PHYSICAL REVIEW E **94**, 052107 (2016)

Enantiodromic effective generators of a Markov jump process with Gallavotti-Cohen symmetry

S. A. A. Terohid,* P. Torkaman,† and F. H. Jafarpour‡

Physics Department, Bu-Ali Sina University, 65174-4161 Hamedan, Iran

(Received 12 July 2016; revised manuscript received 4 October 2016; published 4 November 2016)

This paper deals with the properties of the stochastic generators of the effective (driven) processes associated with atypical values of transition-dependent time-integrated currents with Gallavotti-Cohen symmetry in Markov jump processes. Exploiting the concept of biased ensemble of trajectories by introducing a biasing field s, we show that the stochastic generators of the effective processes associated with the biasing fields s and $E-s$ are enantiodromic with respect to each other where E is the conjugated field to the current. We illustrate our findings by considering an exactly solvable creation-annihilation process of classical particles with nearest-neighbor interactions defined on a one-dimensional lattice.

DOI: 10.1103/PhysRevE.94.052107

I. INTRODUCTION

The appearance of rare events in equilibrium and nonequilibrium many-body systems have become a focus of recent intense research [1–3]. An important question is how these fluctuations arise. In order to answer this question let us consider a classical interacting particles system in its steady state. We assume that this system can be modeled by a Markov jump process in continuous time in or out of equilibrium [4]. In a Markov jump process the system jumps spontaneously from one classical configuration in the configuration space, which will be assumed to be finite-dimensional throughout this paper, to another classical configuration with certain transition rate making a trajectory or path. We are generally interested in measuring a transition-dependent time-integrated observable such as activity or particle current during an extended period of time called the observation time. As we mentioned, we are more specifically interested in the fluctuations of this observable. Imagine that the stationary probability distribution function of this observable can be built by doing some experiments and measuring the observable along the trajectories of the process in its steady state. In principle, both typical and atypical values (or fluctuations) of the observable can be observed. Some of these trajectories are responsible for creating the typical values of the observable while some other trajectories are responsible for creating a specific fluctuation or an atypical value of the observable. Now, if we look at a restricted set of trajectories that is responsible for a specific fluctuation, we are basically dealing with a conditioning. Here the corresponding ensemble is called the path microcanonical ensemble. There are actually many trajectories leading to a specific fluctuation and one might ask if it is possible to describe this by a stochastic Markov process. It has been shown that in the long observation-time limit each specific fluctuation can be described by a specific stochastic Markov process that is nonconditioned and is called the effective or driven process [5].

The stochastic generator associated with the effective process for a specific fluctuation can, in principle, be obtained as follows. We start with a so-called tilted generator [6–13]. The tilted generator can be constructed by multiplying the transition rates of the original stochastic Markov process by an exponential factor that depends on both a biasing field s and the increment of the current during that transition. The parameter s is a real parameter that selects the fluctuation and plays a role similar to the inverse of temperature in the ordinary statistical mechanics. Given that our observable satisfies the large deviation principle with a convex rate function and that the tilted generator has a spectral gap, it has been shown that using a generalization of Doob's h-transform of the tilted generator one can obtain the stochastic generator of the effective process that explicitly depends on s. The path ensemble associated with the effective dynamics is called the path canonical ensemble. It has already been shown that the microcanonical and the canonical path ensembles are equivalent in the long-time limit [14]. It is worth mentioning that the effective process inherits many properties of the original process, including the symmetries; however, the interactions might be very complicated, hence the characterization of them are of great importance [15]. A one-dimensional classical Ising chain that exhibits ferromagnetic ordering in its biased ensemble of trajectories is an example that reveals this feature [5]. Similar examples are also studied in Refs. [16,17].

The question we are aiming to answer in this paper is, for a given transition-dependent time-integrated current, whether the stochastic generators of the effective processes associated with two different fluctuations are related. Let us start with a nonequilibrium Markov jump process in continuous time in the steady-state. We also consider a transition-dependent time-integrated current as an observable that satisfies the large-deviation principle with a convex rate function [18]. Now we assume that this nonequilibrium system is created as a result of applying an external field E conjugated to that current to an equilibrium process. Among different choices, one can achieve this by a special scaling of the reaction rates of the equilibrium system. This will clearly restrict us to study only a very specific class of processes. Moreover, this specific type of scaling results in a rate function that satisfies the Gallavotti-Cohen symmetry with respect to E [19]. We will show that in a given Markov jump process the stochastic generator of the effective process at s is *enantiodromic* with respect to the stochastic generator of the effective process at $E-s$. The concept of the enantiodromy relation has already

*a.terohid@iauh.ac.ir

†p.torkaman@basu.ac.ir

‡farhad@ipm.ac.ir

been introduced and used in the field of stochastic interacting particles [4,20]. In order to show how this property helps us deduce more information about the effective interactions associated with certain fluctuations in the system, we will provide an exactly solvable example. Our example consists of a one-dimensional system of classical particles with nearest-neighbor interactions, which includes creation and annihilation of particles. As we will see, using the enantiodromy relation and knowing the effective interactions at s, one finds exact information about the effective interactions at $E-s$. It turns out that the nature of interactions can be quite different from each other at these two points.

This paper is organized as follows: in Sec. II, after a brief review of the basic mathematical concepts, we will give the main results. Section III is devoted to an application of our findings by providing an exactly solvable coagulation-decoagulation model of classical particles. In Sec. IV we will give the concluding remarks.

II. BASICS CONCEPTS AND RESULTS

Let us start with a Markov process in continuous time defined by a set of configurations $\{c\}$ and transition rates $\omega_{c\to c'}$ between its configurations in a finite-dimensional configuration space. Considering the vectors $\{|c\rangle\}$ as an orthogonal basis of a complex vector space, the probability of being in configuration c at time t is given by $P(c,t)=\langle c|P(t)\rangle$. The time evolution of $|P(t)\rangle$ is governed by the following master equation [4]:

$$\frac{d}{dt}|P(t)\rangle = \hat{\mathcal{H}}|P(t)\rangle, \tag{1}$$

in which the stochastic generator or Hamiltonian $\hat{\mathcal{H}}$ is a square matrix with the following matrix elements:

$$\langle c'|\hat{\mathcal{H}}|c\rangle = (1-\delta_{c,c'})\omega_{c\to c'} - \delta_{c,c'}\sum_{c''\neq c}\omega_{c\to c''}. \tag{2}$$

We assume that in the long-time limit the process reaches its steady-state so that the left-hand side of Eq. (1) becomes zero. We aim to study the fluctuations of an observable over a long observation time in the steady state. As a dynamical observable we consider a transition-dependent time-integrated current. The current is time-extensive and a functional of the trajectory that the system follows in the configuration space during the observation time. This is a sum of the increments $\theta_{c\to c'}$'s every time a jump from c to c' occurs. For particle current in one dimension we have $\theta_{c\to c'}=\pm 1$.

Let us now assume that our original process has a nonequilibrium steady state that has been obtained by applying an external driving field to an equilibrium process. Among different possibilities, a very special choice for making this connection is by considering the following rule:

$$\omega_{c\to c'} = \omega^{\rm eq}_{c\to c'} e^{\frac{E}{2}\theta_{c\to c'}}, \tag{3}$$

in which E is the external driving field conjugated to the current. The transition rates of the equilibrium process and the equilibrium stationary distribution satisfy the detailed balance equations, i.e., $\omega^{\rm eq}_{c\to c'}P_{\rm eq}(c)=\omega^{\rm eq}_{c'\to c}P_{\rm eq}(c')$, which can be written as

$$\omega_{c\to c'} e^{-\frac{E}{2}\theta_{c\to c'}} P_{\rm eq}(c) = \omega_{c'\to c} e^{-\frac{E}{2}\theta_{c'\to c}} P_{\rm eq}(c'). \tag{4}$$

Defining the modified Hamiltonian $\hat{\mathcal{H}}(s)$ (or the tilted generator) with the matrix elements [8–11]

$$\langle c'|\hat{\mathcal{H}}(s)|c\rangle = (1-\delta_{c,c'})e^{-s\theta_{c\to c'}}\omega_{c\to c'} - \delta_{c,c'}\sum_{c''\neq c}\omega_{c\to c''}, \tag{5}$$

one can write Eq. (4) in a matrix form,

$$\hat{\mathcal{H}}(E-s) = P_{\rm eq}\hat{\mathcal{H}}^{\rm T}(s)P_{\rm eq}^{-1}, \tag{6}$$

in which $P_{\rm eq}$ is a diagonal matrix with the matrix elements $\langle c|P_{\rm eq}|c\rangle = P_{\rm eq}(c)$ [21]. Considering the following eigenvalue equations for the modified Hamiltonian,

$$\begin{aligned}\hat{\mathcal{H}}(s)|\Lambda(s)\rangle &= \Lambda(s)|\Lambda(s)\rangle,\\ \langle\tilde{\Lambda}(s)|\hat{\mathcal{H}}(s) &= \Lambda(s)\langle\tilde{\Lambda}(s)|,\end{aligned} \tag{7}$$

it is clear from Eq. (6) that all of the eigenvalues of the modified Hamiltonian have the following symmetry, $\Lambda(s)=\Lambda(E-s)$, which includes its largest eigenvalue, i.e.,

$$\Lambda^*(s) = \Lambda^*(E-s), \tag{8}$$

which is called the Gallavotti-Cohen symmetry [19,21,22]. Finally, the similarity transformation Eq. (6) indicates that

$$|\Lambda(s)\rangle = P_{\rm eq}|\tilde{\Lambda}(E-s)\rangle. \tag{9}$$

As we have already explained, in the long-observation-time limit each specific fluctuation in the system can be described by a stochastic Markov process called the effective process, which is equivalent to the conditioning of the original process on seeing a certain fluctuation. It has been shown that the stochastic generator of this effective stochastic process is given by

$$\hat{\mathcal{H}}_{\rm eff}(s) = U(s)\hat{\mathcal{H}}(s)U^{-1}(s) - \Lambda^*(s), \tag{10}$$

which is a generalization of Doob's h-transform and that $U(s)$ is a diagonal matrix with the matrix element $\langle c|U(s)|c\rangle = \langle\tilde{\Lambda}^*(s)|c\rangle$ [5,14]. The off-diagonal matrix elements of the operator $\hat{\mathcal{H}}_{\rm eff}(s)$ in Eq. (10) are given by

$$\langle c'|\hat{\mathcal{H}}_{\rm eff}(s)|c\rangle = \langle c'|\hat{\mathcal{H}}(s)|c\rangle\frac{\langle\tilde{\Lambda}^*(s)|c'\rangle}{\langle\tilde{\Lambda}^*(s)|c\rangle}, \tag{11}$$

or equivalently

$$\omega^{\rm eff}_{c\to c'}(s) = e^{-s\theta_{c\to c'}}\omega_{c\to c'}\frac{\langle\tilde{\Lambda}^*(s)|c'\rangle}{\langle\tilde{\Lambda}^*(s)|c\rangle}. \tag{12}$$

Defining the diagonal matrix

$$U_{\rm TTI}(s) = |\Lambda^*(s)\rangle\langle\tilde{\Lambda}^*(s)|, \tag{13}$$

the steady-state distribution of $\hat{\mathcal{H}}_{\rm eff}(s)$ is given by

$$P_{\rm TTI}(c,s) = \langle c|P_{\rm TTI}(s)\rangle = \langle c|U_{\rm TTI}(s)|c\rangle, \tag{14}$$

where the subscript TTI is an abbreviation for the time translational invariance regime [5].

Starting from Eq. (8) and using Eq. (10) and after some algebra one finds the following enantiodromy relation:

$$\hat{\mathcal{H}}_{\rm eff}(E-s) = U_{\rm TTI}(s)\hat{\mathcal{H}}^{\rm T}_{\rm eff}(s)U^{-1}_{\rm TTI}(s), \tag{15}$$

in which T means the transpose of the square matrix. The notion of enantiodromy has its origin in the time-reversal of the temporal order, as the transpose matrix describes the motion of the process *backward* in time. The two effective stochastic generators $\hat{\mathcal{H}}_{\rm eff}(s)$ and $\hat{\mathcal{H}}_{\rm eff}(E-s)$ share the same spectrum and also the same stationary distribution [4,21]; however, they describe two different processes. For instance while $\hat{\mathcal{H}}_{\rm eff}(s)$ might contain local and short-range interactions, $\hat{\mathcal{H}}_{\rm eff}(E-s)$ can contain long-range and complicated interactions. The generator $\hat{\mathcal{H}}_{\rm eff}(E-s)$ is called the adjoint generator with respect to $\hat{\mathcal{H}}_{\rm eff}(s)$, which in the absence of detailed balance defines a new process with the same allowed transitions as $\hat{\mathcal{H}}_{\rm eff}(s)$ [21].

Using Eq. (15) we find that the transition rates of these effective processes are related through

$$\omega^{\rm eff}_{c\to c'}(E-s)=\omega^{\rm eff}_{c'\to c}(s)\frac{P_{\rm TTI}(c',s)}{P_{\rm TTI}(c,s)}. \tag{16}$$

It can be seen that the transition rates of the effective process at $E-s$ depend on both the reversed transition rates and the stationary distribution of the effective process at s. It is worth mentioning that Eq. (16) is obtained by assuming that the original process has the Gallavotti-Cohen symmetry in the sense of Eq. (3). Finally, Eq. (16) gives the following constraints on the transition rates of the above-mentioned effective processes:

$$\omega^{\rm eff}_{c\to c'}(E-s)\omega^{\rm eff}_{c'\to c}(E-s)=\omega^{\rm eff}_{c\to c'}(s)\omega^{\rm eff}_{c'\to c}(s). \tag{17}$$

Similar result has already been obtained in Ref. [23] for fluids under continuous shear.

Let us consider $s=E/2$. Because of the Gallavotti-Cohen symmetry, this point is the minimum of the largest eigenvalue $\Lambda^*(s)$. At this point the slope of the eigenvalue is zero, hence the average current is zero. This means that the effective process is in equilibrium. From Eq. (15), one finds

$$\hat{\mathcal{H}}_{\rm eff}\left(\frac{E}{2}\right)=U_{\rm TTI}\left(\frac{E}{2}\right)\hat{\mathcal{H}}^{\rm T}_{\rm eff}\left(\frac{E}{2}\right)U^{-1}_{\rm TTI}\left(\frac{E}{2}\right),$$

which is a self-enantiodromy relation for the stochastic generator of the effective process at $s=E/2$ [4]. Since at $s=E/2$ the effective process is in equilibrium, the detailed-balance condition has to be recovered; i.e.,

$$P_{\rm TTI}\left(c,\frac{E}{2}\right)\omega^{\rm eff}_{c\to c'}\left(\frac{E}{2}\right)=P_{\rm TTI}\left(c',\frac{E}{2}\right)\omega^{\rm eff}_{c'\to c}\left(\frac{E}{2}\right).$$

Using Eqs. (12), (13), and (14) one can readily find

$$\frac{\omega_{c\to c'}e^{-\frac{E}{2}\theta_{c\to c'}}}{\omega_{c'\to c}e^{-\frac{E}{2}\theta_{c'\to c}}}=\frac{\langle\tilde{\Lambda}^*(\frac{E}{2})|c\rangle\langle c'|\Lambda^*(\frac{E}{2})\rangle}{\langle\tilde{\Lambda}^*(\frac{E}{2})|c'\rangle\langle c|\Lambda^*(\frac{E}{2})\rangle}.$$

Comparing this relation with Eq. (4), we also find

$$P_{\rm eq}(c)=\frac{\langle c|\Lambda^*(\frac{E}{2})\rangle}{\langle\tilde{\Lambda}^*(\frac{E}{2})|c\rangle}.$$

Note that we have [5]

$$P_{\rm TTI}\left(c,\frac{E}{2}\right)=\left\langle c\middle|\Lambda^*\left(\frac{E}{2}\right)\right\rangle\left\langle\tilde{\Lambda}^*\left(\frac{E}{2}\right)\middle|c\right\rangle.$$

Finally, at $s=0$, the relation Eq. (16) becomes

$$\omega^{\rm eff}_{c\to c'}(E)=\omega_{c'\to c}\frac{P^*(c')}{P^*(c)}, \tag{18}$$

in which $\omega_{c'\to c}$'s and $P^*(c)$ are the transition rates and the steady-state probability distribution of our original nonequilibrium process.

III. AN EXACTLY SOLVABLE EXAMPLE

In what follows we use Eq. (18) to investigate the effective dynamics at $s=E$ of an exactly solvable coagulation-decoagulation model of classical particles on a one-dimensional lattice of length L with reflecting boundaries. The system evolves in time according to the following reaction rules:

$$\begin{aligned}
A+A&\to\emptyset+A \quad \text{with rate } q^{-1},\\
A+\emptyset&\to\emptyset+A \quad \text{with rate } q^{-1},\\
A+A&\to A+\emptyset \quad \text{with rate } q,\\
\emptyset+A&\to A+\emptyset \quad \text{with rate } q,\\
\emptyset+A&\to A+A \quad \text{with rate } \Delta q,\\
A+\emptyset&\to A+A \quad \text{with rate } \Delta q^{-1},
\end{aligned} \tag{19}$$

where A and $\emptyset$ stand for the presence of a particle and a vacancy at a given lattice-site, respectively. There is no input or output of particles at the boundaries. We assume that $q\geqslant 1$ and $\Delta>0$. The steady-state properties of the model have been studied in detail using different techniques [24–27]. The full spectrum of the stochastic generator and also the density profile of the particle in the steady-state have been calculated exactly [24,25]. The stochastic generator of the system is reducible. The system does not evolve in time if it is completely empty. However, there is a nontrivial steady-state where there is at least one particle on the lattice. It is known that in the nontrivial case the system undergoes a phase transition from a high-density phase ($q^2<1+\Delta$) into a low-density phase ($q^2>1+\Delta$) at $q^2=1+\Delta$. Last but not least, it has been shown that the steady-state of this system can be written as a linear superposition of shocks that perform random walk on the lattice [27].

In Ref. [28] the authors have shown that one can define an entropic reaction-diffusion current in this system with the Gallavotti-Cohen symmetry respecting to the conjugate field $E=\ln q^2$. Assigning an occupation number c_i to the lattice site i, we assume that $c_i=0$ ($c_i=1$) corresponds to the presence of a vacancy (particle) at the lattice-site i. The vector space of each lattice site is two-dimensional with the basis vector $|c_i=1\rangle=\binom{0}{1}$ and $|c_i=0\rangle=\binom{1}{0}$. The configuration space of the system $(\mathbb{C}^2)^{\otimes L}$ is 2^L-dimensional. The modified Hamiltonian $\hat{\mathcal{H}}(s)$ for this current should be written as

$$\hat{\mathcal{H}}(s)=\sum_{k=1}^{L-1}\mathcal{I}^{\otimes(k-1)}\otimes\hat{h}(s)\otimes\mathcal{I}^{\otimes(L-k-1)}, \tag{20}$$

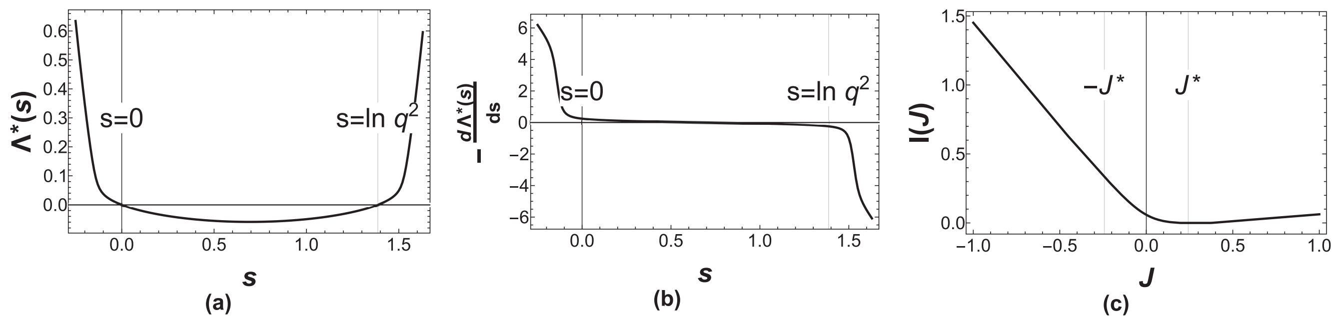

FIG. 1. In (a) we have plotted the largest eigenvalue of Eq. (20) and in (b) its first derivative for $L = 8, q = 2$, and $\Delta = 0.5$. For these values of the reaction rates the system is in the low-density phase. We have also plotted the large deviation function of the current $I(J)$ in (c).

in which $\mathcal{I}$ is a 2×2 identity matrix in the basis (0,1) and that

$$\hat{h}(s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q(\Delta+1) & q^{-1}e^{s} & q^{-1}e^{s} \\ 0 & qe^{-s} & -q^{-1}(\Delta+1) & qe^{-s} \\ 0 & q\Delta e^{-s} & q^{-1}\Delta e^{s} & -q-q^{-1} \end{pmatrix},$$

which is written in the basis (00,01,10,11).

In Fig. 1 we have plotted the numerically obtained largest eigenvalue of Eq. (20) and its derivative and also the large deviation function of the current for a system of size $L = 8$. The quantity $-d\Lambda^*(s)/ds$ at $s = 0$ gives the average current in the steady-state J^*, which can be calculated using a matrix product method quite straightforwardly [25,29]. At $s \neq 0$ the quantity $-d\Lambda^*(s)/ds$ gives the average of the current. At $s = s^* = \ln q^2$ the average current is $-J^*$.

Let us briefly review the basics of the matrix product method [29]. According to this method the steady-state probability distribution of a given configuration $c = \{c_1, \ldots, c_L\}$ is given by

$$P^*(c) = \langle c|P^*\rangle \propto \langle W|\prod_{i=1}^{L}(c_i D + (1-c_i)E)|V\rangle, \qquad (21)$$

in which E and D are noncommuting square matrices while $\langle W|$ and $|V\rangle$ are vectors. These matrices and vectors satisfy a quadratic algebra, which has a four-dimensional matrix representation [25]. Using Eq. (21) one can, in principle, calculate the average of any observable in the steady-state including the reaction-diffusion current explained before. This has actually been done in Ref. [28], hence the average current at s^* is exactly known.

In a separate work [27], it has been found that the process defined by Eq. (19) has the following property: it has an invariant state space under the evolution generated by $\hat{\mathcal{H}}(0)$ in Eq. (20). This state space consists of product shock measures with two shock fronts at the lattice sites i and j, where $0 \leqslant i \leqslant j-1 \leqslant L$. Note that the lattice sites 0 and $L+1$ are auxiliary lattice sites to have a well-defined shock measure. The structure of the product shock measure is

$$|i,j\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes i} \otimes \begin{pmatrix} 1-\rho \\ \rho \end{pmatrix}^{\otimes j-i-1} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes L-j+1}, \qquad (22)$$

in which $\rho = \Delta/(1+\Delta)$, so that

$$\hat{\mathcal{H}}(0)|i,j\rangle = \sum_{i',j'} \chi_{i',j'}|i',j'\rangle. \qquad (23)$$

The coefficients $\chi_{i',j'}$ are explicitly given in Ref. [27]. It has been shown that the shock fronts at two lattice sites i and j perform simple random walk on the lattice. More precisely, while for $q > 1$ the left shock front is always biased to the left, the bias of the right shock front depends on the values of both q and Δ. For $q^2 > 1+\Delta$, the right shock front is biased to the left while for $q^2 < 1+\Delta$ the right shock front is biased to the right. Moreover, using Eq. (23) the steady state of the system can be constructed as a linear superposition of the states of type Eq. (22) with exactly known coefficients. It is clear that $|P^*\rangle = |\Lambda^*(s=0)\rangle$.

It is not difficult to check that at s^* we have

$$\begin{aligned}
\hat{\mathcal{H}}(s^*)|i,j\rangle &= q|i+1,j\rangle + q^{-1}(1+\Delta)|i-1,j\rangle \\
&\quad + q(1+\Delta)|i,j+1\rangle + q^{-1}|i,j-1\rangle \\
&\quad - (q+q^{-1})(2+\Delta)|i,j\rangle, \\
&\quad \text{for} \quad i = 1,\ldots,L-2 \\
&\quad \text{and} \quad j = i+2,\ldots,L, \\
\hat{\mathcal{H}}(s^*)|0,j\rangle &= q(1+\Delta)|0,j+1\rangle + q^{-1}|0,j-1\rangle \\
&\quad - (q+q^{-1}(1+\Delta))|0,j\rangle, \\
&\quad \text{for} \quad j = 2,\ldots,L, \\
\hat{\mathcal{H}}(s^*)|i,L+1\rangle &= q|i+1,L+1\rangle \\
&\quad + q^{-1}(1+\Delta)|i-1,L+1\rangle \\
&\quad - (q^{-1}+q(1+\Delta))|i,L+1\rangle, \\
&\quad \text{for} \quad i = 1,\ldots,L-1, \\
\hat{\mathcal{H}}(s^*)|i,i+1\rangle &= 0, \quad \text{for} \quad i = 0,\ldots,L, \\
\hat{\mathcal{H}}(s^*)|0,L+1\rangle &= 0. \qquad (24)
\end{aligned}$$

These relations mean that as long as the shock fronts are far from the boundaries of the lattice, i.e., $i \neq 0$ and $j \neq L+1$, they show the same simple random walk behavior. However, as soon as the right (left) shock front attaches to the right (left) boundary, it will not detach (reflect) from there. In other words, the following product measure,

$$|0,L+1\rangle = \begin{pmatrix} 1-\rho \\ \rho \end{pmatrix}^{\otimes L}, \qquad (25)$$

is the right eigenvector of $\hat{\mathcal{H}}(s^*)$ with zero eigenvalue that is we have $|\Lambda^*(s^*)\rangle = |0,L+1\rangle$. We can also calculate the elements of the left eigenvector of $\hat{\mathcal{H}}(s^*)$ as follows:

$$\langle \tilde{\Lambda}^*(s^*)|c\rangle = \frac{\langle W|\prod_{i=1}^{L}(c_i D + (1-c_i)E)|V\rangle}{Z\rho^{\sum_{i=1}^{L} c_i}(1-\rho)^{L-\sum_{i=1}^{L} c_i}}, \quad (26)$$

where the normalization factor Z, which is given by $\langle W|(D+E)^L|V\rangle$, can be calculated using the matrix representation of the algebra given in Ref. [25].

From Eq. (15) at $s=0$ one can easily see that $\hat{\mathcal{H}}_{\text{eff}}(s^*)$ and $\hat{\mathcal{H}}_{\text{eff}}(0)$ have exactly the same right eigenvector with zero eigenvalue:

$$|P_{\text{TTI}}(s=0)\rangle = |P_{\text{TTI}}(s=s^*)\rangle.$$

Using Eq. (18) one can easily recognize that the effective transition rates at s^* depend on both the initial and final configurations. On the other hand, they can be calculated exactly using the matrix product method explained earlier. For two arbitrary configurations, $c=\{c_1,\ldots,c_L\}$ and $c'=\{c'_1,\ldots,c'_L\}$, we have

$$\omega^{\text{eff}}_{c\to c'}(\ln q^2) = \omega_{c'\to c}\frac{\langle W|\prod_{i=1}^{L}(c'_i D + (1-c'_i)E)|V\rangle}{\langle W|\prod_{i=1}^{L}(c_i D + (1-c_i)E)|V\rangle}. \quad (27)$$

Because of the local interaction nature of the original process, c and c' are different from each other only in the configurations of two consecutive lattice sites (say, i and $i+1$)

$$\cdots c_i c_{i+1}\cdots \to \cdots c'_i c'_{i+1}\cdots.$$

As we mentioned, E and D do not commute with each other. At the same time, neither the numerator nor denominator of the fraction appeared in Eq. (27) can be decomposed into productive factors so that the final result depend only on the local configurations. This means that the interactions in the effective process at s^* are nonlocal yet the effective transition rates can be calculated exactly. In what follows we will give an explicit example by considering the initial and final configurations as

$$\text{c} \quad \underbrace{\emptyset\ldots\emptyset}_{i-1} A \underbrace{\emptyset\ldots\emptyset}_{j-i-1} A \underbrace{\emptyset\ldots\emptyset}_{L-j}$$

and

$$\text{c}' \quad \underbrace{\emptyset\ldots\emptyset}_{i} A \underbrace{\emptyset\ldots\emptyset}_{j-i-2} A \underbrace{\emptyset\ldots\emptyset}_{L-j},$$

respectively. This indicates the diffusion of the leftmost particle to the right. One can now write

$$\omega^{\text{eff}}_{c\to c'} = q\frac{\langle W|E^i D E^{j-i-2} D E^{L-j}|V\rangle}{\langle W|E^{i-1} D E^{j-i-1} D E^{L-j}|V\rangle}. \quad (28)$$

This expression can be calculated exactly using the matrix representation of the operators and vectors. It turns out that the final result depends on i (or i and j) explicitly. Since the mathematical expression is rather complicated, we have plotted Eq. (28) in Fig. 2 for the system in the low-density phase.

For $q=2$ the diffusion rate to the right in the original process is 0.5. However, as it can be seen in Fig. 2 the diffusion rate to the right can be as large as 2 depending on the lattice-site numbers i and j. In contrast in the high-density phase where $q^2 < 1+\Delta$, the effective diffusion rate to the right is almost constant and independent of i and j except where i and j are very close to each other.

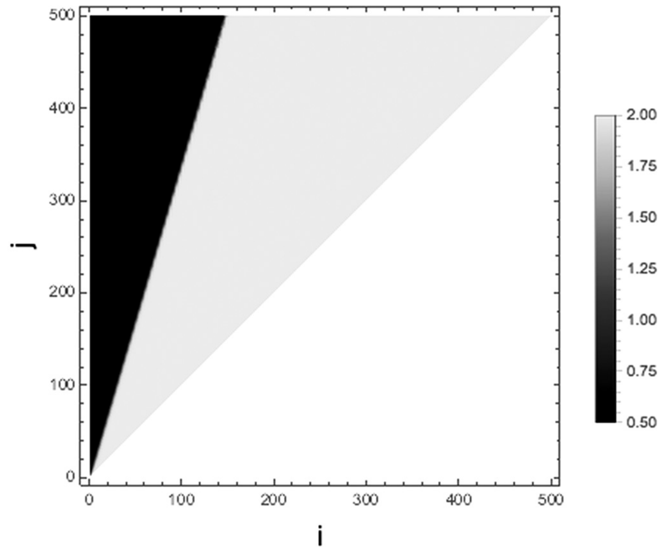

FIG. 2. Density plot of Eq. (28) as a function of i and j for $L=500$, $q=2$, and $\Delta=0.5$.

IV. CONCLUDING REMARKS

In this paper we have investigated the connection between the stochastic generators of effective processes corresponding to two specific atypical values of an entropic current with the Gallavotti-Cohen symmetry. For a specific family of processes we have shown that the effective stochastic generators at the points s and $E-s$ are enantiodromic with respect to each other. These two generators have the same spectrum and the same steady states; however, one of them generates the process backward in time with respect to the other one. It is important to note that the characteristics of the interactions corresponding to these points could be completely different as we have shown in an exactly solvable example with nonconserving dynamics, including coagulation and decoagulation processes on a one-dimensional lattice with reflecting boundaries. Although the original model includes nearest-neighbor interactions, we have shown that the adjoint process consists of nonlocal interactions.

The Gallavotti-Cohen symmetry has been observed in the systems for which the characteristic polynomial of the modified Hamiltonian is symmetric with respect to some external field. However, there are also situations in which only the dominant eigenvalue of the modified Hamiltonian is symmetric. On the other hand, it has been shown that under some conditions on the structure of the configuration space and the reaction rates, the large deviation functions for the probability distributions of time-integrated currents satisfy a so-called Gallavotti-Cohen-like symmetry [30]. It would be interesting to investigate the connections between the effective stochastic generators corresponding to specific fluctuations in these cases.

ACKNOWLEDGMENT

We thank the anonymous referee for his/her enlightening comments and criticisms that were very helpful to improve this manuscript.

[1] S. Albeverio, V. Jentsch, and H. Kantz, *Extreme Events in Nature and Society* (Springer, Berlin, 2006).
[2] S. Auer and D. Frenkel, Nature **409**, 1020 (2001); R. P. Sear, J. Phys. Condens. Matter **19**, 033101 (2007).
[3] E. I. Shakhnovich and A. M. Gutin, Nature **346**, 773 (1990); E. I. Shakhnovich, Chem. Rev. **106**, 1559 (2006); G. D. Rose, P. J. Fleming, J. R. Banavar, and A. Maritan, Proc. Natl. Acad. Sci. USA **103**, 16623 (2006).
[4] G. M. Schütz, *Phase Transitions and Critical Phenomena* (Academic Press, London, 2001), Vol. 193.
[5] R. L. Jack and P. Sollich, Prog. Theor. Phys. Suppl. **184**, 304 (2010).
[6] V. Lecomte, C. Appert-Rolland, and F. van Wijland, J. Stat. Phys. **127**, 51 (2007); J. P. Garrahan, R. L. Jack, V. Lecomte, E. Pitard, K. van Duijvendijk, and F. van Wijland, J. Phys. A **42**, 075007 (2009).
[7] D. Simon, J. Stat. Mech. (2009) P07017.
[8] V. Popkov, G. M. Schütz, and D. Simon, J. Stat. Mech. (2010) P10007.
[9] V. Belitsky and G. M. Schütz, J. Phys. A **46**, 295004 (2013).
[10] V. Belitsky and G. M. Schütz, J. Stat. Phys. **152**, 93 (2013).
[11] R. J. Harris, V. Popkov, and G. M. Schütz, Entropy **15**, 5065 (2013).
[12] O. Hirschberg, D. Mukamel, and G. M. Schütz, J. Stat. Mech. (2015) P11023.
[13] G. M. Schütz, *From Particle Systems to Partial Differential Equations II*, Vol. 129 (Springer International Publishing, Switzerland, 2015), p. 371.
[14] R. Chetrite and H. Touchette, Ann. Henri Poincaré **16**, 2005 (2015).
[15] R. L. Jack and P. Sollich, Eur. Phys. J. Special Topics **224**, 2351 (2015).
[16] V. Popkov and G. M. Schutz, J. Stat. Phys. **142**, 627 (2011).
[17] R. L. Jack and P. Sollich, J. Phys. A **47**, 015003 (2014).
[18] H. Touchette, Phys. Rep. **478**, 1 (2009).
[19] D. J. Evans, E. G. D. Cohen, and G. P. Morriss, Phys. Rev. Lett. **71**, 2401 (1993); D. J. Evans and D. J. Searles, Phys. Rev. E **50**, 1645 (1994); G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. **74**, 2694 (1995); J. Stat. Phys. **80**, 931 (1995).
[20] M. Henkel and G. M. Schütz, J. Phys. A **37**, 591 (2004).
[21] R. J. Harris and G. M. Schütz, J. Stat. Mech. (2007) P07020.
[22] J. L. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999).
[23] A. Baule and R. M. L. Evans, Phys. Rev. Lett. **101**, 240601 (2008); J. Stat. Mech. (2010) P03030.
[24] H. Hinrichsen, K. Krebs, and I. Peschel, Z. Phys. B **100**, 105 (1996).
[25] H. Hinrichsen, S. Sandow, and I. Peschel, J. Phys. A **29**, 2643 (1996).
[26] F. H. Jafarpour, Physica A **339**, 369 (2004).
[27] F. H. Jafarpour and S. R. Masharian, Phys. Rev. E **77**, 031115 (2008).
[28] P. Torkaman and F. H. Jafarpour, Phys. Rev. E **87**, 012138 (2013).
[29] R. A. Blythe and M. R. Evans, J. Phys. A **40**, R333 (2007).
[30] A. C. Barato, R. Chetrite, H. Hinrichsen, and D. Mukamel, J. Stat. Phys. **146**, 294 (2012); A. C. Barato and R. Chetrite, J. Phys. A **45**, 485002 (2012).