

Elephant random walks and their connection to Pólya-type urns

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In this paper, we explain the connection between the elephant random walk (ERW) and an urn model à la Pólya and derive functional limit theorems for the former. The ERW model was introduced in [Phys. Rev. E **70**, 045101 (2004)] to study memory effects in a highly non-Markovian setting. More specifically, the ERW is a one-dimensional discrete-time random walk with a complete memory of its past. The influence of the memory is measured in terms of a memory parameter p between zero and one. In the past years, a considerable effort has been undertaken to understand the large-scale behavior of the ERW, depending on the choice of p . Here, we use known results on urns to explicitly solve the ERW in all memory regimes. The method works as well for ERWs in higher dimensions and is widely applicable to related models.

DOI: [10.1103/PhysRevE.94.052134](https://doi.org/10.1103/PhysRevE.94.052134)**I. INTRODUCTION**

Random walks and, more generally, diffusion processes are widely used in theoretical physics to describe phenomena of traveling motion and mass transport. Due to the fractal structure of nature and space and temporal long-range correlations in particle movements (see, e.g., Refs. [1–5]), so-called anomalous diffusions often appear, where the mean square displacement of a particle is no longer a linear function of time, but is rather given by a power law.

A simple model exhibiting anomalous diffusion is the so-called elephant random walk (ERW) introduced by Schütz and Trimper [6] in 2004, which is the topic of this paper. The ERW model is a one-dimensional discrete-time nearest-neighbor random walk on \mathbb{Z} , which remembers its full history and chooses its next step as follows: First, it selects randomly a step from the past, and then, with probability $p \in [0, 1]$, it repeats what it did at the remembered time, whereas with the complementary probability $1 - p$, it makes a step in the opposite direction. We refer to the next section for the precise definition. The memory parameter $p \in [0, 1]$ allows us to model the willingness of the walker to do the same as in the past. When $p = 1/2$, the memory has no effect on the movement: the model becomes Markovian.

The ERW model and some variations thereof have drawn a lot of attention in the last years; see, e.g., Refs. [6–16] to mention just a few. One of the key questions concerns the influence of the memory on the long-time behavior. Various results and predictions have been obtained, e.g., in Refs. [6–8]. In this paper, we explicitly determine the long-time behavior of the ERW model in all regimes $p \in [0, 1]$. We obtain central limit theorems for the full process of the ERW, with a scaling depending on the choice of p . In the regime $p \leq 3/4$, the limiting process turns out to be Gaussian (with explicit parameters). In the superdiffusive case $p > 3/4$, the limit is non-Gaussian as was already predicted in Refs. [7, 8]. We point

out that our limit theorems are stronger than finite-dimensional convergence of the ERW. In particular, they imply convergence of continuous functionals of the walker.

Our method uses a connection to Pólya-type urns that was already known before in the literature; see, e.g., the works of Harris [14, 15] and also the survey of Pemantle [17] on related random processes with reinforcement. Being robust and simple, the method is neither limited to one-dimensional models nor to the specific ERW model, but rather widely applicable to other random walks with memory. A bit more precisely, given what is known from the theory of urns, we will see that the asymptotic behavior of such models is essentially determined by the spectral decomposition of the (replacement) matrix of the corresponding urn.

Since the ERW is arguably the most natural and simplest model of a one-dimensional random walk with a complete memory, we concentrate in this paper on the basic ERW and leave it mostly to the reader to adapt the method to other walks with memory. However, we outline some possible extensions in Sec. V.

The rest of this paper is structured as follows: After having introduced the exact ERW model in the following section, we describe in Sec. III a particular discrete-time urn model containing balls of two colors, where step by step a new ball is added. We then show in Sec. IV how the known limit results on the composition of the urn can be transferred into statements about the position of the ERW when time goes to infinity. In Sec. V, we discuss various extensions and, in the last part, we summarize our findings.

II. THE MODEL

Let us now introduce the exact model, in the way it was first defined in Ref. [6]. The ERW is a one-dimensional random walk $(S_n, n \in \mathbb{N}_0)$ on the integers starting, say, at zero at time zero, $S_0 = 0$. At time $n \geq 1$, the position of the walk is given by

$$S_n = S_{n-1} + \sigma_n,$$

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where $\sigma_n, n \in \mathbb{N} = \{1, 2, \dots\}$ are random variables taking values in $\{\pm 1\}$, which are specified as follows: First, σ_1 takes the value 1 with some probability $q \in [0, 1]$ and the value -1 with probability $1 - q$. Accordingly, the first step of the ERW goes to the right (left) with probability q ($1 - q$). At any later time $n \geq 2$, we choose a number n' uniformly at random among the previous times $1, \dots, n - 1$ and set

$$\sigma_n = \begin{cases} +\sigma_{n'} & \text{with probability } p \\ -\sigma_{n'} & \text{with probability } 1 - p, \end{cases}$$

where $p \in [0, 1]$ is a memory parameter which is inherent to the model. Note that the case $p = 1/2$ corresponds to simple symmetric random walk: there is no memory effect. Moreover, we remark that $S_n = \sigma_1 + \dots + \sigma_n$. We implicitly agree that the various random choices made in this construction are independent from each other.

In Ref. [6], the question of how the memory of the history influences the position of the walker at large times was investigated. In particular, by writing $\langle \cdot \rangle$ for the expectation operator, it was shown that the mean displacement of the ERW satisfies for $n \gg 1$,

$$\langle S_n \rangle \sim \frac{(2q - 1)}{\Gamma(2p)} n^{2p-1}, \tag{1}$$

while for the second moment, it was proved that

$$\langle S_n^2 \rangle \sim \begin{cases} \frac{n}{3-4p} & \text{for } 0 \leq p < 3/4 \\ n \ln n & \text{for } p = 3/4 \\ \frac{n^{4p-2}}{(4p-3)\Gamma(4p-2)} & \text{for } 3/4 < p \leq 1. \end{cases} \tag{2}$$

The last display entails at $p = 3/4$ a transition from a diffusive ($0 \leq p < 3/4$) to a superdiffusive ($3/4 < p \leq 1$) regime, whereas at $p = 3/4$, the ERW behaves marginally superdiffusive. Using an approximation by a non-Markovian Fokker–Planck equation, the random walk propagator of the ERW model was reported in Ref. [6] to be Gaussian in all regimes (with a time-dependent diffusion constant), an observation which was later adapted in Ref. [7] for the superdiffusive regime $p > 3/4$, where a more precise analysis showed that the random walk propagator is in fact non-Gaussian. Here, the term *propagator* refers to the probability density of the usual continuum limit. See also Ref. [8] for a related work confirming that the Fokker–Planck approximations do not yield adequate results for the ERW model, at least not in the superdiffusive regime. The statistics in the regime $1/2 < p \leq 3/4$ were left open in Ref. [7].

The main purpose of this paper is to affirm the observation of Ref. [7] in the superdiffusive regime and clarify the behavior in the remaining regimes by explicitly calculating the large-scale behavior of the ERW model by using a connection to Pólya-type urns, which we explain next.

III. THE CONNECTION TO PÓLYA-TYPE URNS

Imagine a discrete-time urn with balls of two colors; say, black and red. The composition of the urn at time $n \in \mathbb{N}$ is given by a vector $X_n = (X_n^1, X_n^2)$, where the first component X_n^1 counts the number of black balls at time n , and the second component X_n^2 counts the number of red balls. We restrict ourselves to starting compositions $X_1 = \xi$ for some (possibly

random) vector $\xi = (\xi^1, \xi^2)$ taking values in $\{(1, 0), (0, 1)\}$ almost surely. The urn now evolves according to the following dynamics: At time $n = 2, 3, \dots$, we draw a ball uniformly at random, observe its color, put it back to the urn and add with probability p a ball of the same color, and with probability $1 - p$ a ball of the opposite color. Then we update X_n , so that X_n describes the composition of the urn after the $(n - 1)$ st drawing.

The connection to the ERW model is remarkably simple: If $(S_n, n \in \mathbb{N}_0)$ is the ERW started from $S_0 = 0$ such that $S_1 = \xi^1 - \xi^2$, then

$$(S_n, n \in \mathbb{N}) =_d (X_n^1 - X_n^2, n \in \mathbb{N}), \tag{3}$$

where $=_d$ refers to equality in law. In other words, the difference between the number of black and red balls in the above urn evolves like an ERW with first step equaling $\xi^1 - \xi^2$.

The urn described above fits into a broader setting of so-called generalized Friedman’s or Pólya urns; see Refs. [18–21] for first results (with deterministic replacement rules). Athreya and Karlin [22] proved an embedding of urn schemes into continuous-time multitype Markov branching processes, which includes the treatment of generalized Friedman’s urn processes with randomized replacement rules, as in our case. These techniques were further developed by Janson in Ref. [23], which serves as the main reference for this paper. Many results on urns can also be found in Mahmoud’s book [24], which is, however, more combinatorial in nature.

Key quantities that govern the long-time behavior of the urn process are the eigenvalues and eigenvectors of the so-called mean replacement matrix. In our case, it is given by

$$A = \begin{pmatrix} p & 1 - p \\ 1 - p & p \end{pmatrix}. \tag{4}$$

The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2p - 1$, and the corresponding right and left eigenvectors are $v_1 = \frac{1}{2}(1, 1)'$, $v_2 = \frac{1}{2}(1, -1)'$, $u_1 = (1, 1), u_2 = (1, -1)$, where we write v' for the transpose of v . Here, as in (2.2) and (2.3) of Ref. [23], we have chosen v_1, v_2 and u_1, u_2 such that $u_1 v_1' = u_2 v_2' = 1$ and the L^1 -norm of v_1, v_2 is equal to one.

It is well known (see, e.g., Refs. [22, 23, 25, 26]) that the asymptotics of the urn depends on the position of λ_2/λ_1 with respect to $1/2$ (in the situation of a more general urn, assuming that the largest eigenvalue λ^* is positive and simple, one has to check whether there is an eigenvalue different from λ^* with real part $> \lambda^*/2$). This already explains on a formal level why, for the ERW model, a phase transition occurs at $p = 3/4$.

IV. RESULTS AND PROOFS FOR THE STANDARD ELEPHANT RANDOM WALK MODEL

The paper of Janson [23] contains an exhaustive and very broad treatment of urn schemes and corresponding functional limit theorems. For our purpose, it is most convenient to adapt the general results from there and to translate them into the setting of the ERW model, via Eq. (3).

A. The diffusive case ($0 \leq p < 3/4$)

Our first convergence result deals with a distributional convergence of processes, which holds in the Skorokhod space

$D([0, \infty))$ of right-continuous functions with left-hand limits. We simply recall that distributional convergence in $D([0, \infty))$ to a process without discontinuities at fixed times is stronger than finite-dimensional distributional convergence, and point at Ref. [27] for more background.

Theorem 1. Let $0 \leq p < 3/4$. Then, for n tending to infinity, we have the distributional convergence in $D([0, \infty))$:

$$\left(\frac{S_{\lfloor tn \rfloor}}{\sqrt{n}}, t \geq 0 \right) \implies (W_t, t \geq 0),$$

where $W = (W_t, t \geq 0)$ is a continuous \mathbb{R} -valued Gaussian process specified by $W_0 = 0$, $\langle W_t \rangle = 0$ for all $t \geq 0$, and

$$\langle W_s W_t \rangle = \frac{s}{3 - 4p} \left(\frac{t}{s} \right)^{2p-1}, \quad 0 < s \leq t.$$

We observe that, when $p = 1/2$, W is a standard Brownian motion. Of course, this we already know from Donsker’s invariance principle, since in this case, the ERW behaves as a simple symmetric (Bernoulli) random walk on \mathbb{Z} , except possibly for the first step.

Proof. We apply Theorem 3.31(i) of Ref. [23], which shows that

$$(n^{-1/2}(X_{\lfloor tn \rfloor} - tn\lambda_1 v_1), t \geq 0)$$

converges in distribution towards a continuous \mathbb{R}^2 -valued Gaussian process $V = (V_t, t \geq 0)$ with $V_0 = 0$ and $\langle V_t \rangle = 0$ for all $t \geq 0$. In our case, we have $\lambda_1 = 1$, and the covariance structure of V is closer specified in Remark 5.7 of Ref. [23]. Display (5.6) in that work shows that

$$\langle V_s V_t \rangle = s \Sigma_I e^{\ln(t/s)A}, \quad 0 < s \leq t,$$

with Σ_I being a 2×2 matrix defined under (2.15) of Ref. [23]. An explicit calculation gives

$$\Sigma_I = \frac{1}{4(3 - 4p)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and the matrix exponential reads, in our case,

$$e^{\ln(t/s)A} = P \begin{pmatrix} \frac{t}{s} & 0 \\ 0 & (\frac{t}{s})^{2p-1} \end{pmatrix} P^{-1}, \quad \text{with } P = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Together, we obtain for $0 < s \leq t$,

$$\langle V_s V_t \rangle = \frac{s}{4(3 - 4p)} \left(\frac{t}{s} \right)^{2p-1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

By definition of S_m and the continuous mapping theorem, we then deduce that $(n^{-1/2} S_{\lfloor tn \rfloor}, t \geq 0)$ converges in law in $D([0, \infty))$ to a process $W = (W_t, t \geq 0)$ given by $W_t = V_t^1 - V_t^2$ almost surely, where for $i = 1, 2$, V^i denotes the i th component of V . This proves our claim. ■

Note that the covariance structure of the limit W does not fit the asserted effective diffusion coefficient in Ref. [6], cf. Display (27) there. But the asymptotic behavior of the ERW mean square displacement derived in Ref. [6] [see Display (2) above] is in agreement with the second moment of W .

Moreover, we note that the initial steps of the ERW do not influence its long-time behavior. Indeed, this can easily be derived from the fact that the above urn admits the same Gaussian limit when starting from more general configurations

$\xi = (\xi^1, \xi^2) \in \mathbb{N}_0^2$ with $(|\xi|^2) < \infty$ and $\xi \neq (0, 0)$. Specifying, for example, to the deterministic initial configuration $\xi = (k_1, k_2)$ for some $k_1, k_2 \in \mathbb{N}$, the increment process $(X_n^1 - X_n^2, n = 1, 2, \dots)$ can be seen as an ERW observed from time $k = k_1 + k_2$ on when conditioned to be at position $k_1 - k_2$ at time k . Applying Theorem 3.31(i) of Ref. [23] to the urn when starting from configuration $\xi = (k_1, k_2)$, we deduce that the first k steps do not influence the limiting behavior.

B. The critical case ($p = 3/4$)

In the borderline case $p = 3/4$, part (ii) of Theorem 3.31 of Ref. [23] applies.

Theorem 2. Let $p = 3/4$. Then, for n tending to infinity, we have the distributional convergence in $D([0, \infty))$

$$\left(\frac{S_{\lfloor n^t \rfloor}}{\sqrt{\ln n n^{t/2}}}, t \geq 0 \right) \implies (B_t, t \geq 0),$$

where $B = (B_t, t \geq 0)$ is a standard one-dimensional Brownian motion.

The function space $D([0, \infty))$ is defined as in the diffusive case discussed above.

Proof. According to Theorem 3.31(ii) of Ref. [23],

$$((\ln n)^{-1/2} n^{-t/2} (X_{\lfloor n^t \rfloor} - n^t \lambda_1 v_1), t \geq 0)$$

converges in law towards a continuous \mathbb{R}^2 -valued Gaussian process $V = (V_t, t \geq 0)$ with $V_0 = 0$ and mean $\langle V_t \rangle = 0$ for all $t \geq 0$. The covariance structure of V is given by expression (3.27) of Ref. [23], which simplifies in our case to

$$\langle V_s V_t \rangle = \frac{s}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad 0 < s \leq t.$$

As above, the claim now follows from the continuous mapping theorem. ■

Again, the asymptotics (2) for the second moment of the ERW obtained in Ref. [6] match with the limit. With the same arguments as in the diffusive case, one deduces moreover that the first steps of the walker have no influence on the long-time behavior.

C. The superdiffusive case ($3/4 < p \leq 1$)

In this regime, we can make use of Theorems 3.24 and 3.26 in Ref. [23].

Theorem 3. Set $\alpha = 2p - 1 \in (1/2, 1]$. Then, for n tending to infinity, we have the almost-sure convergence

$$\left(\frac{S_{\lfloor tn \rfloor}}{n^\alpha}, t \geq 0 \right) \longrightarrow (t^\alpha Y, t \geq 0),$$

where Y is some \mathbb{R} -valued random variable different from zero.

Below the proof of the theorem, we give some information on the limiting variable Y .

Proof. We note that, in the notation of Theorem 3.24 of Ref. [23], we have $\Lambda'_{\text{III}} = \{2p - 1\}$. We are therefore in the setting of the last part of the cited theorem and get that

$$(n^{-\alpha}(X_{\lfloor tn \rfloor} - tn\lambda_1 v_1), t \geq 0)$$

converges almost surely to $(t^\alpha \hat{W}, t \geq 0)$, where $\hat{W} = (\hat{W}^1, \hat{W}^2)$ is some nonzero random vector lying in

the eigenspace E_{λ_2} of A , i.e., $\hat{W} \in \{v \in \mathbb{R}^2 : v = \lambda(1, -1) \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}\}$. Since $Y = \hat{W}^1 - \hat{W}^2$ almost surely, the claim follows. ■

In contrast to the regimes discussed in the two previous sections, the distribution of Y does depend on the law of the initial step of the ERW. For example, in the degenerate case $p = 1$, Y has the same distribution as $S_1 = \xi^1 - \xi^2$ [in fact, $(S_{[tn]} = [tn]S_1)$ for all $t \geq 0$ with probability one]. In this regard, see also the remarks in Ref. [23] above Theorem 3.9.

By looking at the skewness and kurtosis of the position of the walker for large n , it was already observed in Ref. [7] that the law of the limit Y cannot be Gaussian, even not when starting from the symmetric initial condition $\mathbb{P}(\xi = (1, 0)) = \mathbb{P}(\xi = (0, 1)) = 1/2$. See also Ref. [8] for a similar observation.

Moreover, we point at Theorem 3.26 of Ref. [23], which can be used to (recursively) calculate the moments of Y . Let us for simplicity assume that $\xi = (1, 0)$. Then, using additionally Theorem 3.10 of Ref. [23], one finds for the first two moments

$$\langle Y \rangle = \frac{1}{\Gamma(2p)}, \quad \langle Y^2 \rangle = \frac{1}{(4p - 3)\Gamma(4p - 2)},$$

as we should have expected from Eqs. (1) and (2). For higher moments, see the remark below Theorem 3.1 of Ref. [23]. We however mention that, even in the case of an urn with deterministic replacement rules, there is in general no closed form for the moments of the limiting variable. See Ref. [26] and further references therein for more on this.

V. EXTENSIONS

It is the purpose of this section to exemplify that the approach via Pólya-type urns is robust and allows extensions and modifications of the ERW model in various directions. We leave it to the reader to perform the exact calculations and rather hint at the urn model one should consider.

A. Higher dimensions

Let us first explain how to obtain limit results for an ERW in higher dimensions. In dimension $d \geq 1$, one should simply consider an urn with $2d$ different colors. More specifically, in $d = 2$, one might want to study the urn $X_n = (X_n^1, X_n^2, X_n^3, X_n^4), n \in \mathbb{N}$, with mean replacement matrix

$$A_2 = \begin{pmatrix} p & (1-p)/3 & (1-p)/3 & (1-p)/3 \\ (1-p)/3 & p & (1-p)/3 & (1-p)/3 \\ (1-p)/3 & (1-p)/3 & p & (1-p)/3 \\ (1-p)/3 & (1-p)/3 & (1-p)/3 & p \end{pmatrix}.$$

The corresponding nearest-neighbor ERW on \mathbb{Z}^2 is given by

$$S_n = (X_n^1 - X_n^2)e'_1 + (X_n^3 - X_n^4)e'_2,$$

with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Starting from $X_1 = (1, 0, 0, 0)$, say, this means that the ERW first visits $(1, 0)$. Then, at any later time $n \geq 2$, the walker chooses a time n' uniformly at random among the previous times $1, \dots, n - 1$ and decides with probability p to perform a step in the same direction as at time n' , and with probability $(1 - p)/3$ each to perform a step in one of the three other coordinate directions.

The expression for S_n in the display above can again be analyzed with the results of Janson [23]. In particular, since the eigenvalues of A_2 are given by $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = (4p - 1)/3$, according to the remarks before Sec. IV, a phase transition from diffusive to superdiffusive behavior occurs at $p = 5/8$.

B. Elephant random walk with reinforced memory

In a different direction, one might want to model an ERW which has a reinforced memory; for example, in the sense that the more often a particular time from the past is remembered, the more likely it is to remember this time again. From the point of view of neural networks, this is certainly a reasonable and desirable assumption on the model. More concretely, one might want to study a random walk with memory where the remembered time n' at the n th step is not chosen uniformly at random among the previous times $1, \dots, n - 1$, but rather proportionally to a weight distribution, with a weight that takes into account the number of previous choices of n' . In this regard, it is interesting to point at the connection observed in Ref. [16] between the ERW model and so-called random (uniform) recursive trees, which can naturally be used to model the memory of the walker. The memory tree of an ERW with a reinforced memory would correspond to a so-called preferential attachment tree; see, e.g., Ref. [28]. In terms of a two-color urn, one might want to consider a “reinforced” mean replacement matrix, for example

$$B = \begin{pmatrix} a + p & 1 - p \\ 1 - p & a + p \end{pmatrix},$$

where $a \in \mathbb{N}_0$ is an additional parameter measuring the strength of the reinforcement. Here, when a ball is drawn, one puts it back to the urn with a additional balls of the same color. In addition, one tosses a coin with probability p for heads and probability $1 - p$ for tails. If a head shows up, one adds another ball of the same color, whereas in case of tails, one puts a ball of the opposite color into the urn. Note that the case $a = 0$ corresponds to the uniform ERW model discussed above.

Again, this urn model fits into the general framework of urns treated in Ref. [23]. The eigenvalues of B are given by $\lambda_1 = a + 1$ and $\lambda_2 = a + 2p - 1$. Hence, provided $a < 3$, a phase transition for the urn occurs at

$$p_a = (3 - a)/4.$$

As above, let us now assume that the starting configuration of the urn is given by a (possibly random) vector ξ taking values in $\{(1, 0), (0, 1)\}$. Regarding the corresponding random walk model $S = (S_n, n \in \mathbb{N}_0)$ (we use the same notation as for the original ERW), there is a little subtlety here: Most naturally, from time 1 on, S should not be defined as the difference $(X_n^1 - X_n^2, n \in \mathbb{N})$ of black and red balls as before, but rather as the difference of black and red balls which were put into the urn as a consequence of the coin tosses, plus the initial difference $\xi^1 - \xi^2$. In other words, one should not take into account the a additional balls of the same color which are put into the urn at every draw for determining the position of the walker. In particular, if $p = 1/2$, except for the first step, S behaves again like a simple symmetric random walk

(but note that $p_a < 1/2$ if $a \geq 2$!). If $p \neq 1/2$, the behavior of the walk S can be traced back to the composition of the urn $((X_n^1, X_n^2), n \in \mathbb{N})$. Namely, writing $\Delta_n = S_{n+1} - S_n$ for the increment of the walker at time n , one finds for its mean conditioned on X_n ,

$$\langle \Delta_n \rangle = (2p - 1) \left(\frac{2X_n^1}{(a+1)n - a} - 1 \right).$$

As to the urn, one can apply the results of Ref. [23] cited above to obtain functional limit theorems, more precisely Theorem 3.31(i) of Ref. [23] in the case $p < p_a$, Theorem 3.31(ii) of Ref. [23] in the case $p = p_a$, and Theorem 3.24 of Ref. [23] in the case $p > p_a$. The usual diffusion approximation now yields corresponding results for the walker S when $p \neq 1/2$; namely, diffusive behavior if $p < p_a$, marginally superdiffusive behavior if $p = p_a$ (with the same rescaling as in Theorem 2), and superdiffusive behavior if $p > p_a$ (with the same rescaling as in Theorem 3).

C. Modified elephant random walk of Harbola, Kumar, and Lindenberg

Harbola, Kumar, and Lindenberg [10] proposed a modified ERW representing a minimal one-parameter model of a random walk with memory, which gives rise to all three possible types of behavior (superdiffusive, diffusive, and subdiffusive). Again, $p \in [0, 1]$ is a memory parameter which is inherent to the model.

In contrast to the original ERW, the random walker moves only to the right, but it may also stay still. More precisely, the modified ERW $(S_n, n \in \mathbb{N}_0)$ starts at $S_0 = 0$, and then, at time $n \geq 1$, the position of the walker is given by

$$S_n = S_{n-1} + \sigma_n,$$

with $\sigma_n, n \in \mathbb{N}$, being $\{0, 1\}$ -valued random variables with the following law: First, for concreteness, we assume that the first step goes deterministically to the right, $\mathbb{P}(\sigma_1 = 1) = 1$ (this is a slight simplification compared with the model considered in Ref. [10]). At any later time $n \geq 2$, we choose a number n' uniformly at random among the previous times $1, \dots, n-1$. If $\sigma_{n'} = 1$, i.e., the walker moved to the right at time n' , we set

$$\sigma_n = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p. \end{cases}$$

If $\sigma_{n'} = 0$, i.e., the walker stood still at time n' , we set $\sigma_n = 0$, so that the walker does again not move at time n .

In the notation of Janson [23], the mean replacement matrix of the corresponding two-color urn (black balls for moving to the right, red balls for standing still) is

$$C = \begin{pmatrix} p & 0 \\ 1-p & 1 \end{pmatrix},$$

where the first (second) column of C is the expected change when a black (red) ball is drawn. We stress that, often in the literature (e.g., in Ref. [1]), rather the transpose C' is considered as the mean replacement matrix.

In words, the dynamics of the urn process is described as follows: Starting from some nontrivial initial condition at time $n = 1$, we draw at time $n = 2, 3, \dots$ a ball uniformly at

random, observe its color and put it back to the urn. If we drew a black ball, we add with probability p another black ball and with the complementary probability $1 - p$ a red ball to the urn, whereas if the observed color was red, we add deterministically another red ball to the urn.

Note that, if we start the urn model with one single black ball, the position S_n of the modified ERW at time n is given by the number of black balls at time n .

The eigenvalues of the above matrix C are $\lambda_1 = 1$ and $\lambda_2 = p$. Here, the results of Ref. [23] are not applicable, since λ_1 does not belong to the dominating class: Indeed, when starting the urn process from a single red ball, the dynamics adds only red balls to the urn, and never a black ball. Such random triangular urn schemes were however treated by Aguech [29], generalizing the results of Janson [30] for triangular urns with deterministic replacement. In particular, Theorem 2(a) of Ref. [29] shows that the right rescaling for the number of black balls at time n is n^p (there is no recentering), and one has almost-sure convergence as n tends to infinity to a nontrivial (non-Gaussian) limit. This is in accordance with the results of Harbola, Kumar, and Lindenberg [10], proving that, in this random walk model, subdiffusive (if $p < 1/2$), diffusive (if $p = 1/2$), and superdiffusive (if $p > 1/2$) behavior does occur.

A slightly more complicated model of a random walker moving to the left, right, and staying put, which also exhibits all three types of behavior, was presented by the same authors earlier in Ref. [9]. There, one should consider an urn with balls of three different colors: one corresponding to a movement to the right, one corresponding to a movement to the left, and one for staying at the same place.

VI. CONCLUSION

In this paper we have explicitly determined the long-time behavior of the one-dimensional ERW model introduced in 2004 by Schütz and Trimper [6]. We used a simple connection to Pólya-type urns and relied on limit results for the latter that were already established before. The ERW belongs to the class of models describing anomalous diffusion and is one of the few non-Markovian models that turns out to be explicitly solvable. However, as we exemplified in this paper, the ERW model (or variants thereof) or, more generally, processes with reinforcement can sometimes be reformulated in terms of urn models, which have been studied for a long time in the mathematical literature and are still objects of active research. In particular, results on urns often lead to a deeper understanding of the corresponding random walk model.

Note added. Recently, a work of Coletti, Gava and Schütz [31] appeared on the arXiv, with related results on the ERW but using a different approach.

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