. 5). For an unambiguous characterization of this behavior and for the identification of possible logarithmic corrections to a consistent power-law slope, simulations on larger systems should be realized. We note that finding a discontinuous phase transition with a power-law cluster size distribution, and not an exponential or Gaussian one, is not an unusual result, as can be seen, for example, in Axelrod's model for social influence [7].

In analogy with the approach used in percolation theory, we can measure the average cluster size S defined as:

$$S = \frac{\sum_s n_s s^2}{\sum_s n_s s},$$

where n_s stands for the number of clusters of size s and the sum run over all possible values of s. For a discontinuous transition, S is expected to scale with L^2 at the transition point [14]. This relation is reasonable verified for our system as can be stated looking at Fig. 5.

In summary, we have presented a numerical study of the naming game model with a noise term on a regular 2D lattice with the intent of clearly characterizing, for the first time, the nature of the transition generated by this system.

Our analysis has shown that the model effectively displays a nonequilibrium phase transition between a consensus state and a phase with coexisting conventions in dependence of the control parameter which represents the efficiency of the communication process. The nature of the phase transition has been studied by means of a finite-size scaling analysis. The variance and the moment ratio show a scaling behavior which can be associated with a discontinuous transition and allow the estimation of the transition point at the thermodynamic limit. Additional confirmations of these results come from the collapse of the scaling plot of $\langle s_{\max} \rangle / L^2$ and its fluctuations which have been obtained using the scaling law expected for a discontinuous transition.

Finally, we have presented some results for the behavior of the cluster distribution near the transition point. The distribution displays a power-law behavior, while the average cluster size scaling with the system size is compatible with a discontinuous transition.

These results are not relevant only for the naming game community, but, more in general, they can be interesting for the community of the statistical physicists which work with discontinuous nonequilibrium phase transitions into absorbing states. In particular, they are related to systems used in the description of opinion formation and which are characterized by the presence of a large number of possible different absorbing states. Considering systems embedded in a 2D space, we can mention Axelrod's model. In such a model, the consensus-fragmentation phase transition is discontinuous if the number of cultural features is greater than two [7].

A nonequilibrium Potts model with q absorbing states [15] is another example. In a 2D space for $q = 2$ it belongs to the DP universality class. In contrast, for $q = 3$ there is clear evidence of a discontinuous phase transition, a behavior which is conjectured to persist for any $q > 3$. In the case of the naming game, an interesting model with three absorbing states was introduced in Ref. [16]. In that case, the phase diagram was depicted in detail, but no results on the nature of the transition were presented. A discontinuous behavior was also described in another version of the naming game model, characterized by the presence of an open-ended reservoir of words [8]. There, the transition is driven by the tuning of a control parameter which limits the degree of variance of the invented words and it was characterized by means of a finite-size scaling analysis. Recently, another mechanism was introduced which generates an analogous transition [17]. It is based on a parameter which controls the preference for selecting multiword agents rather than single-word ones. The study is based on the analysis of the decay of the fraction of agents presenting more than one word, but, unfortunately, it is not able to clearly classify the nature of the transition between the active and absorbing states.

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