Efficiency and its bounds of minimally nonlinear irreversible heat engines at arbitrary power

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The efficiency for minimally nonlinear irreversible heat engines at any arbitrary power has been systematically evaluated, and general lower and upper efficiency bounds under the tight coupling condition for different operating regions have been proposed, which can be seen as the generalization of the bounds $[\eta_C/2 < \eta_{\max P} < \eta_C/(2-\eta_C)]$ on efficiency at maximum power ($\eta_{\max P}$), where η_C means the Carnot efficiency. We have also calculated the universal bounds of the maximum gain in efficiency in different operating regions to give further insight into the efficiency gain with the power away from the maximum power. In the region of higher loads (higher than the load which corresponds to the maximum power), a small power loss away from the maximum power induces a much larger gain in efficiency. As actual heat engines may not work at the maximum power condition, this paper may contribute to operating actual heat engines more efficiently.

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I. INTRODUCTION

Optimization of real-life thermodynamic cycles has drawn increasing attraction due to the depletion of fossil fuel and energy saving. Heat engines are the main facilities to offer the electricity and power for our everyday lives, which generally operate between two heat reservoirs at temperatures T_h and $T_c\,(T_h > T_c)$. The maximum efficiency for a traditional heat engine is constrained by the Carnot efficiency $\eta_C = 1 - T_c/T_h$ according to the second law of thermodynamics [1]. However in achieving the Carnot efficiency, the power output vanishes because of the infinite cycle duration, which is unrealistic for actual applications. Therefore, Carnot heat devices where the processes are quasistatic to achieve the Carnot efficiency must be speeded up to meet the actual demands. By considering the finite cycle duration, Curzon-Ahlborn proposed the upper bound of efficiency ($\eta_{\mathrm{CA}} = 1 - \sqrt{T_c/T_h}$) for heat engines working at the maximum power output condition [2]. It carved a milestone for thermodynamics, that is finite time thermodynamics, to which much effort has been devoted [3–5].

The maximum power output is often adopted as the main criterion for optimizing actual heat engines, and many models have been developed to obtain the general expression of the efficiency at the maximum power (EMP) $\eta_{\max P}$, such as the endoreversible model [2], the low-dissipation model [6], and the irreversible models based on the Onsager relation [7,8]. Based on the low-dissipation model under asymmetric dissipation limits, Esposito *et al.* [6] deduced the general lower and upper EMP bounds $[\eta_C/2 < \eta_{\max P} < \eta_C/(2-\eta_C)]$, which are also obtained through the minimally nonlinear irreversible heat engines under the tight coupling condition [8]. The minimally nonlinear irreversible model is a modification of the linear irreversible model [7] by considering the power loss due to the fraction losses. Izumida and Okuda [8] and Izumida *et al.* [9] demonstrated that the model could describe the low-dissipation models for both heat engines and refrigerators. The further relation between the low-dissipation models and the minimally nonlinear irreversible models under the symmetric dissipation condition has been analyzed in detail [10]. Later, based on some improvements of the above models [11–15], much literature has been focused on the general expression of the EMP and its bounds, and useful results are provided.

Recently, increasing attention has been drawn to the optimization of heat engines which do not operate at the maximum power condition, instead, under the compromise between the energy benefit and the power loss. To evaluate this compromise, the Ω criterion $[\Omega = (2\eta - \eta_C)\dot{Q}_h]$ and ecological criterion ($E = P - T_0\dot{\sigma}$) have been proposed to optimize actual heat engines [16,17], where $\dot{Q}_h$ is the heat absorbed, P is the power output, T_0 is the environmental temperature, and $\dot{\sigma}$ is the entropy production rate. Under these criteria, the general upper and lower bounds on efficiency have been specifically investigated through different models [10,18–21]. In addition, as mentioned in previous literature [22], the actual thermal plants and heat engines should run in a regime with slightly smaller power than the maximum power, yet at a larger efficiency than the EMP. Therefore, it is of great importance to study the efficiency of heat engines at arbitrary power output. The first steps in this direction were performed in Refs. [23–27]. In the present paper, we first introduce the model of minimally nonlinear heat engines in Sec. II and then systemically discuss the efficiency and relative gain in efficiency at any arbitrary power output for minimally nonlinear heat engines in Secs. III and IV, respectively. The general efficiency bounds and the bounds for the relative gain in efficiency at any arbitrary power output have been proposed under the tight coupling condition, and some discussions under the nontight situations have also been presented. Finally, in Sec. V, some important conclusions are drawn.

II. MINIMALLY NONLINEAR IRREVERSIBLE HEAT ENGINES

For heat engines, a certain heat flux $\dot{Q}_h$ is absorbed from the hot reservoir (T_h), and then some of which ($\dot{Q}_c$) is evacuated to the cold reservoir (T_c) at the end of a cycle. Meanwhile, the power ($P = \dot{Q}_h - \dot{Q}_c$) is produced. After a cycle, the working fluid in the heat engine returns to its initial state; therefore, its entropy change per cycle is zero. The total entropy production

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rate $\dot{\sigma}$ of the heat engine can be written as

$$\dot{\sigma} = -\frac{\dot{Q}_h}{T_h} + \frac{\dot{Q}_c}{T_c} = -\frac{P}{T_c} + \dot{Q}_h\left(\frac{1}{T_c} - \frac{1}{T_h}\right), \quad (1)$$

where the dot denotes the quantity per unit time for simultaneous heat engines (working in a time-independent steady state) or the quantity divided by the cycle time duration for sequential heat engines (working in a time-periodic steady state). We assume the system performs work $W = Fx$ against an external force F (e.g., a mechanical, chemical, or electrical force) with thermodynamically conjugate variable x (displacement). The corresponding thermodynamic force is $X_1 = F/T_c$. The thermodynamic flux is $J_1 = \dot{x}$; the dot refers to the time derivative. Therefore $\dot{x}$ can be interpreted as the velocity to move the external load. This treatment is widely adopted in the previous literature [7,8]. The power can be rewritten as $P = -F\dot{x} = -J_1 X_1 T_c$; the other thermodynamic force and its conjugate flux can be defined as $X_2 = 1/T_c - 1/T_h$ and $J_2 = \dot{Q}_h$, respectively. In linear irreversible heat engines, the relations of the thermodynamic fluxes and forces are governed by the linear relations [7]. By adding a nonlinear term to the linear relations to consider the power loss due to the fraction losses, the minimally nonlinear irreversible model was proposed, and the extended Onsager relations that describe the heat engines read [8]

$$J_1 = L_1 X_1 + L_{12} X_2, \quad (2)$$

$$J_2 = L_{21} X_1 + L_{22} X_2 - \gamma_h J_1^2. \quad (3)$$

Comparing to the linear irreversible model, the addition of a new quadratic nonlinear term can be seen as a natural extension of the linear irreversible heat engine. Here we assume the dissipation is still weak and does not make the processes far from the equilibrium that is to say the model is still in the near equilibrium state. Hence the characteristics of the Onsager coefficients in Refs. [7,28] still hold in Eqs. (2) and (3). Hence the Onsager coefficients (L_{ij}) with the reciprocity $L_{12} = L_{21}$ satisfy the relations $L_{11} \geqslant 0$, $L_{22} \geqslant 0$, and $L_{11}L_{22} - L_{12}L_{21} \geqslant 0$. The nonlinear term $-\gamma_h J_1^2$ indicates part of the power loss due to the fraction losses is transformed into the heat and is then transferred to the hot reservoir, where γ_h denotes its strength ($\gamma_h > 0$). The heat evacuated to the cold reservoir can be calculated as

$$\dot{Q}_c = \dot{Q}_h - P = \dot{Q}_h + J_1 X_1 T_c \equiv J_3. \quad (4)$$

According to Eq. (2), Eqs. (3) and (4) can be rewritten as

$$J_2 = \frac{L_{21}}{L_{11}} J_1 + L_{22}(1 - q^2) X_2 + \gamma_c J_1^2, \quad (5)$$

$$J_3 = \frac{L_{21}}{L_{11}} \frac{T_c}{T_h} J_1 + L_{22}(1 - q^2) X_2 + \gamma_c J_1^2, \quad (6)$$

where $q = L_{12}/\sqrt{L_{11}L_{22}}$ is the dimensionless coupling strength ($|q| \leqslant 1$ [7]). Equation (5) shows that the heat absorbed $\dot{Q}_h$ depends on J_1. As $P = -J_1 X_1 T_c$, J_1 is linked to the power. The fact that $\dot{Q}_h$ depends on J_1 means the heat absorbed by the heat engine has a relation with the power output, which is common in our everyday life, such as the electricity generation in the thermal power plant. The nonlinear term $\gamma_c J_1^2$, characterizes part of the power loss due to the fraction losses, is transformed into the heat and is then transferred to the cold reservoirs, where $\gamma_c = T_c/L_{11} - \gamma_h$ and denotes its strength ($\gamma_c > 0$). The existence of the two nonlinear terms results in the decrease in the heat absorbed by the working fluid and the increase in the heat released to the cold reservoir. We call the heat engine described by Eqs. (5) and (6) the minimally nonlinear irreversible model. Minimally implies that we take into account only the fraction losses (quadratic nonlinear terms). Due to $J_1^2 = x_1^2$, the irreversibility takes in the form of J_1^2 which has the dimension of energy. Quadratic nonlinear terms represent the reduction of the mechanical energy that can be seen as the friction losses, which are transformed to the heat, and are then transferred to the heat reservoirs (hot and cold reservoirs). The nonlinearity can be measured by the friction torque of the rotating axis. Equations (5) and (6) indicate the heat absorbed from the hot reservoir and released to the cold reservoir, respectively. The term $L_{22}(1 - q^2)X_2$ appears simultaneously in the above equations. And the power [Eq. (7)] is not impacted by the term $L_{22}(1 - q^2)X_2$. Therefore $L_{22}(1 - q^2)X_2$ means just the direct heat transfer from the hot heat reservoir to the cold one [8], indicating a heat leakage between the hot and the cold reservoirs. The coupling strength can be interpreted as a quantity evaluating the heat leak from the hot reservoir to the cold reservoir. When the couple strength (q) equals 1, the heat leak does not exist. And $q \neq 1$ means the heat leak is considered.

The power can be rewritten as

$$P = \frac{L_{21}}{L_{11}} \eta_C J_1 - \frac{T_c}{L_{11}} J_1^2, \quad (7)$$

and the efficiency is given by

$$\eta = \frac{P}{\dot{Q}_h} = \frac{\frac{L_{21}}{L_{11}} \eta_C J_1 - \frac{T_c}{L_{11}} J_1^2}{\frac{L_{21}}{L_{11}} J_1 + L_{22}(1 - q^2) X_2 - \gamma_h J_1^2}. \quad (8)$$

For deriving the efficiency at the maximum power condition, the Onsager coefficients and the temperatures are fixed. As shown in Eq. (7), the power only depends on the flux J_1. So we can maximize the power with respect to J_1. By taking the derivative of P with respect to J_1, we let $\partial P/\partial J_1 = 0$ at $J_1 = J_{1,\max P}$. Furthermore, the second derivative of P with respect to J_1 at $J_1 = J_{1,\max P}$ satisfies $\partial^2 P/\partial {J_1}^2 < 0$, which means P achieves its maximum value at $J_{1,\max P}$. Then, we have

$$J_{1,\max P} = \frac{L_{21}\eta_C}{2T_C}. \quad (9)$$

Substituting Eq. (9) into Eq. (2) yields

$$X_{1,\max P} = -\frac{L_{21}\eta_C}{2L_{11}T_C}. \quad (10)$$

To step further, we derive the maximum power output and the corresponding efficiency,

$$P_{\max} = \frac{q^2 L_{22} \eta_C^2}{4T_C}, \quad (11)$$

$$\eta_{\max P} = \frac{\eta_C}{2} \frac{q^2}{2 - q^2\left[1 + \frac{\eta_C}{2(1+\gamma_c/\gamma_h)}\right]}. \quad (12)$$

To study the performance of heat engines operating beyond the maximum power regime at an arbitrary P, the relative deviation from the regime of maximum power δP and the relative gain in efficiency $\delta\eta$ can be defined as [25,29]

$$\delta P = \frac{P - P_{\max}}{P_{\max}}, \quad \delta\eta = \frac{\eta - \eta_{\max\ P}}{\eta_{\max\ P}}, \tag{13}$$

where $-1 \leqslant \delta P \leqslant 0$, which means any arbitrary power could not surpass the maximum power.

After some algebra, the ratio of any arbitrary power and the maximum power can be written as

$$\frac{P}{P_{\max}} = \frac{-J_1 X_1 T_C}{-J_{1,max\ P} X_{1,\max\ P} T_C} = \left(2 - \frac{X_1}{X_{1,\max\ P}}\right)\frac{X_1}{X_{1,\max\ P}}. \tag{14}$$

Similarly, the ratio of efficiency at any arbitrary power and that corresponding to the maximum power can be written as

$$\frac{\eta}{\eta_{\max\ P}} = \frac{P}{P_{\max}}\frac{J_{2,\max\ P}}{J_2} = \left(2 - \frac{X_1}{X_{1,\max\ P}}\right)\frac{X_1}{X_{1,\max\ P}} \times \frac{1 - q^2\left(\frac{1}{2} + A\right)}{1 - q^2\left[\frac{1}{2}\frac{X_1}{X_{1,\max\ P}} + \left(2 - \frac{X_1}{X_{1,\max\ P}}\right)^2 A\right]}, \tag{15}$$

where $J_{2,\max\ P}$ represents the heat absorbed under the maximum power condition and $A \equiv \frac{\eta_C}{4(1+\gamma_c/\gamma_h)}$.

Equation (14) is independent of temperatures and γ_c/γ_h. The temperatures and γ_c/γ_h in Eq. (15) determine the form of A. As $\eta_C < 1$ and $0 < \gamma_c/\gamma_h < \infty$, $0 < A < \frac{1}{4}$. Therefore in the range of $0 < X_1/X_{1,\max\ P} < 2$, $\eta/\eta_{\max\ P} > 0$. Further analysis reveals that $\eta/\eta_{\max\ P}$ is a concave function of $X_1/X_{1,\max\ P}$ when $q^2 < 1$. Meanwhile $\eta/\eta_{\max\ P} = 0$ if $X_1/X_{1,\max\ P} = 2$. When $q^2 = 1$, Eq. (15) reads

$$\frac{\eta}{\eta_{\max\ P}} = \frac{X_1}{X_{1,\max\ P}} \frac{\frac{1}{2} - A}{\frac{1}{2} - \left(2 - \frac{X_1}{X_{1,\max\ P}}\right)A}. \tag{16}$$

Equation (16) is a monotonous increasing function with $X_1/X_{1,\max\ P}$. We can see that $\eta/\eta_{\max\ P} \neq 0$ if $X_1/X_{1,\max\ P} \neq 0$. Furthermore, when $X_1/X_{1,\max\ P} = 1$, the expression $\eta/\eta_{\max\ P} = 1$ is recovered whether $q^2 = 1$ or $q^2 \neq 1$. What is more, the values of A do not impact the shape of the curve of $\eta/\eta_{\max\ P}$ with $X_1/X_{1,\max\ P}$. Based on Eqs. (14) and (15), the performance characteristics of the heat engines for one special case ($\eta_C = 0.2$ and $\gamma_c/\gamma_h = 1$) are depicted in Fig. 1. According to the aforementioned analysis, the selected case can represent the general trend. As shown in Fig. 1 under the tight coupling condition ($q^2 = 1$) when the eternal force is lower than $X_{1,\max\ P}$ ($X_1 < X_{1,\max\ P}$), decreasing the power from its maximal value reduces the engine efficiency. When the external force is larger than $X_{1,\max\ P}$ ($X_1 > X_{1,\max\ P}$), the efficiency increases as the power decreases. Compared with the results under the tight coupling condition, the presence of nontight coupling strength ($|q| < 1$) decreases the efficiency when the external force is larger than $X_{1,\max\ P}$, however, increases the efficiency at lower external forces ($X_1 < X_{1,\max\ P}$). Furthermore, it also brings an optimal external force leading to the maximum efficiency, which is located in the region of $X_1 > X_{1,\max\ P}$, as the second terms in Eqs. (5) and (6) indicate the heat leakage from the hot reservoir to the cold one, and it is often the case in the irreversible models with heat leakage.

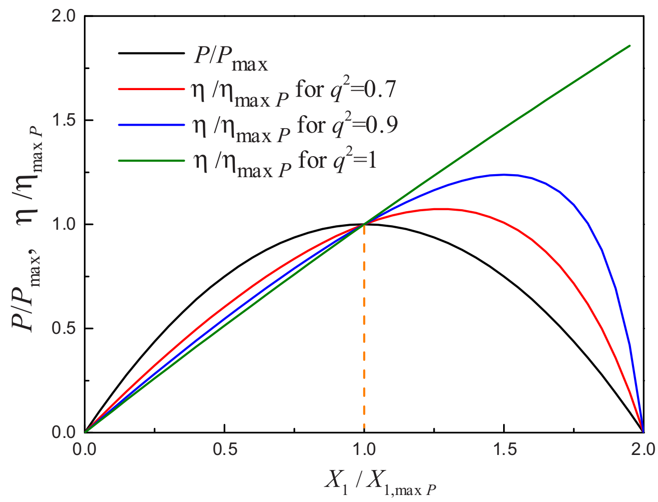

FIG. 1. The relation of relative power and efficiency with the relative thermodynamic force with different coupling strengths where the Carnot efficiency is 0.2 and the dissipation ratio (γ_c/γ_h) equals 1.

Furthermore, combining Eq. (13) and (14), we have

$$\frac{X_1}{X_{1,\max\ P}} = 1 \pm \sqrt{-\delta P}, \tag{17}$$

where the plus sign in the above equation corresponds to the favorable case when the external force is increased and the enhancement of efficiency occurs. The minus sign describes the opposite branch. Furthermore, Eq. (15) can be rewritten as

$$\frac{\eta}{\eta_{\max\ P}} = (1+\delta P)\frac{1 - q^2\left[\frac{1}{2} + \frac{\eta_C}{4(1+\gamma_c/\gamma_h)}\right]}{1 - q^2\left[\frac{1\pm\sqrt{-\delta P}}{2} + \frac{(1\mp\sqrt{-\delta P})^2}{4(1+\gamma_c/\gamma_h)}\eta_C\right]}. \tag{18}$$

According to Eqs. (12) and (18), we have arrived at the efficiency at any arbitrary power as the function of the relative power loss (δP),

$$\eta(P) = \frac{\eta_C}{4}(1+\delta P)\frac{q^2}{1 - q^2\left[\frac{1\pm\sqrt{-\delta P}}{2} + \frac{(1\mp\sqrt{-\delta P})^2}{4(1+\gamma_c/\gamma_h)}\eta_C\right]}. \tag{19}$$

III. EFFICIENCY BOUNDS UNDER THE TIGHT COUPLING CONDITION

Under the tight coupling condition ($q^2 = 1$), Eq. (19) can be written as

$$\eta(P) = \frac{\eta_C}{2}\frac{1 \pm \sqrt{-\delta P}}{1 - \frac{1\mp\sqrt{-\delta P}}{2(1+\gamma_c/\gamma_h)\eta_C}}, \tag{20}$$

where the plus sign corresponds to the region of enhanced efficiency $X_1 > X_{1,\max\ P}$; the minus sign corresponds to the region of lowered efficiency $X_1 < X_{1,\max\ P}$.

According to Eq. (20), under the asymmetric dissipation limits $\gamma_c/\gamma_h \to \infty$ and $\gamma_c/\gamma_h \to 0$, we obtain the lower and upper bounds on the efficiency at any finite

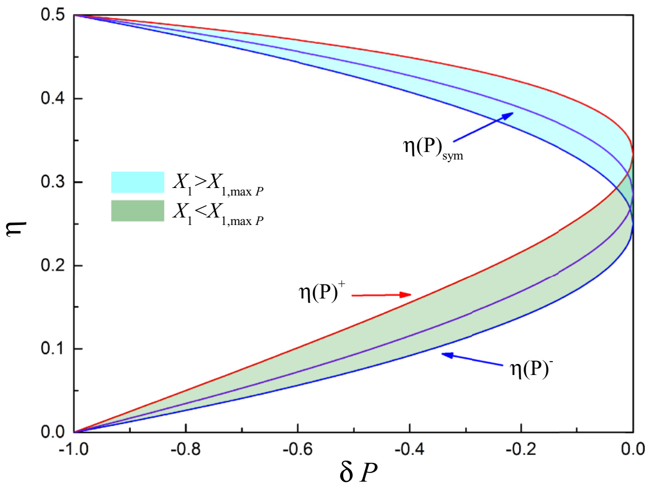

FIG. 2. The efficiency bounds at any arbitrary power. The efficiency under the symmetric dissipation condition also is plotted based on Eq. (25) where the Carnot efficiency is 0.5.

power,

$$\eta(P)^- = \frac{\eta_C}{2}(1 \pm \sqrt{-\delta P}), \tag{21}$$

$$\eta(P)^+ = \eta_C \frac{1 \pm \sqrt{-\delta P}}{2 - (1 \mp \sqrt{-\delta P})\eta_C}. \tag{22}$$

The expressions in Eq. (21) have also been obtained under the tight coupling condition for linear irreversible heat engines [29]. In Ref. [29], they are the upper and lower bounds, respectively, of the efficiency, which, however, are just the lower bounds under different regions of external forces in the present model.

To be more specific, in the region of $X_1 > X_{1,\max P}$, the efficiency at any arbitrary power is located as

$$\frac{\eta_C}{2}(1+\sqrt{-\delta P}) < \eta(P)_{X_1>X_{1,\max P}} < \eta_C \frac{1+\sqrt{-\delta P}}{2-(1-\sqrt{-\delta P})\eta_C}. \tag{23}$$

The expressions in Eq. (23) are also deduced through the low-dissipation heat engines under the same region [22], which coincides with the fact that the low-dissipation model can be described by the minimally nonlinear model with tight coupling strength [8,10].

Similarly, in the region of $X_1 < X_{1,\max P}$, we have

$$\frac{\eta_C}{2}(1-\sqrt{-\delta P}) < \eta(P)_{X_1<X_{1,\max P}} < \eta_C \frac{1-\sqrt{-\delta P}}{2-(1+\sqrt{-\delta P})\eta_C}. \tag{24}$$

Furthermore, under the symmetric dissipation condition $\gamma_c/\gamma_h = 1$, the efficiency is given by

$$\eta(P_{\text{sym}}) = \eta_C \frac{1 \pm \sqrt{-\delta P}}{2 - (1 \mp \sqrt{-\delta P})\eta_C/2}. \tag{25}$$

The above efficiency bounds at any arbitrary power are plotted in Fig. 2. In the region of $X_1 < X_{1,\max P}$, as the power increases (δP increases from -1 to 0), the upper and lower bounds of the efficiency also increase, whose difference also increases. In the region of $X_1 > X_{1,\max P}$, the power decreases away from the maximum power, but the efficiency increases, and so do the upper and lower bounds of the efficiency, however, whose difference decreases. When the power vanishes, the efficiency turns out to be the Carnot efficiency.

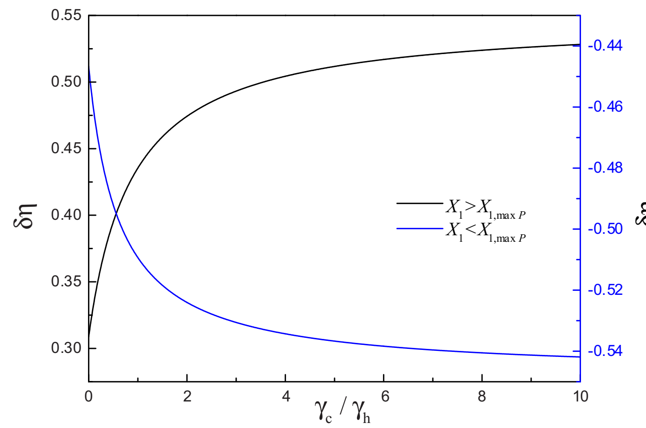

FIG. 3. The relative gain in efficiency as a function of the dissipation ratio where the Carnot efficiency is 0.5 and $\delta P = -0.3$.

In addition, for $\delta P \to 0$, Eqs. (23) and (24) become the general efficiency bounds under the maximum power through the low-dissipation and minimally nonlinear irreversible heat engines under the asymmetric dissipation limits [6,8],

$$\frac{\eta_C}{2} \leqslant \eta_{\max P} \leqslant \frac{\eta_C}{2 - \eta_C}. \tag{26}$$

According to Eq. (13), the relative gain in efficiency ($\delta\eta$) is

$$\delta\eta = \frac{(\pm\sqrt{-\delta P})\left(1 - \frac{\pm\sqrt{-\delta P}}{(1+\gamma_c/\gamma_h)}\right)}{1 - \frac{1\mp\sqrt{-\delta P}}{2(1+\gamma_c/\gamma_h)}\eta_C}. \tag{27}$$

Figure 3 displays the relative gain in efficiency as a function of the dissipation ratio (γ_c/γ_h). According to Eq. (27), as $0 < \eta_C/(1 + \gamma_c/\gamma_h) < 1$, the sign of $\delta\eta$ depends on that of $\pm\sqrt{-\delta P}$. A further derivative of $\delta\eta$ with respect to γ_c/γ_h shows that $\delta\eta$ is a monotonic function with γ_c/γ_h at small values of γ_c/γ_h then reaches a plateau at large γ_c/γ_h. When $\gamma_c/\gamma_h \to \infty$, $\delta\eta \to \pm\sqrt{-\delta P}$. Therefore, the cases selected here can represent the trends. From Fig. 3, we can see that, in the region with higher external loaders ($X_1 > X_{1,\max P}$), the relative gain in efficiency achieves its minimum and maximum values at asymmetric dissipation limits $\gamma_c/\gamma_h \to 0$ and $\gamma_c/\gamma_h \to \infty$, respectively. However in the region with lower external loaders ($X_1 < X_{1,\max P}$), the relative gain in efficiency achieves its minimum and maximum values at asymmetric dissipation limits $\gamma_c/\gamma_h \to \infty$ and $\gamma_c/\gamma_h \to 0$, respectively. Therefore under the asymmetric dissipation limits, Eq. (27) is reduced to

$$\delta\eta = \pm\sqrt{-\delta P}, \quad \text{at } \gamma_c/\gamma_h \to \infty, \tag{28}$$

and

$$\delta\eta = \frac{2(1-\eta_C)(\pm\sqrt{-\delta P})}{2 - (1 \mp \sqrt{-\delta P})\eta_C}, \quad \text{at } \gamma_c/\gamma_h \to 0. \tag{29}$$

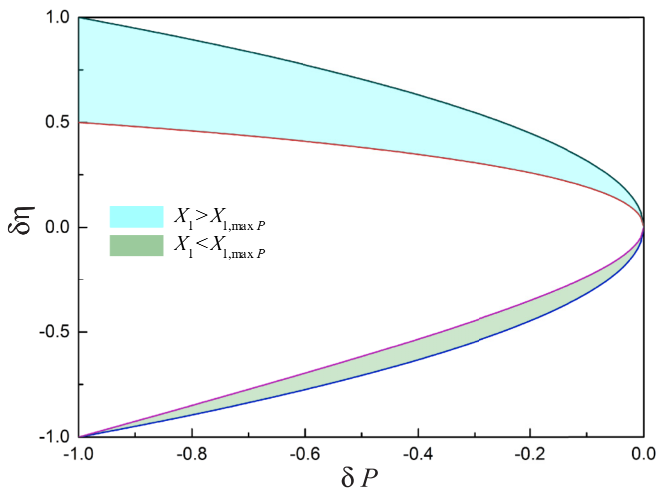

FIG. 4. The bounds [Eqs. (30) and (31)] of the relative gain in efficiency at any arbitrary power where the Carnot efficiency is 0.5.

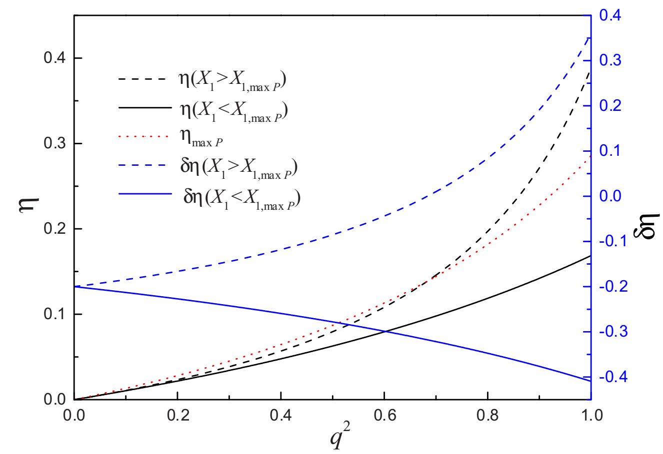

FIG. 5. The efficiency and relative gain in efficiency under different coupling strengths where the Carnot efficiency is 0.5, $\delta P = -0.2$, and $\gamma_c/\gamma_h = 1$. In addition, the efficiency corresponding to the maximum power is also plotted in this figure.

To step further, we can arrive at the bounds of the relative efficiency gain at any arbitrary power output,

$$\frac{2(1-\eta_C)\sqrt{-\delta P}}{2-(1-\sqrt{-\delta P})\eta_C} < \delta\eta < \sqrt{-\delta P}, \quad \text{if } X_1 > X_{1,\max P}, \tag{30}$$

$$-\sqrt{-\delta P} < \delta\eta < \frac{2(1-\eta_C)(-\sqrt{-\delta P})}{2-(1+\sqrt{-\delta P})\eta_C}, \quad \text{if } X_1 < X_{1,\max P}. \tag{31}$$

The expressions in Eq. (30) also are deduced through the low-dissipation heat engines [22]. Based on Eqs. (30) and (31), the bounds for relative gain in efficiency at any arbitrary power are plotted in Fig. 4. In the region of $X_1 < X_{1,\max P}$, as the power increases (δP increases from -1 to 0), the relative gain in efficiency increases, so the upper and lower bounds also increase, meanwhile, whose difference first increases then decreases. In the region of $X_1 > X_{1,\max P}$, the power decreases away from the maximum power, and the relative gain in efficiency increases, however, the difference in the upper and lower bounds increases. The bounds of the relative gain in efficiency coincide at the maximum power condition, i.e., $\delta P \to 0$ where the derivative $\delta\eta$ with δP does not exist. It means that the gain in efficiency when working near the maximum power region is much larger than the power loss and that it is desirable to operate the heat engine at a slightly lower power than at the maximum one in the region of $X_1 > X_{1,\max P}$ where the engine attains significantly larger efficiency enhancement.

IV. EFFICIENCY BOUNDS UNDER THE NONTIGHT COUPLING CONDITIONS

According to Eq. (19), the efficiency at any arbitrary power increases with increasing q^2. When $|q| \neq 1$, the existence of the second terms in Eqs. (5) and (6) illustrates the heat leakage from the hot reservoir to the cold one, which does not impact the power output, however, decreases the efficiency. As $|q| \leqslant 1$, the upper bound of the efficiency is established for $q^2 = 1$, that is, the tight coupling condition. Hence the upper bound $\eta(P)^+$ illustrated in Eq. (22) applies for all the coupling conditions. For a given power output ($\delta P = -0.2$), the efficiency and relative gain in efficiency under different coupling strengths are presented in Fig. 5. The general efficiency and the efficiency under the maximum power condition both increase with increasing coupling strengths. In the region with higher external loads of $X_1 > X_{1,\max P}$, the relative gain in efficiency increases, however, decreases in the region with lower external loads of $X_1 < X_{1,\max P}$. It indicates that compared to a relative gain in energy, it is more appealing to run the heat engine in the region with higher external loaders of $X_1 > X_{1,\max P}$ under larger coupling strengths where we can see a sharp gain in efficiency.

V. CONCLUSIONS

The universality of the efficiency under the maximum power output has been extensively investigated. However the actual heat engines may not work at the maximum power condition [10,18,20] but rather in the regime with slightly small power and considerably large efficiency, which induces an urgency to study the efficiency bounds at some given power output. In this paper, the efficiency bounds of minimally nonlinear irreversible heat engines at arbitrary power have systematically been researched. The lower and upper efficiency bounds under the tight coupling condition for different operating regions have been deduced. In order to further illustrate the efficiency gain with the power away from the maximum power, we calculated the bounds of the maximum gain in efficiency in different operating regions. Furthermore, under different coupling strengths, the universal upper bound of the efficiency at any arbitrary power has been proposed, and the relative gain in efficiency for the given power output has been analyzed. In the region of higher loads, a small power loss away from the maximum power induces a large gain in efficiency. Hence it is more favorable to operate the heat engine in that region, which is a slight deviation from the maximum power state to achieve a higher efficiency.

For quantum systems, a detailed study on thermoelectric quantum heat engines under arbitrary power is presented in Refs. [22,23], and the efficiency bound has been deduced, which is $\eta^+ = \eta_C[1 - 0.478\sqrt{(1-\eta_C)(1+\delta P)}]$. The upper bound on efficiency equals Carnot efficiency at zero power output but decays with increasing power output. Furthermore, as a counterpart, the performance of refrigerators at a given cooling power should also be analyzed to meet the actual demand, and there exist such models describing refrigerators as the general endoreversible model with nonisothermal processes, the low-dissipation model, the minimally nonlinear irreversible one, and even quantum refrigerators, etc. [9,30–36], which offer the possibility to specifically investigate the coefficient of performance of the refrigerator running at any arbitrary cooling power, thus to provide a guidance for efficiently running actual refrigerators.

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