Noisy oscillator: Random mass and random damping

Stanislav Burov* and Moshe Gitterman†
Physics Department, Bar-Ilan University, Ramat Gan 52900, Israel

(Received 21 July 2016; revised manuscript received 10 November 2016; published 28 November 2016)

The problem of a linear damped noisy oscillator is treated in the presence of two multiplicative sources of noise which imply a random mass and random damping. The additive noise and the noise in the damping are responsible for an influx of energy to the oscillator and its dissipation to the surrounding environment. A random mass implies that the surrounding molecules not only collide with the oscillator but may also adhere to it, thereby changing its mass. We present general formulas for the first two moments and address the question of mean and energetic stabilities. The phenomenon of stochastic resonance, i.e., the expansion due to the noise of a system response to an external periodic signal, is considered for separate and joint action of two sources of noise and their characteristics.

DOI: 10.1103/PhysRevE.94.052144

I. INTRODUCTION

One of the most general and most widely used models in physics is the damped linear harmonic oscillator, which is described by the following equation:

$$m\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega^2x=0. \tag{1}$$

This model has been applied in many fields, ranging from quarks to cosmology. The ancient Greeks already had a general idea of oscillations and used them in musical instruments. Many applications have been found in the last 400 years [1]. The solution of Eq. (1) depends on the parameters γ/m and ω^2/m. For a solution of the type $x=\exp(\alpha t)$, one obtains $\alpha=-\frac{\gamma}{2m}\pm\sqrt{\frac{\gamma^2}{4m^2}-\frac{\omega^2}{m}}$. For $(\gamma/m)^2\geqslant 4(\omega^2/m)$, α is real and negative, i.e., for $t\to\infty$, x monotonically goes to zero, as required for a stable system. However, for $(\gamma/m)^2<4(\omega^2/m)$, α is complex, which means that the approach of x to zero takes place with periodically decreasing amplitude.

Equation (1) describes a pure mechanical system in the classical sense, i.e., zero temperature, while for quantum description the fluctuations persist even in the zero-temperature limit. For nonzero temperature, the deterministic equation (1) has to be supplemented by thermal noise $\eta(t)$,

$$m\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega^2x=\eta(t), \tag{2}$$

where $\eta(t)$ is a random variable with zero mean $\langle\eta(t)\rangle=0$ and a two-point correlation function $\langle\eta(t)\eta(t')\rangle=2D\delta(t-t')$, which for thermal noise must satisfy the fluctuation-dissipation theorem [2] $\langle\eta^2(t)\rangle=4\gamma\kappa T$, where κ is the Boltzmann constant. For $m=0$ and $\omega=0$, Eq. (2) describes an overdamped Brownian particle, first introduced by Einstein more than 100 years ago.

Another generalization of Eq. (1) consists in adding external noise, which enters the equation of motion multiplicatively. For example, random damping yields

$$m\frac{d^2x}{dt^2}+[1+\xi(t)]\gamma\frac{dx}{dt}+\omega^2x=\eta(t). \tag{3}$$

This equation was first used for the problem of water waves influenced by a turbulent wind field [3]. By replacing the coordinate x and time t by the order parameter and coordinate, respectively, Eq. (1) can be transformed into the stationary linearized Ginzburg-Landau equation with a convective term, which describes phase transitions in moving systems [4]. There are an increasing number of problems in which particles advected by the mean flow pass through the region under study. These include problems of phase transition under shear [5], open flows of liquids [6], Rayleigh-Bénard and Taylor-Couette problems in fluid dynamics [7], dendritic growth [8], chemical waves [9], and the motion of vortices [10].

There is also a different type of Brownian motion, in which the surrounding molecules are capable not only of colliding with the Brownian particle, but they also adhere to it for some random time, thereby changing its mass [11]. Such a process is described by the following stochastic equation:

$$m[1+\xi(t)]\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega^2x=\eta(t). \tag{4}$$

There are many situations in chemical and biological solutions in which the surrounding medium contains molecules which are capable of both colliding with the Brownian particle and also adhering to it for a random time. There are also some applications of a variable-mass oscillator [12]. Modern applications of such a model include a nanomechanical resonator which randomly absorbs and desorbs molecules [13]. The diffusion of clusters with randomly growing masses has also been considered [14]. There are many other applications of an oscillator with a random mass [15], including ion-ion reactions [16,17], electrodeposition [18], granular flow [19], cosmology [20,21], film deposition [22], traffic jams [23,24], and the stock market [25,26].

In this paper we further generalize Eq. (1) to include the case of all three previously mentioned sources of noise, the additive part of Eq. (2) and the multiplicative parts of Eqs. (3) and (4). Such an equation will describe a coarse-grained situation when a particle is affected by random kicks from its nearby environment (additive noise), adhesion of the molecules in the environment (random mass), and changes in the nearby environment (random friction). While additive random noise is usually taken to be a Gaussian δ correlated (i.e., white) noise, this is not the case for multiplicative noise.

*stasbur@gmail.com
†moshe.gitterman@biu.ac.il

It is natural to include correlations for the multiplicative part, since for example it can take some time for the attached molecule to return to the environment. Another complication is the value of the noise. While the random additive kick can be of any magnitude and sign (i.e., $\pm$), the multiplicative noise does not have such luxury. Indeed, for the random-mass case, a large negative value of the noise would imply a nonphysical negative mass. Although friction can attain negative magnitude, it is much more common for friction to be strictly positive. To overcome such restrictions, we use exponentially correlated dichotomous noise for multiplicative noises [1]. A noise $\xi(t)$ is called dichotomous when it randomly jumps between two states and its correlation function $\langle\xi(t')\xi(t'')\rangle$ decays exponentially. The advantage of such a choice for the noise is that it is not only correlated and bounded, it is also simple enough to serve as a test case for more complicated noise [27].

The paper is structured as follows. In Sec. II, we introduce the generalization of Eq. (1) for the case of random mass and random damping. The specific noise and the main mathematical tool (Shapiro-Loginov formula) are described. Section III is devoted to the calculation of the first and second moments of x. For each moment, two stability criteria are discussed, using the roots of an appropriate characteristic polynomial. The question of response to an external time-dependent periodic driving force is addressed in Sec. IV. We use examples of strictly random mass and strictly random friction to explain various types of observed stochastic resonances.

II. RANDOM MASS AND RANDOM DAMPING

We start with the generalization of the equation of a linear damped oscillator as previously described. In our generalization the noise perturbs both the mass of the oscillator and the friction:

$$m(1+\xi_1(t))\frac{d^2x}{dt^2}+\gamma(1+\xi_2(t))\frac{dx}{dt}+\omega^2x=\eta(t). \quad (5)$$

The additive noise is taken to be zero average, δ correlated $\langle\eta(t_1)\eta(t_2)\rangle = 2D\delta(t_1-t_2)$ and it is uncorrelated with the multiplicative noise terms $\langle\eta(t_1)\xi_1(t_2)\rangle = \langle\eta(t_1)\xi_2(t_2)\rangle = 0$. The multiplicative noise terms are both assumed to be symmetrical dichotomous noise with two-point correlation function:

$$\langle\xi_1(t_1)\xi_1(t_2)\rangle = \sigma_1^2\exp(-\lambda_1|t_1-t_2|),$$
$$\langle\xi_2(t_1)\xi_2(t_2)\rangle = \sigma_2^2\exp(-\lambda_2|t_1-t_2|). \quad (6)$$

We further assume that the multiplicative noise terms are uncorrelated: $\langle\xi_1(t_1)\xi_2(t_2)\rangle = 0$. An advantage of treating the noise as symmetrical dichotomous noise is that it allows one to obtain results for the case of white noise. In the limit $\lambda_1\to\infty$ (with constant $\sigma_1^2/\lambda = D_1$), the noise ξ_1 transforms to white (i.e., δ correlated) noise (a similar transformation holds for ξ_2). Before turning to the calculation of the moments of x, we mention the central tool we apply to obtain a solution. For an exponentially correlated stochastic process ξ [i.e., Eq. (6)] and some general function of the process $g(\xi)$, the following relation holds:

$$\left(\frac{d}{dt}+\lambda\right)^n\langle\xi g\rangle = \left\langle\xi\frac{d^ng}{dt^n}\right\rangle, \quad (7)$$

where n is a positive integer. Equation (7) is the Shapiro-Loginov formula [28] and its generalization for the case of two sources of noise is $(d/dt+(\lambda_1+\lambda_2))^n\langle\xi_1\xi_2 g\rangle = \langle\xi_1\xi_2 d^n g/dt^n\rangle$.

III. CALCULATION OF THE MOMENTS

A. Behavior of the mean

We perform four operations upon Eq. (5): (i) averaging with respect to the noise, (ii) multiplying by $\xi_1(t)$ and averaging, (iii) multiplying by $\xi_2(t)$ and averaging, and (iv) multiplying by $\xi_1(t)\xi_2(t)$ and averaging. By exploiting the property of dichotomous noise $\xi_1(t)\xi_1(t)=\sigma_1^2$ and $\xi_2(t)\xi_2(t)=\sigma_2^2$ and applying the Shapiro-Loginov formula [as given by Eq. (7)] we obtain

$$\mathbf{a}\left(\frac{d}{dt}\right)\cdot\begin{pmatrix}\langle\xi_1x\rangle\\ \langle\xi_2x\rangle\\ \langle\xi_1\xi_2x\rangle\\ \langle x\rangle\end{pmatrix}=0, \quad (8)$$

where

$$\mathbf{a}\left(\frac{d}{dt}\right)=\begin{pmatrix}0 & \left(b_2^2+\frac{\gamma}{m}b_2+\frac{\omega^2}{m}\right) & b_3^2 & \sigma_2^2\frac{\gamma}{m}\frac{d}{dt}\\ \left(b_1^2+\frac{\gamma}{m}b_1+\frac{\omega^2}{m}\right) & 0 & \frac{\gamma}{m}b_3 & \sigma_1^2\frac{d^2}{dt^2}\\ b_1^2 & \frac{\gamma}{m}b_2 & 0 & \left(\frac{d^2}{dt^2}+\frac{\gamma}{m}\frac{d}{dt}+\frac{\omega^2}{m}\right)\\ \sigma_2^2\frac{\gamma}{m}b_1 & \sigma_1^2b_2^2 & \left(b_3^2+\frac{\gamma}{m}b_3+\frac{\omega^2}{m}\right) & 0\end{pmatrix}. \quad (9)$$

In Eq. (9) $b_1=(d/dt+\lambda_1)$, $b_2=(d/dt+\lambda_2)$, and $b_3=(d/dt+(\lambda_1+\lambda_2))$. The well known Cramer rule yields

$$\left|\mathbf{a}\left(\frac{d}{dt}\right)\right|\langle x\rangle = 0. \quad (10)$$

Substituting the expressions for a_{ij} yields a differential equation of eighth order with constant coefficients $\sum_{i=0}^{i=8}c_{8-i}\frac{d^i\langle x\rangle}{dt^i}=0$.

Seeking a solution of the form $e^{\alpha t}$, we obtain that α is a solution of $|\mathbf{a}(\alpha)|=0$. The expressions for various α can only be found numerically. The stability of the system can be explored by studying the asymptotic behavior of $\langle x\rangle$. The behavior will be stable if $\langle x(t)\rangle\to 0$ as $t\to\infty$. The general criteria for stability is the condition that for all α, which satisfy $|\mathbf{a}(\alpha)|=0$, the value of α has a negative real part. The Routh-Hurowitz theorem [29] provides the condition for all the roots of a polynomial to have a negative real part. The condition involves the calculation of the determinants of matrices up to 15×15 and is rather cumbersome. Instead, one can plot the various roots α on the complex plane and investigate their positions for various values of the parameters γ/m, ω^2/m, λ_1, λ_2, σ_1, and σ_2. In Fig. 1, two examples are

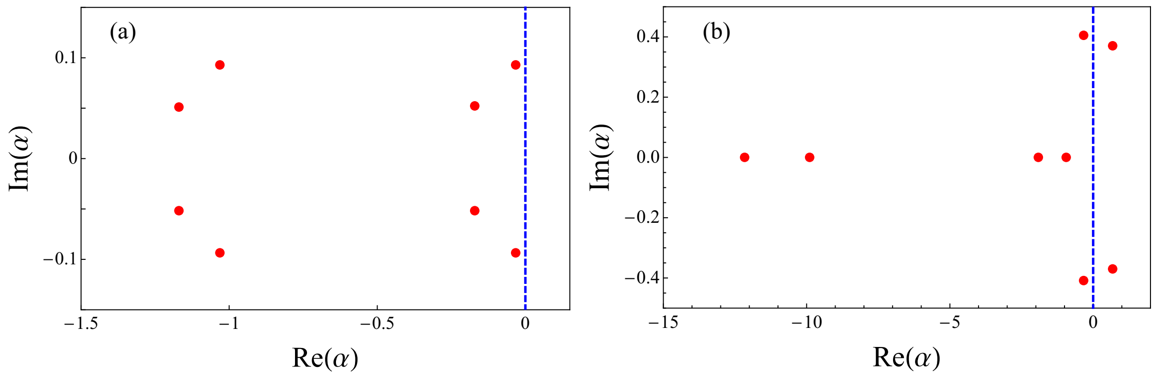

FIG. 1. Values of α's which satisfy $|\mathbf{a}(\alpha)| = 0$, plotted on the complex plane for two different sets of parameters (each dot represents different α): (a) $\gamma/m = 0.1$, $\omega^2/m = 0.1$, $\lambda_1 = 1$, $\lambda_2 = 0.1$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.9$; (b) $\gamma/m = 4$, $\omega^2/m = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\sigma_1 = 0.1$, and $\sigma_2 = 1.5$. The dashed line separates the complex plane into $\mathrm{Re}[\alpha] < 0$ and $\mathrm{Re}[\alpha] > 0$.

presented. In Fig. 1(a) the configuration of the roots is such that for all eight α, $\mathrm{Re}[\alpha] < 0$ and eventually $\langle x \rangle$ decays to zero. When there is at least one α for which $\mathrm{Re}[\alpha] \geqslant 0$, i.e., Fig. 1(b), $\langle x \rangle$ does not converge to zero and the behavior is not stable in the mean sense. We note that the transition to instability can be achieved in various ways. There are various configurations of parameters for which, exactly at the transition point, $\langle x \rangle$ will exhibit stable oscillations. Specifically, this occurs for $\gamma/m = 1$, $\omega^2/m = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\sigma_1 = 1/10$, $\sigma_2 = 1.612443\ldots$. In Fig. 2 the behavior of $\langle x(t) \rangle$ is plotted as a function of time for the mentioned parameters and three different values of σ_2. Below the transition to instability ($\sigma_2 = 1.45$), decaying oscillations occur. At the instability

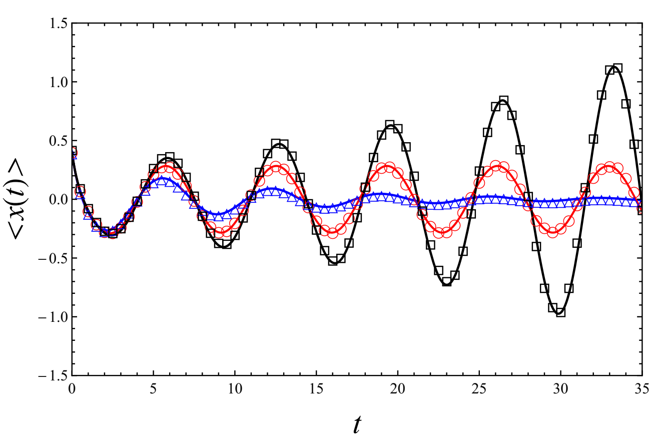

FIG. 2. Temporal behavior of $\langle x(t) \rangle$ for three different values of the random damping noise strength σ_2 while other parameters are kept constant: $\gamma/m = 1$, $\omega^2/m = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, and $\sigma_1 = 1/10$. The thick lines are the solutions of Eq. (10) while the symbols are obtained from numerical simulation of the process. Triangles ($\sigma_2 = 1.45$) are below the transition to instability, circles ($\sigma_2 = 1.612\ldots$) at the transition, and squares ($\sigma_2 = 1.7$) above the transition. The numerical data (symbols) were obtained by simulating 1×10^6 realizations of the process, each simulation performed by drawing the random times between switches of $1 \pm \sigma_1$ from an exponential distribution and similarly drawing random times between the switches of $1 \pm \sigma_2$. During the instances when neither of the noises switched, the system was forwarded in time by exact integration.

($\sigma_2 = 1.612\ldots$) the oscillations are stable, and above the transition ($\sigma_2 = 1.7$) the oscillations are diverging. Those results were obtained both by solution of Eq. (10) and by numerical simulation of the stochastic process.

B. Behavior of $\langle x^2 \rangle$

The stability criteria in the mean sense, as described in the previous section, can be rather unsatisfying. Indeed, the convergence of the mean to zero in the long run does not provide any certainty that the process x [as described by Eq. (5)] will be in the vicinity of zero. For example, the simple random walk starting from zero will on average be at zero, but the divergence of the second moment of a simple random walk produces very long excursions towards $\pm\infty$. It is thus preferable to obtain conditions for stability based on the behavior of the second moment $\langle x^2 \rangle$. Generally the divergence of specific moment $\langle x^n(t) \rangle$ depends on the properties of the tail of the time-dependent distribution of x, $P(x,t)$. The case when $P(x,t)$ decays as $|x|^{-1-z}$, with $1 < z < 2$, produces a stable solution for the mean but divergence of the second comment. The ability to compute the full distribution $P(x,t)$ is beyond the scope of this study (or any other study to the best of our knowledge) and we therefore proceed to the exploration of the second comment. We note that in the literature [30,31] the instability based on the behavior of the second moment is addressed as an energetic instability. In order to obtain the various possible behaviors of $\langle x^2 \rangle$, we now turn to Eq. (5) and proceed similarly to what was done for $\langle x \rangle$.

We rewrite Eq. (5) in the following form:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\frac{\gamma}{m}\frac{1+\xi_2}{1+\xi_1}y - \frac{\omega^2}{m}\frac{1}{1+\xi_1}x - \frac{m}{1+\xi_1}\eta(t), \end{aligned} \tag{11}$$

and then obtain from Eq. (11) three equations after multiplying them by x and by y and summing up the mixed terms (i.e., $y dx/dt + x dy/dt$):

$$\begin{aligned}\frac{dx^2}{dt} &= 2xy,\\ \frac{dy^2}{dt} &= -\frac{2\gamma}{m}\frac{1+\xi_2}{1+\xi_1}y^2 - \frac{2\omega^2}{m}\frac{1}{1+\xi_1}xy + \frac{2}{m(1+\xi_1)}y\eta(t),\\ \frac{dxy}{dt} &= -\frac{\gamma}{m}\frac{1+\xi_2}{1+\xi_1}xy + y^2 - \frac{\omega^2}{m}\frac{1}{1+\xi_1}x^2 + \frac{1}{m(1+\xi_1)}x\eta(t).\end{aligned} \tag{12}$$

First we average Eq. (12) with respect to η. Since the multiplicative noise terms ξ_1, ξ_2 are uncorrelated with η, we treat them as constants and only need to compute the correlators $\langle x\eta(t)\rangle_\eta$ and $\langle y\eta(t)\rangle_\eta$. The symbol $\langle\cdots\rangle_\eta$ means average only with respect to η. Since $\eta(t)$ is a Gaussian δ correlated noise we can invoke the Novikov theorem [32] for the correlators. The theorem states that for a vector $\mathbf{u} = (u_1,u_2,\ldots,u_n)$ of dimension n and Gaussian δ correlated noise $\eta(t)$ which satisfies the relation

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}) + \mathbf{g}(\mathbf{u})\eta(t), \tag{13}$$

where $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), \ldots, f_n(\mathbf{u}))$ and $\mathbf{g}(\mathbf{u}) = (g_1(\mathbf{u})g_2(\mathbf{u}), \ldots, g_n(\mathbf{u}))$, the correlators satisfy

$$\langle g_i(\mathbf{u})\eta(t)\rangle_\eta = D\sum_{j=1}^{n}\left\langle \frac{\partial g_i(\mathbf{u})}{\partial u_j} g_j(\mathbf{u})\right\rangle_\eta . \tag{14}$$

From Eq. (12), we define $\mathbf{u} = (x^2,y^2,xy)$ and $\mathbf{g}(\mathbf{u}) = (0, \frac{2}{m(1+\xi_2)}\sqrt{y^2}, \frac{1}{m(1+\xi_2)}\sqrt{x^2})$. Applying the Novikov theorem yields

$$\langle y\eta\rangle_\eta = \frac{D}{m(1+\xi_1)}, \quad \langle x\eta\rangle_\eta = 0. \tag{15}$$

Averaging Eq. (12) with respect to η and inserting Eq. (15) for the correlators, we obtain

$$\begin{aligned}\frac{d\langle x^2\rangle_\eta}{dt} - 2\langle xy\rangle_\eta &= 0,\\ (1+\xi_1)^2\frac{d\langle y^2\rangle_\eta}{dt} + \frac{2\gamma}{m}(1+\xi_2)(1+\xi_1)\langle y^2\rangle_\eta + \frac{2\omega^2}{m}(1+\xi_1)\langle xy\rangle_\eta - \frac{2D}{m^2} &= 0,\\ (1+\xi_1)\frac{d\langle xy\rangle_\eta}{dt} + \frac{\gamma}{m}(1+\xi_2)\langle xy\rangle_\eta - (1+\xi_1)\langle y^2\rangle_\eta + \frac{\omega^2}{m}\langle x^2\rangle_\eta &= 0.\end{aligned} \tag{16}$$

Equation (16) is then treated in the same fashion as Eq. (5) in Sec. III A. Four operations are performed upon each line in Eq. (16): (i) averaging with respect to the noises, (ii) multiplying by $\xi_1(t)$ and averaging, (iii) multiplying by $\xi_2(t)$ and averaging, and (iv) multiplying by $\xi_1(t)\xi_2(t)$ and averaging. Since all sources of noise are uncorrelated we can switch the order of averaging. The outcome of the averaging order switching is that we may treat $\langle x^2\rangle_\eta$, $\langle y^2\rangle_\eta$, $\langle xy\rangle_\eta$ as x^2, y^2, xy and after applying the Shapiro-Loginov procedure [Eq. (7)], only terms of the type $(\langle x^2\rangle,\langle y^2\rangle,\langle xy\rangle,\langle\xi_1 x^2\rangle,\ldots)$ remain. The final result of the averaging is written in matrix form:

$$\mathbf{M}\cdot\vec{X} = \vec{X}_0, \tag{17}$$

where $\mathbf{M}$ is given by

$$\mathbf{M}\left(\frac{d}{dt}\right) = \begin{pmatrix} \frac{d}{dt} & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & b1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & b1 & 0 & -2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b1 & 0 & -2\\ 0 & (1+\sigma_1^2)\frac{d}{dt}+\frac{2\gamma}{m} & \frac{2\omega^2}{m} & 0 & 2b_1+\frac{2\gamma}{m} & \frac{2\omega^2}{m} & 0 & \frac{2\gamma}{m} & 0 & 0 & \frac{2\gamma}{m} & 0\\ 0 & 2\sigma_1^2\frac{d}{dt}+\frac{2\gamma}{m}\sigma_1^2 & \frac{2\omega^2}{m}\sigma_1^2 & 0 & (1+\sigma_1^2)b1+\frac{2\gamma}{m} & \frac{2\omega^2}{m} & 0 & \frac{2\gamma}{m}\sigma_1^2 & 0 & 0 & \frac{2\gamma}{m} & 0\\ 0 & \frac{2\gamma}{m}\sigma_2^2 & 0 & 0 & \frac{2\gamma}{m}\sigma_2^2 & 0 & 0 & (1+\sigma_1^2)b_2+\frac{2\gamma}{m} & \frac{2\omega^2}{m} & 0 & 2b_3+\frac{2\gamma}{m} & \frac{2\omega^2}{m}\\ 0 & \frac{2\gamma}{m}\sigma_1^2\sigma_2^2 & 0 & 0 & \frac{2\gamma}{m}\sigma_2^2 & 0 & 0 & 2\sigma_1^2 b_2+\frac{2\gamma}{m}\sigma_1^2 & \frac{2\omega^2}{m}\sigma_1^2 & 0 & (1+\sigma_1^2)b_3+\frac{2\gamma}{m} & \frac{2\omega^2}{m}\\ \frac{\omega^2}{m} & -1 & \frac{d}{dt}+\frac{\gamma}{m} & 0 & -1 & b_1 & 0 & 0 & \frac{\gamma}{m} & 0 & 0 & 0\\ 0 & -\sigma_1^2 & \sigma_1^2\frac{d}{dt} & \frac{\omega^2}{m} & -1 & b_1+\frac{\gamma}{m} & 0 & 0 & 0 & 0 & 0 & \frac{\gamma}{m}\\ 0 & 0 & \frac{\gamma}{m}\sigma_2^2 & 0 & 0 & 0 & \frac{\omega^2}{m} & -1 & b_2+\frac{\gamma}{m} & 0 & -1 & b_3\\ 0 & 0 & 0 & 0 & 0 & \frac{\gamma}{m}\sigma_2^2 & 0 & -\sigma_1^2 & \sigma_1^2 b_2 & \frac{\omega^2}{m} & -1 & b_3+\frac{\gamma}{m} \end{pmatrix}, \tag{18}$$

where $\vec{X} = (\langle x^2\rangle,\langle y^2\rangle,\langle xy\rangle,\langle\xi_1 x^2\rangle,\langle\xi_1 y^2\rangle,\langle\xi_1 xy\rangle,\langle\xi_2 x^2\rangle, \langle\xi_2 y^2\rangle,\langle\xi_2 xy\rangle,\langle\xi_1\xi_2 x^2\rangle,\langle\xi_1\xi_2 y^2\rangle,\langle\xi_1\xi_2 xy\rangle)$ and $\vec{X}_0 = (0,0,0,0,2D/m^2,0,0,0,0,0,0,0)$. Cramer's rule implies

$$\left|\mathbf{M}\left(\frac{d}{dt}\right)\right|\langle x^2\rangle = \left|\mathbf{M}_{1,5}\left(\frac{d}{dt}\right)\right|\frac{2D}{m^2}, \tag{19}$$

where $|\mathbf{M}_{1,5}|$ is the $\{1,5\}$ minor of matrix $\mathbf{M}$, i.e., the determinant of matrix $\mathbf{M}$ where the first column and fifth row were removed from the matrix. The determinants on both sides of Eq. (19) are differential operators and since $|\mathbf{M}_{1,5}(d/dt)|$ operates on a constant it can be replaced by $|\mathbf{M}_{1,5}(0)|$. The stable solution is

$$\langle x_s^2 \rangle = \frac{|\mathbf{M}_{1,5}(0)|}{|\mathbf{M}(0)|}(2D/m^2). \tag{20}$$

From Eq. (20), it is clear that when $|\mathbf{M}(0)| = 0$, the system is not stable and the second moment diverges. As was the case for $\langle x \rangle$, we can write a more general condition. We search a solution of $|\mathbf{M}(\frac{d}{dt})|\langle x^2 \rangle = 0$ [i.e., the homogeneous part of Eq. (20)] in the form of $\exp(\alpha t)$. This solution will be stable if $\forall\alpha$ [such that $|\mathbf{M}(\alpha)| = 0$] $\mathrm{Re}[\alpha] < 0$. Then this is the stability criterion and it includes the special case of $\alpha = 0$ that zeros $|\mathbf{M}|$. The search for the criteria of a negative real part of $|\mathbf{M}(\alpha)| = 0$ can be performed by plotting different values α on the complex plane and searching for situations where $\mathrm{Re}[\alpha] \geqslant 0$. Specifically for the mentioned case when $\langle x \rangle$ is stable ($\gamma/m = 1$, $\omega^2/m = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\sigma_1 = 1/10$, $\sigma_2 = 1.45$) the second moment $\langle x^2 \rangle$ will diverge.

IV. RESPONSE TO EXTERNAL DRIVING TERM

We would like to address the question of a response of a noisy oscillator with random mass and random damping to an external time-dependent driving term. The external driving term is taken to be a simple sinusoidal form $A_0 \cos(\Omega t)$. Our general Eq. (5) then becomes

$$m(1 + \xi_1(t))\frac{d^2x}{dt^2} + \gamma(1 + \xi_2(t))\frac{dx}{dt} + \omega^2 x = \eta(t) + A_0 \cos(\Omega t). \tag{21}$$

Repeating the steps of Sec. III A and using the fact that $A_0 \cos(\Omega t)$ and the multiplicative sources of noise are uncorrelated, i.e., $\langle \xi_1(t)\cos(\Omega t)\rangle = \langle \xi_2(t)\cos(\Omega t)\rangle = \langle \xi_1(t)\xi_2(t)\cos(\Omega t)\rangle = 0$, we obtain

$$\mathbf{a}\left(\frac{d}{dt}\right) \cdot \begin{pmatrix} \langle \xi_1 x\rangle \\ \langle \xi_2 x\rangle \\ \langle \xi_1\xi_2 x\rangle \\ \langle x\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A_0\cos(\Omega t) \\ 0 \end{pmatrix}, \tag{22}$$

where $\mathbf{a}(d/dt)$ is defined by Eq. (9). The behavior of $\langle x \rangle$ is given by Cramer's rule:

$$\left|\mathbf{a}\left(\frac{d}{dt}\right)\right|\langle x\rangle = -\left|\mathbf{a}_{4,3}\left(\frac{d}{dt}\right)\right| A_0 \cos(\Omega t), \tag{23}$$

where $|\mathbf{a}_{4,3}(d/dt)|$ is the {4,3} minor of $\mathbf{a}(d/dt)$.

In the limit $t \to \infty$, when a stable solution for $|\mathbf{a}(\frac{d}{dt})|\langle x\rangle = 0$ exists and is equal to zero, $\langle x \rangle$ is given by

$$\langle x\rangle = A\cos(\Omega t + \phi) \tag{24}$$

with

$$A/A_0 = \sqrt{\frac{|\mathbf{a}_{4,3}(-i\Omega)||\mathbf{a}_{4,3}(i\Omega)|}{|\mathbf{a}(-i\Omega)||\mathbf{a}(i\Omega)|}} \tag{25}$$

and

$$\tan(\phi) = \frac{|\mathbf{a}_{4,3}(-i\Omega)||\mathbf{a}(i\Omega)| + |\mathbf{a}_{4,3}(i\Omega)||\mathbf{a}(-i\Omega)|}{|\mathbf{a}_{4,3}(-i\Omega)||\mathbf{a}(i\Omega)| - |\mathbf{a}_{4,3}(i\Omega)||\mathbf{a}(-i\Omega)|}i. \tag{26}$$

The response of $\langle x \rangle$ to the external driving term is equals A/A_0 [Eq. (25)] when a stable solution exists.

A. Various aspects of response

The expression for the response A/A_0 depends on seven parameters of the system and Ω. In order to obtain insight into the various possible types of behavior, we first treat the two simpler cases where only one source of multiplicative noises is present, i.e., (i) random damping [Eq. (4)] or (ii) random mass [Eq. (3)]. The equation describing the case of a random mass and random damping, i.e., Eq. (5), reduces to case (i) by taking σ_2 and λ_2 to zero and to case (ii) by taking σ_1 and λ_1 to zero. Therefore, the response to an external periodic driving term for both simpler cases is provided by A/A_0 in Eq. (25) by setting the appropriate parameters to zero. We note that both of these simpler cases were previously treated [1]. In the following mainly the behavior of A/A_0 as a function of Ω is presented. The behavior of A/A_0 as a function of σ_1 and σ_2 is presented in the Appendix.

1. Random mass

The response for the case of a random mass is presented in Figs. 3(a)–3(c). In Fig. 3(a) a resonance is found for quite small values of noise strength ($\sigma_1^2 = 0.01$). Increasing the noise strength while keeping the correlation parameter λ_1 constant produces an additional maximum for A/A_0, as shown in Fig. 3(b). This second resonance is due to the splitting of the first peak and decreasing its height. Such splitting occurs while the value of λ_1 is quite small, i.e., large correlation times of ξ_1.

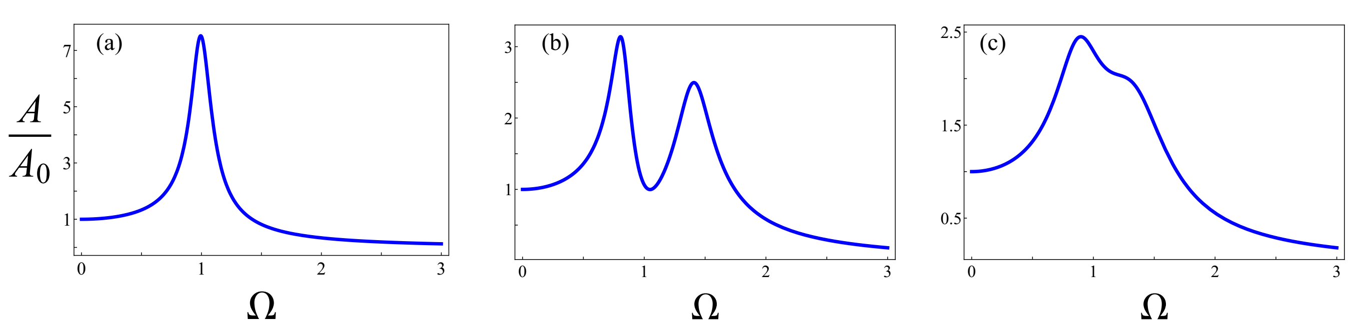

FIG. 3. Response A/A_0 as a function of angular frequency (Ω) of the periodic external driving force as given in Eq. (25) for the case of only a random mass. A maximum of A/A_0 for specific parameters of the system describes a resonance between the behavior of x and the external driving force: (a) $\gamma/m = 0.1$, $\omega^2/m = 1$, $\lambda_1 = 0.1$, $\sigma_1 = 0.1$, and $\sigma_2 = 0$; (b) the same parameters as in (a) except that $\sigma_1 = 0.5$; (c) the same parameters as in (b) except that $\lambda_1 = 0.35$.

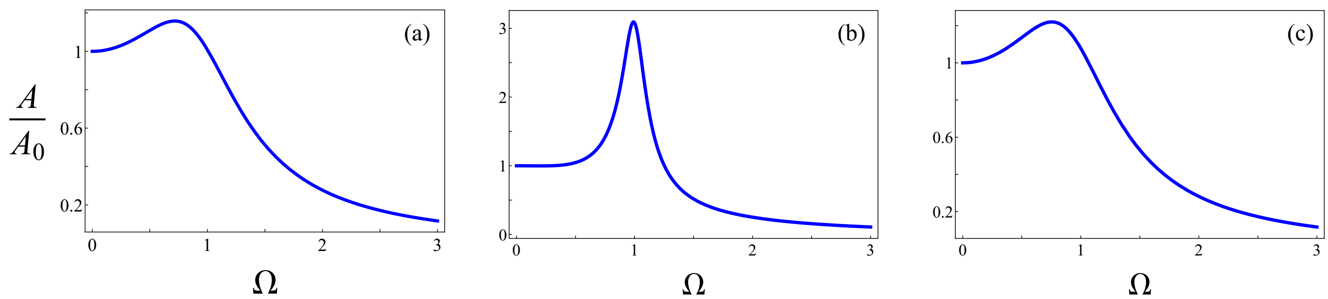

FIG. 4. Response A/A_0 as a function of angular frequency (Ω) of the periodic external driving as given in Eq. (25) for the case of only random damping. A maximum of A/A_0 for specific parameters of the system describes a resonance between the behavior of x and the external driving force: (a) $\gamma/m = 0.1$, $\omega^2/m = 1$, $\lambda_2 = 0.1$, $\sigma_1 = 0$, and $\sigma_2 = 0.1$; (b) the same parameters as in (a), except that $\sigma_2 = 0.9$; (c) the same parameters as in (b), except that $\lambda_2 = 10$.

In order to understand the observed effect we notice the fact that random noise ξ_1 produces two mass values and creates two intrinsic states for the oscillator. In each of the states the oscillator behaves as a simple oscillator with additive noise. Existence of a resonance will depend on specific parameters of the state: m_i, γ_i, and ω (subscript i runs over possible state indices). The resonant frequency Ω_R (if it exists) is provided by the well known formula [33]

$$\Omega_R = \sqrt{\frac{\omega^2}{m_i} - \frac{\gamma_i^2}{2m_i^2}}. \tag{27}$$

In the case of random mass, $m_1 \neq m_2$ and $\gamma_1 = \gamma_2$. If the oscillator can attain a resonance in both of the states, and the frequencies of those resonances are sufficiently distinct, we expect to observe two resonant frequencies as described in Fig. 3. Each of the resonant frequencies will correspond to an intrinsic regime or state of the oscillator and the splitting effect artificially resembles splitting of states in a quantum system. The existence of two states for the oscillator is not sufficient for appearance of two resonant frequencies; the oscillator must also spend a sufficient amount of time (on average) in each of these states in order to attain a resonance. Since the oscillator is constantly jumping from one state to the other, the time to build up a "proper" response to an external field might be insufficient. The oscillator will jump to the other state, where a different response will start to build up. It is thus important that the noise correlation time will be long enough. Indeed, this effect is shown in Fig. 3(c). While keeping the strength of the noise the same as in Fig. 3(b), λ_1 was increased and the collapse of the two resonances was obtained. The case of a random mass can thus contribute to the existence of a single stochastic resonance, but it can also split a single resonance into two resonances (when the correlation time of the noise is sufficiently long). The appearance of multiple resonances was also observed in different noisy representations of the harmonic oscillator [31,34].

2. *Random damping*

The response for the case when only random damping exists is presented in Figs. 4(a)–4(c). Figure 4(a) shows a resonance for a small strength of the multiplicative noise ξ_2, $\sigma_2 = 0.01$. In Fig. 4(b), the value of σ_2^2 was taken to be 0.9, yielding a threefold increase in the peak value of A/A_0. The effect of resonant frequency splitting, similar to the random mass case, is not observed. The oscillator attains two intrinsic states with $\gamma_1 \neq \gamma_2$ and $m_1 = m_2$. The functional form of Eq. (27) allows two different resonant frequencies for two states with specific values of ω and damping. But in contrast to the random mass case the difference between two resonant frequencies is not sufficient ($0 < \sigma_2 < 1$). Random transitions between two states and the differences in response for each intrinsic state (i.e., decrease in response of one state while there is an increase of the other) will smear the presences of two maxima if the maxima frequencies are not sufficiently separated. It seems that for random damping the frequency separation is not sufficient and no splitting is observed. The increase in the resonance strength due to increase in the damping noise can be explained as a pronounced resonance in a state where the damping is very low [i.e., $\gamma(1-\sigma_2)$]. This response increase is expected to disappear when the time the oscillator spends in a given state will decrease, as explained for the random mass case. Indeed when we decrease this time by increasing λ_2 the effect disappears. Figure 4(c) shows the disappearance of the threefold increase of the peak value of the resonance after a significant decrease in the damping noise correlation time, $\lambda_2 \to 10$.

3. *Random mass and damping*

When both sources of noise (random mass and random damping) are present, we expect a mixture of the previously discussed cases to take place. In Fig. 5(a), A/A_0 exhibits a resonance for specific Ω, while the strengths of the sources of noise are quite small: $\sigma_1 = 0.1$ and $\sigma_2 = 0.1$. Increasing the strength of the random mass noise, while leaving the strength of the random damping noise constant, splits the resonance. Figure 5(b) shows two maxima for A/A_0 and the effect is similar to the case of only a random mass, as described in Fig. 3(b). The presence of a small noise term for the damping does not qualitatively change the effect. But if in addition to increasing the strength of ξ_1, one also increases the strength of ξ_2 (i.e., random damping), a nonsymmetric effect occurs. For the case of only random damping, an increase of noise strength expands the size of the resonance [Fig. 4(b)]. In Fig. 5(c), we see that, as the strength of ξ_2 increases, it does not affect the values of the maxima in the same fashion. While the second maxima that appeared in Fig. 5(b) expanded significantly, the first maxima grew only slightly. This asymmetry arises due to

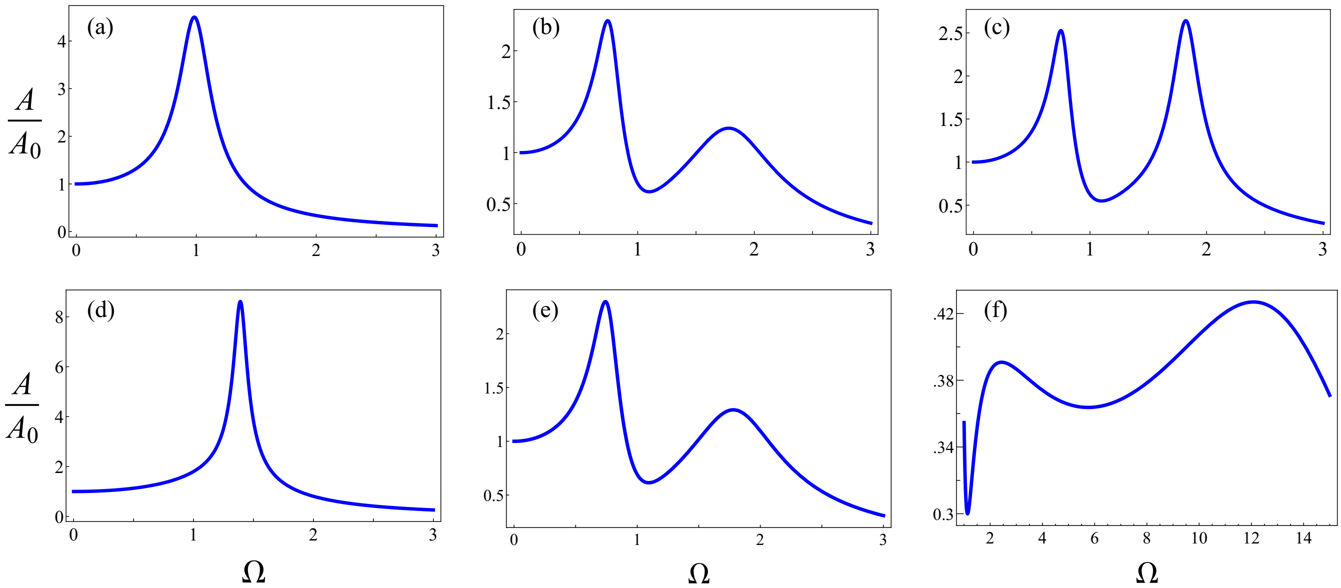

FIG. 5. Response A/A_0 as a function of angular frequency (Ω) of the periodic external driving force as given in Eq. (25). A maximum of A/A_0 for specific parameters of the system describes a resonance between the behavior of x and the external drive: (a) $\gamma/m = 0.2$, $\omega^2/m = 1$, $\lambda_1 = 0.1$, $\lambda_2 = 0.1$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.1$; (b) the same parameters as in (a), except that $\sigma_1 = 0.7$; (c) the same parameters as in (b), except that $\sigma_2 = 0.95$; (d) the same parameters as in (c), except that $\lambda_1 = 10$; (e) the same parameters as in (c), except that $\lambda_2 = 10$; and (f) the same parameters as in (c), except that $\sigma_1 = 0.995$ and $\sigma_2 = 0.75$.

the asymmetry of the resonant frequencies (as a function of mass and damping) at each intrinsic state of the oscillator [i.e., Eq. (27)]. m appears in the denominator and affects Ω_R more violently than γ that appears in the numerator. Due to this fact a significant effect is expected for the state with smaller m and small γ. The temporal correlation must be long enough in order to observe the mentioned effect and indeed increasing either λ_1 [Fig. 5(d)] or λ_2 [Fig. 5(e)] reverses the response to previously observed cases.

In the case of random damping, the presence of two states does not lead to the appearance of resonant splitting. Interestingly enough, when both random damping and random mass are present, an additional resonance splitting can occur. By keeping the temporal correlation of both sources of noise sufficiently long, $\lambda_1 = \lambda_2 = 0.1$, we take the limit of very large strength of a random mass noise ($\sigma_1 = 0.995$) and large strength of random damping noise ($\sigma_2 = 0.7$). The result of additional resonance is presented in Fig. 5(f). Obviously, the simplistic approach that describes each resonant frequency as a frequency that corresponds to a resonance for one of the states of the oscillator fails here.

In order to study this effect further we present the behavior of the resonant frequency Ω_R. In Fig. 6(a) the behavior of the resonant frequency is presented for the case of random mass without random damping and compared to the predictions of Eq. (27). The second resonant frequency appears only when the frequencies of the two states are sufficiently distinct, and in general the behavior of the noisy case follows the predictions for the two different states. Even the nonmonotonicity of Ω_R for random mass is a consequence of the nonmonotonicity

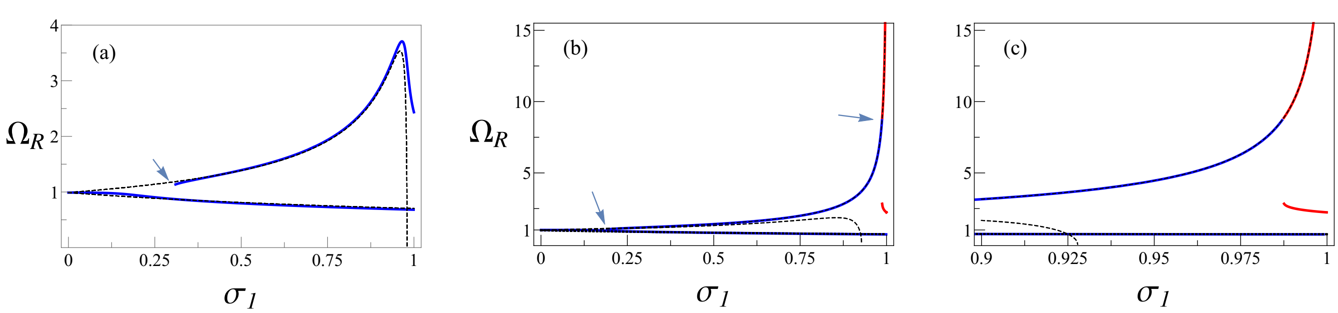

FIG. 6. Resonant frequencies Ω_R as a function of ξ_1 strength, for the case of random mass and random damping (thick lines) and for the specific states of the oscillator (dashed lines). (a) Only random mass noise is present: $\gamma/m = 0.2$, $\omega^2/m = 1$, $\lambda_1 = 0.1$, $\lambda_1 = 0.0$, and $\sigma_2 = 0$. The arrow points to an emergence of a second resonance. (b) Both noises are present: $\gamma/m = 0.2$, $\omega^2/m = 1$, $\lambda_1 = 0.1$, $\lambda_1 = 0.1$, and $\sigma_2 = 0.9$. The left arrow shows the position of emergence of the second resonance while the right arrow shows the position of emergence of the third resonance. Four different dashed lines are presented, and two bottom lines almost coincide for the whole range of σ_1. The different color of the thick line is plotted for the part when three resonances occur. (c) Zoom into the range $0.9 \leqslant \sigma_1 \leqslant 1$ of panel (b).

of Ω_R in Eq. (27). When also random damping is present the situation is quite similar while σ_1 is small enough. In Fig. 6(b), behavior similar to that in Fig. 6(a) is presented. The four different states appear as two states (the dashed lines are very close to one another) and generally there is almost no obvious effect of the additional noise. For large enough σ_1 Eq. (27) predicts disappearance of resonance for one of the states of the oscillator (one of the dashed lines drops to zero). Inside this region where only two states with resonance exist there suddenly appears additional resonance for the noisy case (lower red line). We cannot attribute this resonance to a resonance in an intrinsic state of the oscillator, since this intrinsic resonance does not exist for this range of parameters.

While for a majority of the cases we managed to describe the response behavior in terms of response of the intrinsic states of the oscillator, there are exceptional situations. In those situations the appearance of an additional resonance must be interpreted as an interference between various intrinsic states of the oscillator and not as an attribute of a response in a single state. The noises in our oscillator model are not only capable of creating an intrinsic state that will attain a proper response. An effective coupling between transitions manages to create a preferable response to an external filed. Further study of such coupling is needed.

V. CONCLUSIONS

We considered an oscillator with two multiplicative random forces, which define the random damping and the random mass. The random mass means that the molecules of the surrounding medium not only collide with an oscillator, but also adhere to it for a random time, thereby changing the oscillator mass. We calculated the first and the second moments of the oscillator coordinates by considering these two moments in the form of the damped exponential functions of time, $\exp(\alpha t)$. The signs of α, which are obtained numerically, define the mean and energetic stability of the system. Stable solutions of the moments were represented by determinants of appropriate matrices. We brought references to many applications of such calculations to physics, chemistry, and biology. Specifically we have shown that for the mean, stable oscillations persist at the transition to instability.

The last section described the stochastic resonance phenomenon; that is, the noise increased the applied periodic signal by helping the system to absorb more energy from the external force [35]. We presented the stochastic resonance as the function of the frequency Ω of the applied periodic signal, first separately for a random mass and random damping, and then for the case of joint action of both these sources of noise. For most cases we managed to describe the observed phenomena in terms of simple intrinsic states of the oscillator and presence or nonpresence of resonance for those states. A description by the means of underlying intrinsic states might become useful in experimental situations where the intrinsic states are explored by the means of response to an external field, e.g., biomolecule folding and unfolding experiments [36,37] where distinct folded and unfolded states are explored by external pulling. While the description by the means of response of the intrinsic states holds for a majority of the cases, we found exceptions to this simple description. Specifically, we argue that the appearance of additional resonant frequency at a regime where intrinsic resonance frequency dies out occurs due to transitions between states and not the presence of a single preferable response in an intrinsic state. It is the regime

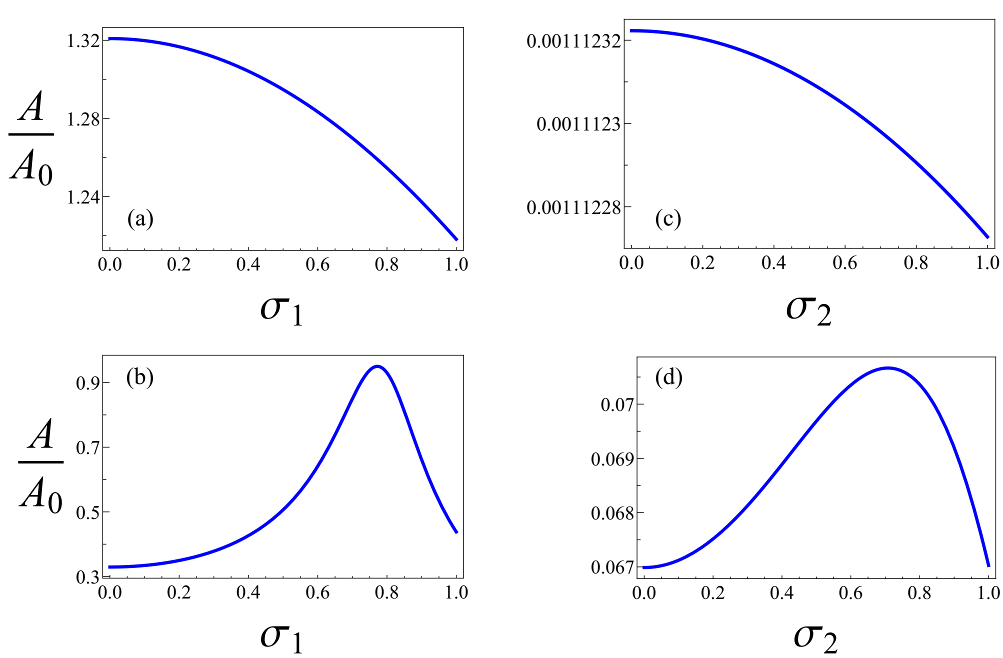

FIG. 7. Response A/A_0 as a function of noise strength (σ_1 and σ_2) of the periodic external driving force as given in Eq. (25) for specific values of Ω: (a) $\gamma/m = 0.2$, $\omega^2/m = 1$, $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\sigma_2 = 0.5$, and $\Omega = 0.5$; (b) the same parameters as in (a), except that $\Omega = 2$; (c) $\gamma/m = 0.2$, $\omega^2/m = 1$, $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\sigma_1 = 0$, and $\Omega = 30$; and (d) the same parameters as in (c), except that $\sigma_1 = 0.994$.

where the interference between states creates a preferable response.

APPENDIX

In the main text we presented the response A/A_0 as a function of Ω. In this Appendix we present the response as a function of noise strength σ_1 and σ_2. In general the dependence of A/A_0 on the noise strength, for specific value of Ω, is associated with the chosen Ω. Nonmonotonic behavior is expected in regions of Ω where the resonant frequency Ω_R will be shifted when changing the noise strength (σ_1 or σ_2). If Ω_R will coincide with the chosen Ω for some value $0 < \sigma_1 < 1$ (or σ_2), a maxima of A/A_0 will appear for this specific value of σ_1 (or σ_2). When such a crossover does not occur, the behavior of A/A_0 is monotonic as displayed in Figs. 7(a) and 7(c). When a crossover of Ω_R occurs, a modest maxima will be observed, as described in Figs. 7(b) and 7(d). The appearance of maxima as a function of σ_2 occurs for nonzero values of σ_1. In the main text we described situations when both σ_1 and σ_2 are nonzero, where two maxima of A/A_0 appear (as a function of Ω). The existence of two (or even three) Ω_R suggest that when those resonant frequencies are shifted one might observe also two maxima for A/A_0 as a function of the noise strength. Due to the fact that the maxima of A/A_0 (as a function of Ω) are well separated (in Ω) we were unable to find parameters where this phenomenon might occur.

[1] M. Gitterman, *The Noisy Oscillator: Random Mass, Frequency, Damping* (World Scientific, Singapore, 2013).
[2] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, London, 1958).
[3] B. J. West and V. Seshadri, J. Geophys. Res. **86**, 4293 (1981).
[4] M. Gitterman, Phys. Rev. E **70**, 036116 (2004).
[5] A. Onuki, J. Phys. Condens. Matter **9**, 6119 (1997).
[6] J. M. Chomaz and A. Couarion, Phys. Fluids **11**, 2977 (1999).
[7] T. E. Faber, *Fluid Dynamics for Physicists* (Cambridge University Press, Cambridge, 1995).
[8] F. Helot and A. Libchaber, Phys. Scr. **T9**, 126 (1985).
[9] A. Saul and K. Showalter, *Oscillations and Travel Waves in Chemical Systems* (Wiley-Interscience, New York, 1985), p. 419.
[10] M. Gitterman, B. Y. Shapiro, and I. Shapiro, Phys. Rev. B **65**, 174510 (2002).
[11] M. Gitterman, J. Phys.: Conf. Ser. **248**, 012049 (2010).
[12] M. S. Abdalla, Phys. Rev. A **34**, 4598 (1986).
[13] M. I. Dykman, M. Khasin, J. Portman, and S. W. Shaw, Phys. Rev. Lett. **105**, 230601 (2010).
[14] J. Łuczka, M. Niemiec, and P. Hänggi, Phys. Rev. E **52**, 5810 (1995).
[15] M. Ausloos and R. Lambiotte, Phys. Rev. E **73**, 011105 (2006).
[16] A. Gadomski and J. Siódmiak, Cryst. Res. Technol. **37**, 281 (2002).
[17] A. Gadomski, J. Siódmiak, I. Santamarìa-Holek, J. M. Rubì, and M. Ausloos, Acta Phys. Polon. B **36**, 1537 (2005).
[18] A. T. Pérez, D. Saville, and C. Soria, Europhys. Lett. **55**, 425 (2001).
[19] I. Goldhirsch and G. Zanetti, Phys. Rev. Lett. **70**, 1619 (1993).
[20] W. Benz, Spatium **6**, 3 (2000).
[21] S. J. Weidenschilling, D. Spaute, D. R. Davis, F. Marzari, and K. Ohtsuki, Icarus **128**, 429 (1997).
[22] N. Kaiser, Appl. Opt. **41**, 3053 (2002).
[23] T. Nagatani, J. Phys. Soc. Jpn. **65**, 3386 (1996).
[24] E. Ben-Naim, P. L. Krapivsky, and S. Redner, Phys. Rev. E **50**, 822 (1994).
[25] M. Ausloos and K. Ivanova, Eur. Phys. J. B **27**, 177 (2002).
[26] M. Ausloos and K. Ivanova, *The Application of Econophysics*, edited by H. Takayasu (Springer-Verlag, Berlin, 2004), pp. 117–124.
[27] I. Bena, Int. J. Mod. Phys. B **20**, 2825 (2006).
[28] V. E. Shapiro and V. M. Loginov, Physica A **91**, 563 (1978).
[29] Q. Rahman and G. Schmeisser, *Analytic Theory of Polynomials* (Oxford University Press, New York, 2002).
[30] V. Méndez, W. Horsthemke, P. Mestres, and D. Campos, Phys. Rev. E **84**, 041137 (2011).
[31] R. Mankin, K. Laas, T. Laas, and E. Reiter, Phys. Rev. E **78**, 031120 (2008).
[32] E. Novikov, Sov. Phys. JETP **20**, 1290 (1965).
[33] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, London, 1960).
[34] Y. Peleg and E. Barkai, Phys. Rev. E **80**, 030104 (2009).
[35] F. Yung and E. Marchesoni, Chaos **21**, 047516 (2011).
[36] J.-D. Wen, M. Manosas, P. T. X. Li, S. B. Smith, C. Bustamante, F. Ritort, and I. Tinoco, Jr., Biophys. J. **92**, 2996 (2007).
[37] M. Hidalgo-Soria, A. Perez-Madrid, and I. Santamaria-Holek, Phys. Rev. E **92**, 062708 (2015).