Comment on "Flow-induced arrest of spatiotemporal chaos and transition to a stationary pattern in the Gray-Scott model"

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In this Comment, we review the results of pattern formation in a reaction-diffusion-advection system following the kinetics of the Gray-Scott model. A recent paper by Das [Phys. Rev. E **92**, 052914 (2015)] shows that spatiotemporal chaos of the intermittency type can disappear as the advective flow is increased. This study, however, refers to a single point in the space of kinetic parameters of the original Gray-Scott model. Here we show that the wealth of patterns increases substantially as some of these parameters are changed. In addition to spatiotemporal intermittency, defect-mediated turbulence can also be found. In all cases, however, the chaotic behavior is seen to disappear as the advective flow is increased, following a scenario similar to what was reported in our earlier work [I. Berenstein and C. Beta, Phys. Rev. E **86**, 056205 (2012)] as well as by Das. We also point out that a similar phenomenon can be found in other reaction-diffusion-advection models, such as the Oregonator model for the Belousov-Zhabotinsky reaction under flow conditions.

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In a recently published paper, Das [1] discusses the influence of advection on the spatiotemporal dynamics of the Gray-Scott model, which is a prototypical model for autocatalytic chemical reactions. This model reads

$$\frac{\partial \alpha}{\partial t} + \sigma \nabla \alpha = 1 - \alpha - \mu \alpha \beta^2 + D_\alpha \nabla^2 \alpha,$$
$$\frac{\partial \beta}{\partial t} + \sigma \nabla \beta = \beta_0 - \phi \beta + \mu \alpha \beta^2 + D_\beta \nabla^2 \beta.$$

In these equations, α and β represent the concentration of two reactants with D_α and D_β their respective diffusion coefficients. The model also includes kinetic parameters (β_0, μ, and ϕ) and a flow rate σ. All these variables and parameters are dimensionless. More specifically, the study in [1] focuses on the case of one-dimensional systems and for values of parameters such that the advection-free ($\sigma = 0$) system displays spatiotemporal chaos in the form of spatiotemporal intermittency. Das shows that by increasing σ, chaos disappears by stabilization of the unstable state from which the chaotic dynamics emerges. After a second transition, the system displays a specific stationary pattern known as flow-distributed oscillations. In this Comment we would like to point out that the results published in [1] are an instance of a much more general trend observed and explained for various reaction-diffusion-advection models.

First we will recall some of the basic dynamical features of the Gray-Scott model. The author considers in [1] the case $\beta_0 = 0$, for which the system can present up to three homogeneous steady states. One of these states corresponds to $(\alpha, \beta) = (1,0)$ and is always a stable node. Two additional steady states appear after a saddle-node bifurcation at $\mu_s = 4\phi^2$. One of them is a saddle $S^s = (\alpha^-, \beta^-)$ and the other one an unstable focus $S^f = (\alpha^+, \beta^+)$, with

$$\alpha^\pm = \frac{1 \mp \sqrt{1 - 4\phi^2/\mu}}{2}, \quad \beta^\pm = \frac{1 \pm \sqrt{1 - 4\phi^2/\mu}}{2\phi}.$$

Just after crossing the saddle-node bifurcation, the state (1,0) is the only attractor in the system. However, as μ is further increased it can cross a critical value μ_A corresponding to a homoclinic bifurcation, after which a limit cycle associated with the unstable focus can also be found. This transition is known as an Andronov homoclinic bifurcation [2]. The focus S^f eventually becomes stable for $\mu > \mu_H$, where $\mu_H = \phi^4/(\phi - 1)$ denotes the location of a Hopf bifurcation. This bifurcation is subcritical for $2 < \phi < 4$, while it is supercritical for $\phi > 4$ [3].

In the case of spatially extended systems with no advection, two types of chaos can be observed in the above model [4]. Spatiotemporal intermittency occurs when the (1,0) stable steady state coexists with the saddle and the unstable focus and is the only attractor of the system, i.e., whenever $\mu_s < \mu < \mu_A$. Note that the results shown and discussed in [1] correspond to this regime. For $\mu_A < \mu < \mu_H$ a limit cycle and a stable node coexist, however, the limit cycle alone displays a form of chaos known as defect-mediated turbulence.

The effect of advection on a system with a single unstable steady state that shows defect-mediated turbulence has been studied earlier [5]. The model used in [5] is the two-variable Oregonator model of the Belousov-Zhabotinsky reaction and what was found is that below a certain velocity for the advection term, the system remains in the defect-mediated turbulence state. After a first threshold, the system displays traveling waves. After a second threshold in velocity of advection, the originally unstable steady state becomes stable. Linear stability analysis is able to predict this transition [5]. Then there are two further transitions, first to damped flow-distributed oscillations, seen near the Dirichlet boundary (the inlet of the system) that decay into the now stable steady state, and then to fully developed flow-distributed oscillations.

Figure 1(a) shows the possible regimes without advective flow ($\sigma = 0$) for the Gray-Scott model. At low ϕ and intermediate values of μ, only spatiotemporal intermittency can be observed. At intermediate and large values of ϕ, a

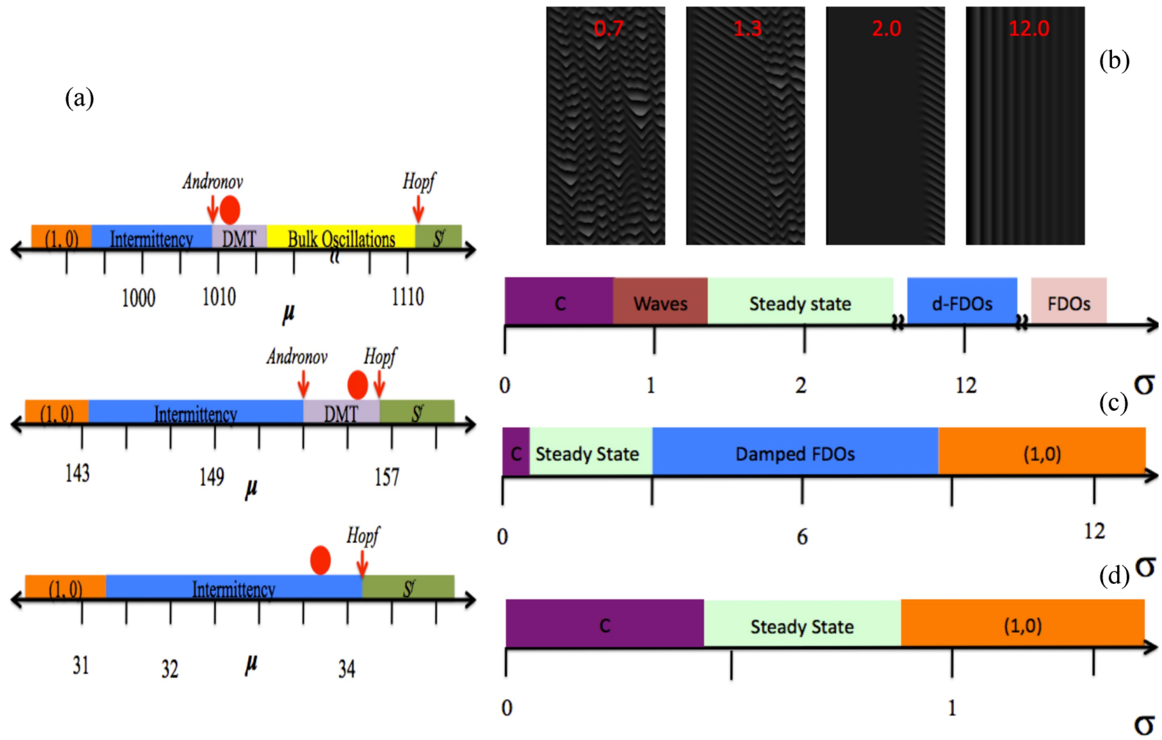

FIG. 1. (a) Phase diagram of patterns at $\sigma = 0$, where the Hopf and Andronov bifurcations are pointed out. Beyond the Hopf bifurcation, S^f denotes the stable focus. Here DMT corresponds to defect-mediated turbulence. The red marks correspond to conditions for which the effect of advection is analyzed. (b) Space time plots [size (100 space units) $\times$ (20 time units)], where the number at the top corresponds to the velocity of advection σ for $\phi = 10$ and $\mu = 1010$; the corresponding phase diagram is at the bottom. (c) Phase diagram of patterns for $\phi = 5$ and $\mu = 155$ and (d) phase diagram for $\phi = 2.8$ and $\mu = 33.7$. In (b)–(d), C corresponds to spatiotemporal chaos [DMT in (b) and (c) and intermittency in (d)], d-FDO to damped flow-distributed oscillations, and FDO to flow-distributed oscillations.

transition from spatiotemporal intermittency to defect-mediated turbulence can be observed by changing μ.

In the regime of defect-mediated turbulence ($\mu_A < \mu < \mu_H$) at high ϕ, the effect of advection is similar to the case seen in the Oregonator model, and as the velocity of advection is augmented, the same succession of spatiotemporal patterns is observed: defect-mediated turbulence, waves, stable steady state, damped flow-distributed oscillations, and flow-distributed oscillations [Fig. 1(b)]. Not surprisingly, for systems showing spatiotemporal intermittency (i.e., whenever $\mu_s < \mu < \mu_A$) the scenario is qualitatively the same. At high ϕ, there is also a small region where, using no-flux boundary conditions, homogenous oscillations can be observed [4]. However, by adding a Dirichlet boundary, the system turns into the defect-mediated state and the system shows the same scenario of patterns as the velocity of the advection is increased.

The regime studied by Das [1] corresponds to a system where, in the absence of advection, spatiotemporal chaos of the intermittency type should be observed for intermediate ϕ. We note, however, that the value used for μ in [1] places the system just at the boundary between the intermittent and the defect-mediated regimes. The patterns observed by Das [1] are actually the same as the patterns observed for $\phi = 5$ and $\mu = 155$ in our study [6], where the spatiotemporal chaos corresponds to defect-mediated turbulence. The succession of patterns as σ is augmented is shown in Fig. 1(c). The transition to the (1,0) state depends on the boundary condition at the inlet so it may or may not appear [6]. In the regime of spatiotemporal intermittency at low ϕ, the advection does not produce the regime of damped flow-distributed oscillations [6] [Fig. 1(d)].

In his paper Das points out that the transition from absolute to convective instability appears as the transition to the steady state [1]. However, for higher μ, the transition from absolute to convective instability, for the chaotic state, is seen as a transition to waves, which come from an instability that keeps on being absolute. The transition to the stabilized steady state occurs when the waves become convectively unstable, as seen in the Oregonator model [5].

To conclude, for high ϕ, the effect of advection on pattern formation in the Gray-Scott model is the same as seen for a system with only one unstable state that generates a limit cycle and develops defect-mediated turbulence (e.g., the Oregonator model). As the velocity of advection is augmented, the patterns observed are spatiotemporal chaos, waves, stable steady state, damped flow-distributed oscillations, and flow-distributed oscillations. For the Gray-Scott model, spatiotemporal chaos can take the form of defect-mediated turbulence or spatiotemporal intermittency, which does not influence the type of patterns seen by adding advection. For intermediate ϕ, the succession of patterns as σ is augmented is spatiotemporal chaos (intermittency [1] or defect-mediated turbulence [6]), stable steady state, and damped flow-distributed oscillations. For low ϕ, only spatiotemporal intermittency is observed, and as σ increases, there is a transition that stabilizes the unstable state.

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