Recovery of couplings and parameters of elements in networks of time-delay systems from time series

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We propose a method for the recovery of coupling architecture and the parameters of elements in networks consisting of coupled oscillators described by delay-differential equations. For each oscillator in the network, we introduce an objective function characterizing the distance between the points of the reconstructed nonlinear function. The proposed method is based on the minimization of this objective function and the separation of the recovered coupling coefficients into significant and insignificant coefficients. The efficiency of the method is shown for chaotic time series generated by model equations of diffusively coupled time-delay systems and for experimental chaotic time series gained from coupled electronic oscillators with time-delayed feedback.

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I. INTRODUCTION

The study of networks of interacting oscillators has attracted a lot of attention in diverse fields of science and engineering. The network topology and intensity of couplings between the network elements define the features of collective dynamics, such as synchronization [1–4]. Moreover, the architecture of couplings has important consequences on the network responses to external perturbations. Therefore, to understand the functionality of the network, it is important to determine the network connectivity. However, in many real-world systems, direct measurement of the connectivity is not possible. Because of this, estimation of the network connectivity from the time series of the network elements has become an active area of research in recent years.

Several approaches have been developed for the recovery of couplings between the network elements. A method for estimating the network connectivity has been proposed that exploits the response properties of the network to external driving signals [5]. A limitation of this approach is the necessity of disturbing oscillators in the network. Some methods exploit an auxiliary response network with the same intrinsic dynamics of the individual nodes as the drive network under study [6–8]. These methods use adaptive feedback control for synchronizing the drive and response networks, thus estimating the network connectivity. However, these methods assume that parameters of nodes are known *a priori*. Several methods for the recovery of couplings between the network elements are based on the phase modeling approach [9–11], Granger causality approach [12–14], or other techniques [15–18].

Most methods for estimating the network connectivity are valid only for networks with known parameters of nodes. However, in many practical situations, the system parameters are not known beforehand. Therefore, it is important to develop methods for estimating the network connectivity and node parameters simultaneously.

The problem of network reconstruction becomes more difficult if the network consists of time-delay systems. Delays are inherent in many real-world systems [19–21] and time-delayed networks are widely used for modeling various realistic multielement systems with delays [22–26]. For the recovery of time-delay systems from time series, a variety of methods have been proposed [27–53]. However, the majority of these methods were applied to the recovery of a single time-delay system.

The problem of reconstruction of networks consisting of coupled time-delay systems has received much less attention. The methods for the recovery of both the architecture of couplings and node parameters in networks of time-delay systems have been proposed recently in Refs. [54,55]. Although these methods have certain merits, they are not devoid of drawbacks. For example, the method considered in Ref. [54] requires invertibility of the node functions, the absence of noise, and the choice of initial conditions for unknown delays in the neighborhood of true values. The method we proposed in Ref. [55] exploits a separate procedure [42,48] for the reconstruction of delay times in the elements and an iteration algorithm for the recovery of couplings, which requires significant time for computation. Moreover, the result of the coupling architecture recovery may depend on the choice of the method parameters.

In the present paper, we propose a method for the reconstruction of networks composed of coupled time-delay systems, which is devoid of the above-mentioned shortcomings. The method is based on the minimization of a special objective function for each element in the network and on the employment of different algorithms for separating the recovered coupling coefficients into significant and insignificant coefficients.

The paper is organized as follows. In Sec. II, the idea of the method is presented. Using the proposed method, in Sec. III, we recover the parameters of the elements and coupling architecture in various networks of time-delay systems from both numerical and experimental time series. The results are summarized in Sec. IV.

II. METHOD

We consider a network consisting of D coupled nodes, with each node being an oscillating time-delay system described by the following delay-differential equation:

$$\varepsilon_i \dot{x}_i(t) = -x_i(t) + f_i(x_i(t-\tau_i)) + \sum_{j=1(j\neq i)}^{D} k_{i,j}[x_j(t) - x_i(t)], \tag{1}$$

where $i = 1,\dots,D$. Here, $x_i(t)$ denotes the state variable of the ith node. A nonlinear function f_i describes the intrinsic dynamics of the ith node. The delay time and parameter of inertia of the ith node are denoted as τ_i and ε_i, respectively. The coupling coefficients $k_{i,j}$ characterize the coupling strength of the link from the jth element to the ith element. In the general case, a bidirectional coupling takes place between any two oscillators in the network.

Let us assume that we have the simultaneously recorded time series $\mathbf{x}_i = \{x_i(n)\}_{n=1}^{N}$, where n is the time index and N represents the total number of samples for each oscillator. We can define the discrete delay time as $\theta_i = \tau_i/\Delta t$, where Δt is the sampling time. Then we can rewrite Eq. (1) in the following form:

$$f_i(x_i(n)) = \varepsilon_i \dot{x}_i(n+\theta_i) + x_i(n+\theta_i) - \sum_{j=1(j\neq i)}^{D} k_{i,j}[x_j(n+\theta_i) - x_i(n+\theta_i)], \tag{2}$$

where $n = 1,\dots,N-\theta_i$. For each oscillator, we sort the values of $x_i(n)$ in ascending order and denote this sorting as transformation Q, which assigns a point with the index $Q(\mathbf{x}_i,n)$ in the sorted time series to a point with the index n in the original time series. We denote the inverse transformation as Q^{-1}, which assigns a point with the index n in the original time series to a point with the index $Q(\mathbf{x}_i,n)$ in the sorted time series. Then we have $n = Q^{-1}(Q(n))$. For brevity, we omit the dependence of Q and Q^{-1} on $\mathbf{x}_i$.

In the case in which some point has the index n in the original time series and the index $Q(n)$ in the sorted time series, the neighbor point to the right of this point in the sorted time series has the index $Q(n)+1$. The index of this neighbor point in the original time series is $p_n = Q^{-1}(Q(n)+1)$. Note that p_n is not close to n in the general case. Since the points with the indices $Q(n)+1$ and $Q(n)$ in the sorted time series are neighbors, the values of the dynamical variable in these points are close. Consequently, the values of function f_i in these points are also close, under the assumption that f_i is a continuous function. We denote the absolute value of the difference between the values of function f_i in these points as $\delta_i(n)$:

$$\delta_i(n) = |f_i(x_i(p_n)) - f_i(x_i(n))|. \tag{3}$$

Using Eq. (2), Eq. (3) may be written as follows:

$$\delta_i(n) = \left| \left\{ x_i(p_n+\theta_i) - \sum_{j=1(j\neq i)}^{D} k_{i,j}[x_j(p_n+\theta_i) - x_i(p_n+\theta_i)] + \varepsilon_i \dot{x}_i(p_n+\theta_i) \right\} - \left\{ x_i(n+\theta_i) - \sum_{j=1(j\neq i)}^{D} k_{i,j}[x_j(n+\theta_i) - x_i(n+\theta_i)] + \varepsilon_i \dot{x}_i(n+\theta_i) \right\} \right|. \tag{4}$$

We introduce new notations and rewrite Eq. (4) as follows:

$$\delta_i(n) = \left| \Delta x_i(n) - \sum_{j=1(j\neq i)}^{D} k_{i,j}[\Delta x_j(n) - \Delta x_i(n)] - (-\varepsilon_i)\Delta\dot{x}_i(n) \right|, \tag{5}$$

$$\Delta x_i(n) = x_i(p_n+\theta_i) - x_i(n+\theta_i), \tag{6}$$

$$\Delta\dot{x}_i(n) = \dot{x}_i(p_n+\theta_i) - \dot{x}_i(n+\theta_i). \tag{7}$$

We denote the length of a line connecting successive points of the nonlinear function f_i in the sorted time series as S_i. The square of S_i is defined as

$$S_i^2 = \sum_{n=1}^{N-\theta_i-1} \left\{ [x_i(p_n) - x_i(n)]^2 + \delta_i^2(n) \right\}. \tag{8}$$

Note that S_i^2 may be considered a function of parameters θ_i, $k_{i,j}$, and ε_i, which are *a priori* unknown. The value of S_i^2 is less at the true choice of these parameters than at their incorrect choice. This is explained by the fact that, for a wrong choice of θ_i, $k_{i,j}$, and ε_i, the distances (4) are not small, even for the neighbor points in the sorted time series.

The unknown parameter values can be found by minimization of S_i^2. Since the differences $x_i(p_n) - x_i(n)$ in Eq. (8) cannot be optimized, it is convenient to minimize the simpler measure L_i^2, rather than S_i^2, characterizing the sum of squares of only the vertical components of distances between the points of the nonlinear function:

$$L_i^2 = \sum_{n=1}^{N-\theta_i-1} \delta_i^2(n) = \sum_{n=1}^{N-\theta_i-1} \left\{ \Delta x_i(n) - \sum_{j=1(j\neq i)}^{D} k_{i,j}[\Delta x_j(n) - \Delta x_i(n)] - (-\varepsilon_i)\Delta\dot{x}_i(n) \right\}^2. \tag{9}$$

For a fixed θ_i, the minimization of L_i^2 can be carried out using the linear least squares method, where $k_{i,j}$ and $(-\varepsilon_i)$ are the required coefficients, $\Delta x_j(n) - \Delta x_i(n)$ and $\Delta\dot{x}_i(n)$ are the basis functions, and $\Delta x_i(n)$ are the values to be approximated. In this case, L_i^2 may be considered the objective function. The finding of its extremum is a standard problem that can be solved in a nonrecursive way.

Since the delay time θ_i is *a priori* unknown, the minimization of the objective function (9) can be carried out for different trial discrete delay times $\theta_i{}'$ chosen in some range. The minimum of the dependence $L_i^2(\tau_i{}')$, where $\tau_i{}' = \theta_i{}'\Delta t$, will be observed at the true delay time τ_i.

With the increase of N, the number of terms in the sum (9) increases in proportion to N. At the same time, the increase of N leads to the decrease of distances between the points in the sorted time series and, as a consequence, leads to the decrease of $\delta_i(n)$. On the average, $\delta_i(n)$ decreases in proportion to $1/N$. Therefore, each term $\delta_i^2(n)$ in the sum (9) decreases in proportion to $(1/N)^2$ with the increase of N. Consequently, $L_i^2 \to 0$ as $N \to \infty$. Thus, for $N \to \infty$, the proposed method is asymptotically accurate and gives asymptotically unbiased estimations of parameters.

The proposed algorithm is described in the general case, where a bidirectional coupling is present between any two oscillators in the network. Such a situation is not typical in practice. If there is no influence of the jth element upon the ith element, the corresponding coupling coefficient $k_{i,j}$ in the model equation (1) should be set to zero. However, application of the procedure of reconstruction described above always gives us $D-1$ nonzero coupling coefficients $k'_{i,j}$ for each element in the network. If some of the potential links in the network are not present, then some of the recovered coefficients $k'_{i,j}$ are redundant. For separating the coefficients $k'_{i,j}$ into significant and insignificant coefficients, the K-means clustering [56] can be used.

We carry out clustering of the recovered coupling coefficients $k'_{i,j}$ in one-dimensional space by separating them into two clusters consisting of significant and insignificant coefficients. The maximal and minimal values of $k'_{i,j}$ are chosen as the initial centers of the clusters. Since the significant coefficients in the general case are much greater than the insignificant coefficients, it is convenient to conduct clustering on a logarithmic scale. After determination of insignificant coupling coefficients, we remove the basis functions corresponding to these insignificant coefficients from the objective function (9) and recover all remaining $k_{i,j}$ once again in order to increase the accuracy of reconstruction.

This approach allows us to recover the architecture of couplings in the network. It should be noted that the considered method operates much more quickly than the iteration method we proposed in Ref. [55]. The iteration method employs a successive trial elimination of coupling coefficients from the model equation for testing the significance of links [55]. In our method, all insignificant couplings are eliminated at once from the model equation.

The method of detecting the insignificant couplings, based on the K-means clustering, operates well in the absence of noise in the case of a comparable number of significant and insignificant coupling coefficients. However, if these conditions are not fulfilled, the boundaries of clusters become close to each other. As a result, the accuracy of the method decreases, and it can detect spurious couplings or miss existing ones. In such cases, for accurate recovery of coupling architecture, one should use more than two clusters in the K-means algorithm. Unfortunately, in the analysis of experimental data, the number of connections between the network elements is *a priori* unknown, and it is difficult to recommend the appropriate number of clusters.

We propose another method for separating the recovered coupling coefficients $k'_{i,j}$ into significant and insignificant coefficients. We sort the coefficients $k'_{i,j}$ in descending order by the absolute value. Then we consider the network containing only one connection between D nodes (1). The coupling coefficient that describes this connection has the greatest absolute value among the recovered coefficients $k'_{i,j}$. For this network, we calculate the following quantity:

$$\Lambda = \sum_{i=1}^{D} L_i^2 \big/ (N - \theta_i - 1). \tag{10}$$

Note that Λ is normalized to $N - \theta_i - 1$. Without this normalization, the contribution of different oscillators to Λ will be different because they have different delay times θ_i and, as a consequence, will have a different number of terms in the objective function (9). After that, we enter the coupling coefficients into the model one by one in descending order of their absolute value and calculate Λ at each step. Finally, we plot the dependence of Λ on M, where $M = 1, \ldots, D(D-1)$ is the number of coupling coefficients in the model equations.

As the model takes into account a greater number of existing couplings, it becomes more accurate. In this case, the quantities (5) characterizing the distance between the points of the reconstructed nonlinear functions decrease, as do the quantities (9) and (10). The decrease of Λ in the $\Lambda(M)$ plot will practically stop at $M = W$, where W is the number of couplings in fact existing in the network. Further adding the coefficients $k'_{i,j}$, which correspond to nonexistent couplings, into the model has almost no influence on the value of Λ. The dependence $\Lambda(M)$ will remain almost constant for $M \geqslant W$. The figures that clearly illustrate this approach are presented in Sec. III. Note that this approach is not as fast as the method that separates the coupling coefficients into significant and insignificant coefficients based on the K-means clustering. In practice, one can use both considered methods. If they give the same result, it supports the conclusion that the architecture of couplings between the network elements is recovered accurately.

III. RESULTS

A. Examples of networks consisting of coupled time-delay systems

We have applied the proposed method to the reconstruction of the network connectivity and node parameters in model networks consisting of standard time-delay systems, such as the Mackey-Glass system and Ikeda system and an experimental network consisting of electronic oscillators with time-delayed feedback.

First, we consider a network of coupled Mackey-Glass equations. Each element in this network is described by Eq. (1)

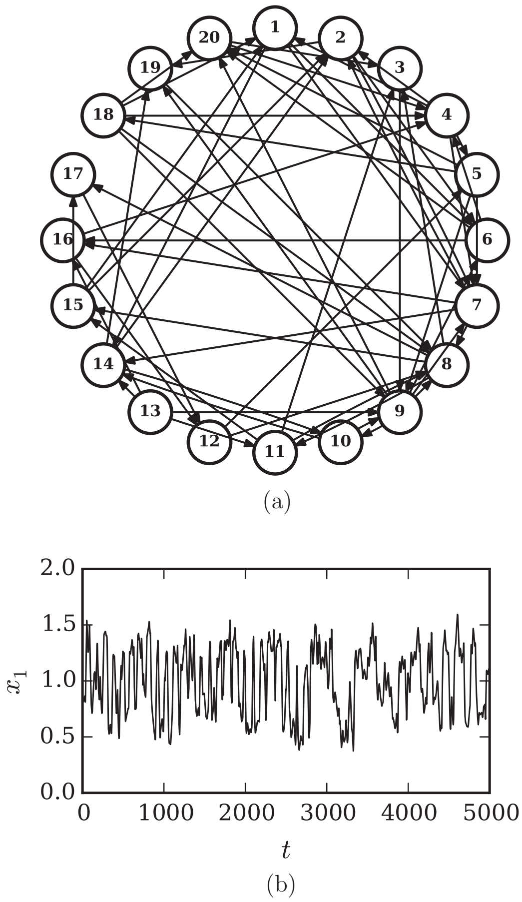

FIG. 1. (a) Coupling architecture in a network of 20 elements. (b) The chaotic time series of $x_1(t)$ corrupted by Gaussian noise with $\sigma_i = 0.004$ in the network of Mackey-Glass equations.

with the following nonlinear function:

$$f_i(x_i(t-\tau_i)) = \frac{a_i x_i(t-\tau_i)}{b_i\left[1+x_i^c(t-\tau_i)\right]}. \qquad (11)$$

The Mackey-Glass equation has been introduced in Ref. [57] as a model of blood production. This equation has been studied in detail in Ref. [58] under variation of delay time τ_i and the fixed values of other parameters in function (11), in which $a_i = 0.2$, $b_i = 0.1$, and $c = 10$. The high-dimensional chaotic attractor of a single Mackey-Glass system was studied at $\tau_i = 300$ [58]. Subsequently, the same set of parameters of the Mackey-Glass equation has been used by many authors in numerical studies of chaotic time-delay systems.

We consider a network consisting of 20 coupled Mackey-Glass equations with a random architecture of couplings. Any two nodes in this network can be uncoupled, coupled in one direction, or coupled in both directions. Figure 1(a) shows the case in which 60 couplings from 380 possible couplings are present. All elements in the network are nonidentical. The parameters of function (11) in the Mackey-Glass equations are assigned arbitrary values in the vicinity of the above-mentioned parameter values used in Ref. [58]. We choose $\tau_i \in [250,400]$, $a_i \in [0.20,0.25]$, $\varepsilon_i \in [7.5,12.5]$, and $c = 10$ corresponding to chaotic oscillations of the elements. Note that $\varepsilon_i = 1/b_i$ for the Mackey-Glass equation. The coupling coefficients are chosen such that $k_{i,j} \in [0.02,0.06]$.

Each time series contains $N = 10\,000$ points recorded with the sampling time $\Delta t = 0.5$. We consider the case of the absence of noise and the case in which each time series is corrupted by additive independent Gaussian noise $\xi_i(t)$ with a zero mean and standard deviation $\sigma_i = 0.004$. In this case, the additive noise has a standard deviation of about 1%–2% of the standard deviation of the data without noise (the signal-to-noise ratio is about 34–40 dB). The elements in the network have different amplitudes of oscillations because of the nonidentical parameters. Thus, the level of noise differs for different oscillators. Figure 1(b) shows the time series of the first element with the parameters $\tau_1 = 263$, $\varepsilon_1 = 12.32$, $a_1 = 0.218$, $k_{1,4} = 0.0475$, $k_{1,15} = 0.0294$, $k_{1,18} = 0.0292$, and $k_{1,j} = 0$, $j \neq 4,15,18$.

As a second example, we study a network consisting of coupled Ikeda equations described by Eq. (1) with $\varepsilon_i = 1$ and the following function:

$$f_i(x_i(t-\tau_i)) = \mu_i \sin[x_i(t-\tau_i) - x_{0i}], \qquad (12)$$

where the parameter μ_i characterizes the amplitude of oscillations and where x_{0i} is a constant. The Ikeda equation [59], which describes the dynamics of an optical bistable resonator, is another standard equation widely used in the simulation of time-delay systems.

We consider a network composed of 16 Ikeda equations with 50 randomly generated couplings between them. The total number of possible couplings in this network is equal to 240. All elements are nonidentical. Their parameters are assigned arbitrary values in the following ranges: $\tau_i \in [2,5]$, $\mu_i \in [15,25]$, $x_{0i} \in [0,2\pi]$, and $k_{i,j} \in [0.1,0.5]$. With these parameters, all elements oscillate in a chaotic regime. The length of each time series is 10 000 points. The sampling time is $\Delta t = 0.01$. Each time series is corrupted by additive independent Gaussian noise $\xi_i(t)$ with a zero mean and standard deviation $\sigma_i = 0.003$, which is about 0.1%–0.2% of the standard deviation of the data without noise (the signal-to-noise ratio is about 54–60 dB). This level of noise is rather small.

At last, we examine a network consisting of coupled experimental electronic oscillators with time-delayed feedback. Each oscillator represents a ring system containing an analog low-pass first-order RC filter, digital delay line, and digital nonlinear device. A block diagram of such an oscillator is presented in Fig. 2. The analog and digital elements of the oscillator are connected with the help of analog-to-digital and digital-to-analog converters.

Each oscillator in the network is described by Eq. (1), where $x_i(t)$ and $x_i(t-\tau_i)$ are the voltages at the delay line input and output, respectively, and $\varepsilon_i = R_i C_i$, where R_i is the resistance and C_i is the capacitance. The digital nonlinear devices in our scheme provide a quadratic transformation f_i. To connect the oscillators in the network, we use the resistors $R_{i,j}$ and voltage followers. It allows us to provide unidirectional coupling between the oscillators in contrast to our paper [55], where the coupling in the experimental scheme could be only bidirectional. The coupling coefficients in Eq. (1) are calculated as $k_{i,j} = R_i/R_{i,j}$.

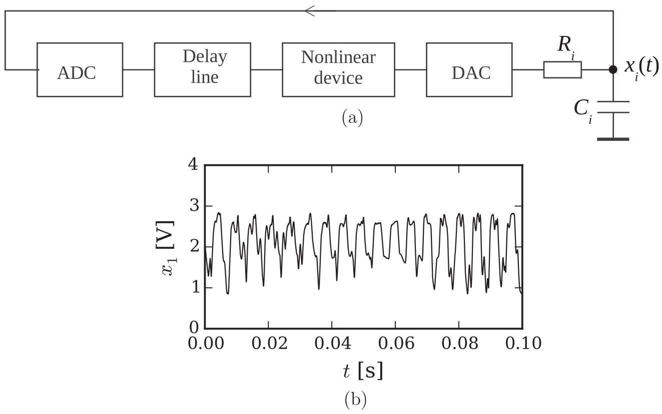

FIG. 2. (a) Block diagram of the ring oscillator with time-delayed feedback. The analog-to-digital converter and the digital-to-analog converter are denoted as ADC and DAC, respectively. (b) The experimental chaotic time series of voltage $x_1(t)$ in the first oscillator.

We consider a network representing a chain of ten unidirectionally coupled experimental oscillators. The chain comprises nonidentical chaotic oscillators whose parameters take the values in the following ranges: $\tau_i \in [2.50,4.75]$ ms, $\varepsilon_i \in [203,536]\,\mu$s, and $k_{i,j} \in [0.10,0.23]$. For $j \neq i+1$, $k_{i,j} = 0$. We simultaneously record the signals $x_i(t)$ using a ten-channel analog-to-digital converter with the sampling frequency $f_s = 100$ kHz. The time series of voltage $x_1(t)$ is shown in Fig. 2(b) for the first oscillator with the parameters $\tau_1 = 2.5$ ms, $\varepsilon_1 = 203\,\mu$s, $k_{1,2} = 0.21$, and $k_{1,j} = 0$ where $j \neq 2$.

B. Reconstruction of delay time in the network elements

First, we illustrate the results of application of our method to the reconstruction of delay times inherent in the intrinsic dynamics of the network oscillators. For each element in the network, we calculate the objective function (9) under variation of the delay time and construct the $L_i^2(\tau_i{}')$ plot. The time derivatives necessary for calculating the objective function (9) are estimated from the time series by applying a local parabolic approximation.

Figure 3 shows the dependences $L_i^2(\tau_i{}')$ for each of the 20 elements in the network of coupled Mackey-Glass equations for the cases of the absence and presence of noise. The L_i^2 values are normalized to $N - \theta_i - 1$. The global minima of all $L_i^2(\tau_i{}')$ in Fig. 3 take place at the true delay times of the network oscillators. In the absence of noise, these minima are deeper and narrower than in the case of the presence of noise.

Figure 4 depicts the dependences $L_i^2(\tau_i{}')$ normalized to $N - \theta_i - 1$ for each of the 16 elements in the network of coupled Ikeda equations in the presence of small noise. The absolute minima of all $L_i^2(\tau_i{}')$ are very well pronounced. They are observed exactly at the true delay times of the network elements.

The dependences $L_i^2(\tau_i{}')$ for each of the ten elements in the chain of coupled experimental electronic oscillators are presented in Fig. 5. The L_i^2 values are normalized to $N - \theta_i - 1$. For nine oscillators, the global minima of $L_i^2(\tau_i{}')$ take place at the true delay times. For only one oscillator, the absolute

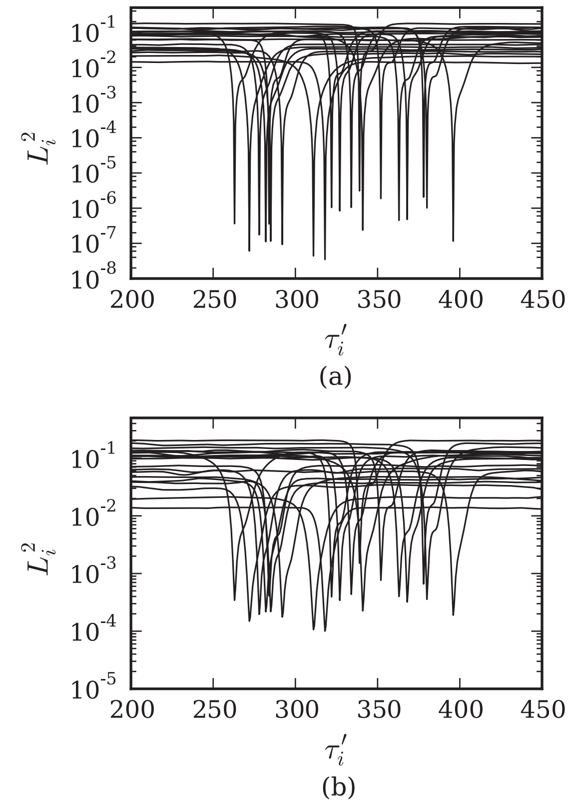

FIG. 3. Dependences $L_i^2(\tau_i{}')$ for each of the 20 elements in the network of Mackey-Glass equations in the absence of noise (a) and in the presence of Gaussian noise with $\sigma_i = 0.004$ (b).

minimum of $L_i^2(\tau_i{}')$ is shifted from the true delay time by the time $t_s = 10\,\mu$s, which is equal to the sampling time.

C. Recovery of couplings in the network

We now illustrate the efficiency of the method for the recovery of the network connectivity. As a result of the reconstruction of model equation (1) for each oscillator in the network, we obtain $D-1$ nonzero coupling coefficients $k'_{i,j}$. Some of these coefficients are redundant. At first, we define the significant coefficients $k'_{i,j}$ using the K-means clustering.

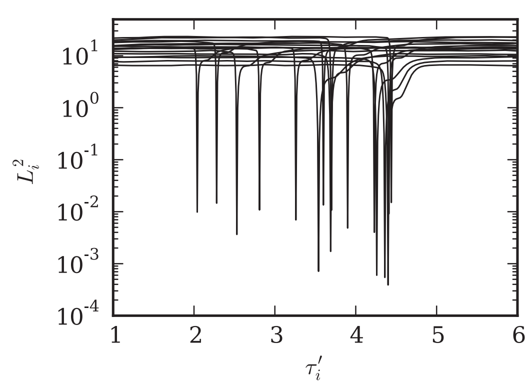

FIG. 4. Dependences $L_i^2(\tau_i{}')$ for each of the 16 elements in the network of Ikeda equations in the presence of small Gaussian noise with $\sigma_i = 0.003$.

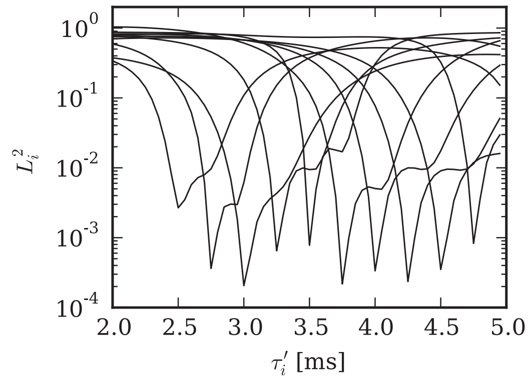

FIG. 5. Dependences $L_i^2(\tau_i{}')$ for each of the ten elements in the chain of experimental electronic oscillators with time-delayed feedback.

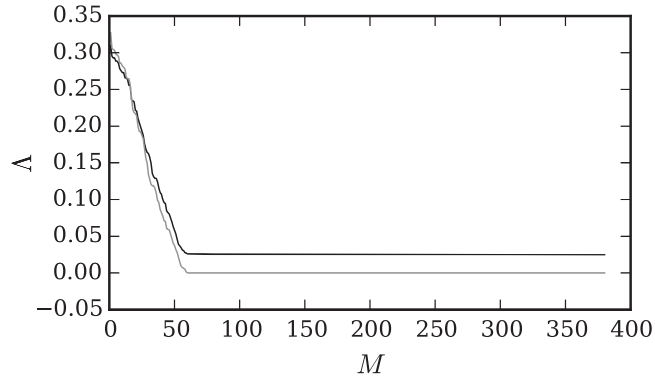

FIG. 7. Dependences $\Lambda(M)$ for the network of Mackey-Glass equations. $\Lambda(M)$ is shown in gray in the absence of noise and in black in the presence of Gaussian noise with $\sigma_i = 0.004$.

Since the insignificant coefficients $k'_{i,j}$ are close to zero and may take negative values, we use the absolute values of $k'_{i,j}$ for separation of coefficients into significant and insignificant coefficients.

In Fig. 6, the distribution of $|k'_{i,j}|$ values for all 20 elements in the network of Mackey-Glass equations is plotted in a logarithmic scale along the horizontal axis. As seen in Fig. 6, $|k'_{i,j}|$ are clearly separated into two clusters consisting of significant (at the right) and insignificant (at the left) coefficients. The presence of noise deteriorates the separation of coefficients. This can be explained by the fact that noise reduces the accuracy of the parameter reconstruction. Since the insignificant coupling coefficients in Fig. 6 are several orders of magnitude less than the significant coefficients, their absolute values undergo greater changes in the presence of noise than the absolute values of the significant coupling coefficients. With the increase of noise above some critical level, the absolute values of some insignificant coefficients $k'_{i,j}$ become comparable with the values of significant coefficients $k'_{i,j}$. As a result, the method begins to detect spurious couplings or miss some of the existing couplings.

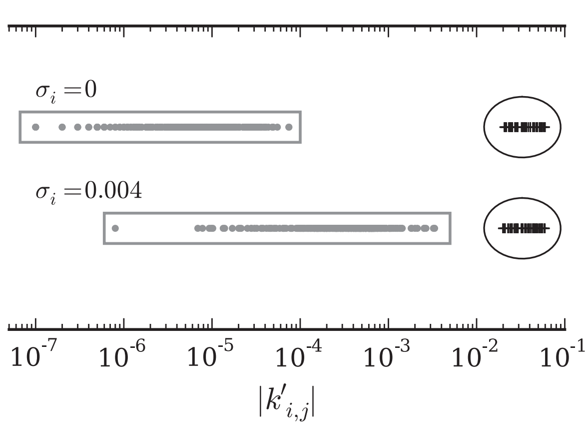

FIG. 6. Distribution of the absolute values of the recovered coupling coefficients for all elements in the network of Mackey-Glass equations in the absence of noise (at the top) and in the presence of noise (at the bottom). The significant coefficients are shown by black crosses located inside the circle. The insignificant coefficients are shown by gray circles located inside the rectangle.

After detecting the insignificant coupling coefficients, we remove them from Eq. (9) and recover $k_{i,j}$ once again. It allows us to recover the coupling architecture accurately [see Fig. 1(a)] in both the absence and presence of noise. The values of the coupling coefficients are reconstructed with good accuracy. For example, for the first element in the network, in the presence of noise, we obtain the following values of recovered coupling coefficients: $k'_{1,4} = 0.0467$, $k'_{1,15} = 0.0288$, and $k'_{1,18} = 0.0287$. The recovered parameter of inertia takes the value $\varepsilon'_1 = 11.29$. The true values of the first element parameters are given in Sec. III A. In the absence of noise, the accuracy of their reconstruction is higher.

Then we separate the recovered coupling coefficients into significant and insignificant coefficients using the method based on the construction of the dependence $\Lambda(M)$. In Fig. 7, the dependences $\Lambda(M)$ are constructed for both the absence and presence of noise. The quantity Λ practically reaches the minimal value at $M = 60$ and remains almost constant as M increases. Note that, in the absence of noise, Λ is close to zero for $M \geqslant 60$. We leave only 60 coupling coefficients, which have the greatest absolute values among the recovered coefficients, and remove all the other $k'_{i,j}$, which are insignificant. This approach, as well as the previous one, allows us to reconstruct the coupling architecture accurately. Thus, both employed methods for the recovery of the network connectivity give the same result.

Next, we recover the couplings in the network of Ikeda equations. Again, at first, we apply the K-means algorithm for clustering $k'_{i,j}$ into significant and insignificant coefficients. In Fig. 8, the distribution of $|k'_{i,j}|$ values for all 16 elements is plotted in a logarithmic scale. Two clusters of $|k'_{i,j}|$ consisting of significant (at the right) and insignificant (at the left) coefficients are clearly seen in Fig. 8. We remove the insignificant coupling coefficients and recover $k_{i,j}$ one more time. Such an approach allows us to reconstruct the architecture of couplings in the network accurately.

The accuracy of estimation of the coupling coefficients is good enough. For example, for the first element in the network, we obtain the following values of the recovered parameters: $k'_{1,7} = 0.2276$, $k'_{1,8} = 0.3308$, $k'_{1,9} = 0.4478$,

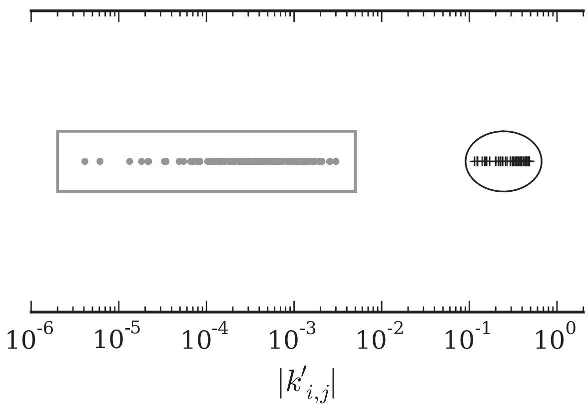

FIG. 8. Distribution of absolute values of the recovered coupling coefficients for all elements in the network of Ikeda equations in the presence of small Gaussian noise with $\sigma_i = 0.003$. The significant coefficients are shown by black crosses located inside the circle. The insignificant coefficients are shown by gray circles located inside the rectangle.

and $k'_{1,10} = 0.3102$. These values are close to the true values $k_{1,7} = 0.2293$, $k_{1,8} = 0.3321$, $k_{1,9} = 0.4492$, and $k_{1,10} = 0.3114$. For the other elements, the accuracy of the reconstruction of parameters is about the same. Inaccuracy of the parameter estimation is mainly caused by the presence of noise. It increases with the increase of the level of noise. The recovered parameter of inertia for the first element is $\varepsilon'_1 = 1.002$. Note that the parameter ε_i is absent in an explicit form in the Ikeda equation.

Figure 9 depicts the dependence $\Lambda(M)$. The quantity Λ practically reaches the minimal value at $M = 50$ and remains almost constant for $M \geqslant 50$. Leaving only 50 coupling coefficients, which have the greatest absolute values among the recovered coefficients, and removing all the other insignificant $k'_{i,j}$, we obtain the accurate recovery of the coupling architecture. Thus, both methods of reconstructing the network connectivity give the same result as well as in the case of the coupled Mackey-Glass equations considered above.

We investigated the efficiency of the proposed method in the case in which all oscillators in the network of Ikeda equations are in a periodic regime. The delay time of oscillators is recovered accurately in this case. This is explained by the fact that the delay time plays a crucial role in the system dynamics. Even a small error in the delay time estimation substantially increases the objective function (9). However, the accuracy of the estimation of the coupling coefficients is an order of magnitude less in a periodic regime than in a chaotic regime. Moreover, the method missed several links in the case of periodic time series. These errors are mainly caused by two reasons. The first reason is the small amount of information in the periodic time series in comparison with the chaotic time series. The second reason is the presence of synchronization between the periodic oscillators with close parameter values.

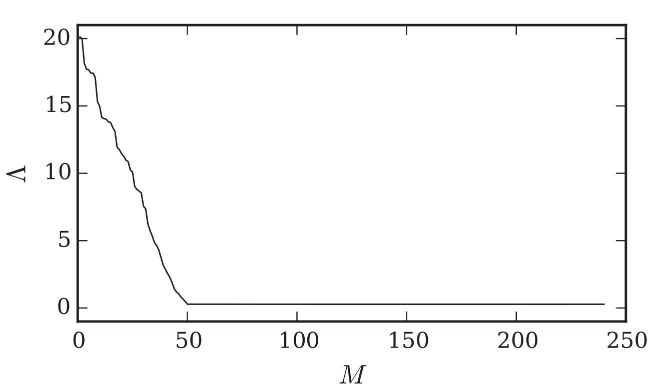

FIG. 9. Dependence $\Lambda(M)$ for the network of Ikeda equations in the presence of small Gaussian noise with $\sigma_i = 0.003$.

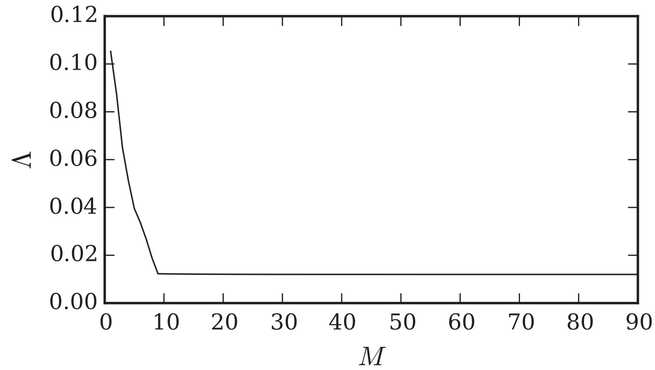

FIG. 10. Dependence $\Lambda(M)$ for the chain of experimental oscillators with time-delayed feedback.

As the third example, we illustrate the recovery of couplings in the chain of ten unidirectionally coupled experimental oscillators. Nine oscillators in the chain are affected by only one neighboring oscillator and are described by Eq. (1) with only one coupling coefficient $k_{i,i+1}$. The tenth oscillator at the edge of the chain is an autonomous one, and its model equation contains no coupling coefficients. However, the minimization of the objective function (9) gives us nine recovered coupling coefficients $k'_{i,j}$ for each oscillator in the chain. Most of these coefficients are redundant. Since the number of significant coefficients is an order of magnitude less than the number of insignificant coefficients, the K-means clustering is not efficient for separation of $k'_{i,j}$.

Figure 10 shows the dependence $\Lambda(M)$. The quantity Λ practically reaches the minimal value at $M = 9$ and remains almost constant for $M \geqslant 9$. We leave only nine coupling coefficients, which have the greatest absolute values among the recovered coefficients, and remove all the other insignificant $k'_{i,j}$. This approach allows us to reconstruct the architecture of couplings in the chain accurately.

For the first oscillator in the chain, the recovered coupling coefficient $k'_{1,2} = 0.22$ is close to the true value $k_{1,2} = 0.21$. The recovered parameter of inertia $\varepsilon'_1 = 204\,\mu$s is also very close to the true value $\varepsilon_1 = 203\,\mu$s. The accuracy of the reconstruction of parameters for the other oscillators in the chain is about the same.

D. Reconstruction of the nonlinear function of the network elements

After the recovery of the delay time, the parameter of inertia, and the coupling coefficients for each of the network oscillators described by Eq. (1), we can reconstruct all the

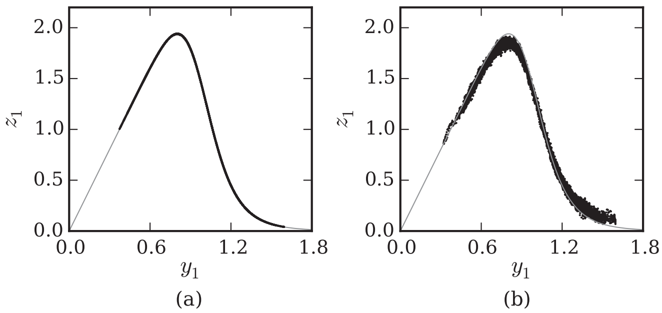

FIG. 11. Function f_1 (black) recovered in the plane (y_1,z_1), where $y_1 = x_1(t-\tau'_1)$ and $z_1 = \varepsilon'_1\dot{x}_1(t) + x_1(t) - \sum_{j=2}^{20} k'_{1,j}[x_j(t) - x_1(t)]$, for the network of Mackey-Glass equations in the absence of noise (a) and in the presence of Gaussian noise with $\sigma_i = 0.004$ (b). The true function f_1 is shown in gray in the background in (a) and in the foreground in (b).

nonlinear functions f_i. We write Eq. (1) as

$$f_i(x_i(t-\tau_i)) = \varepsilon_i\dot{x}_i(t) + x_i(t) - \sum_{j=1(j\neq i)}^{D} k_{i,j}[x_j(t) - x_i(t)]. \tag{13}$$

We denote $x_i(t-\tau_i)$ as y_i and the right-hand side of Eq. (13) as z_i. To recover the function f_i, we plot the dependence of z_i on y_i. Instead of the unknown parameters τ_i, ε_i, and $k_{i,j}$, we use their estimations τ'_i, ε'_i, and $k'_{i,j}$, respectively.

Figure 11 shows the recovered nonlinear function f_1 (in black) of the first element in the network of coupled Mackey-Glass equations for the cases of the absence and presence of noise. In the absence of noise, the recovered function coincides closely with the true nonlinear function of the Mackey-Glass equation [Fig. 11(a)]. The true function f_1 is shown in gray in Fig. 11. In the case of the presence of noise, the quality of the nonlinear function recovery is worse [Fig. 11(b)]. The nonlinear functions of the other elements are reconstructed in a similar way.

Figure 12 shows the reconstructed nonlinear function f_1 of the first element in the network of coupled Ikeda equations in the presence of small noise. It coincides closely with the true nonlinear function of the Ikeda equation that is also presented in Fig. 12. The true and recovered functions are practically indistinguishable in Fig. 12. However, for higher levels of noise, the quality of the reconstruction of the nonlinear function becomes worse. The approximation of the recovered function f_1 with a first-order trigonometric polynomial gives us the following estimation of the function (12) parameters: $\mu'_1 = 14.97$ and $x'_{01} = 2.5631$. These values are very close to the true values $\mu_1 = 15$ and $x_{01} = 2.5629$.

The nonlinear function f_1 of the first oscillator in the chain of coupled experimental electronic oscillators is reconstructed in Fig. 13. It provides a sufficiently good approximation of the true transfer function of the nonlinear device in the first oscillator. In a similar way, we reconstruct the nonlinear functions of other oscillators in the chain.

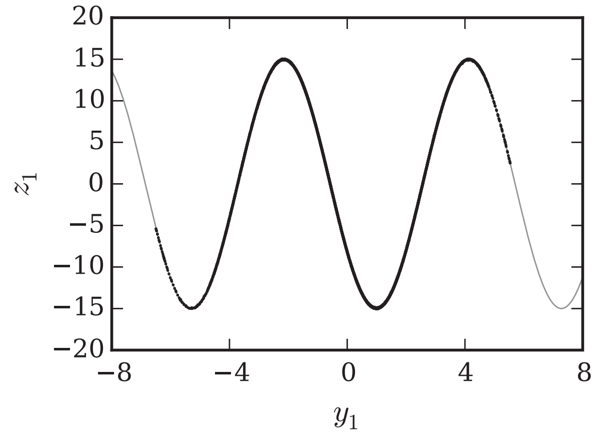

FIG. 12. Function f_1 (black) recovered in the plane (y_1,z_1), where $y_1 = x_1(t-\tau'_1)$ and $z_1 = \varepsilon'_1\dot{x}_1(t) + x_1(t) - \sum_{j=2}^{16} k'_{1,j}[x_j(t) - x_1(t)]$, for the network of Ikeda equations in the presence of small Gaussian noise with $\sigma_i = 0.003$. The true function f_1 is shown in the background in gray.

IV. CONCLUSION

We have proposed the method that allows one to reconstruct from time series the architecture of couplings and parameters of elements in networks consisting of coupled oscillators described by delay-differential equations. The method is based on minimizing the objective function for each element in the network using the least squares method. The introduced objective function characterizes the distance between the points of the recovered nonlinear function of the network element. The proposed approach allows one to reconstruct the delay times, parameters of inertia, nonlinear functions, and coupling coefficients for all elements in the network with good accuracy. The method operates considerably more quickly than the iteration method we exploited earlier for the recovery of coupling architecture [55].

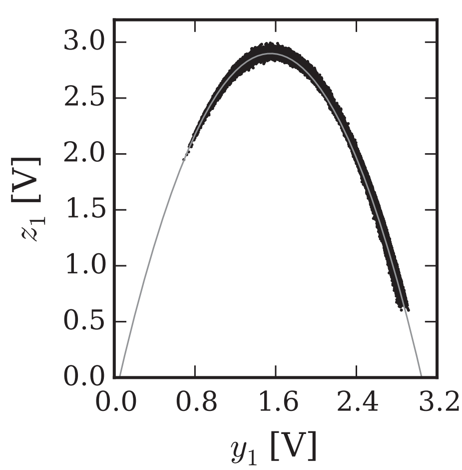

FIG. 13. Function f_1 (black) reconstructed in the plane (y_1,z_1), where $y_1 = x_1(t-\tau'_1)$ and $z_1 = \varepsilon'_1\dot{x}_1(t) + x_1(t) - \sum_{j=2}^{10} k'_{1,j}[x_j(t) - x_1(t)]$, for the chain of experimental oscillators with time-delayed feedback. The true function f_1 is shown in the foreground in gray.

We used two algorithms for separating the recovered coupling coefficients into significant and insignificant coefficients. The first of them is based on the K-means clustering. The second algorithm is based on constructing the dependence of the sum of objective functions for all elements in the network on the number of coupling coefficients in the model equations.

The proposed method can be applied to networks consisting of nonidentical time-delay systems with different coupling architectures. In particular, any two nodes in the analyzed network can be uncoupled, coupled in one direction, or coupled in both directions. The efficiency of the method is shown for chaotic time series of the model and experimental networks described by diffusively coupled delay-differential equations. The considered method can be applied to periodic time series of time-delay systems, but in this case, the accuracy of the network reconstruction is worse. It is possible to extend the proposed method to the recovery of networks with other types of coupling between the elements. For example, the method can be applied to networks consisting of time-delay systems coupled by derivatives of dynamical variables or coupled through the mean field.

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