PHYSICAL REVIEW E **94**, 043110 (2016)

Stability of thin liquid curtains

E. S. Benilov,* R. Barros,† and S. B. G. O'Brien‡
Department of Mathematics, University of Limerick, Limerick V94 T9PX, Ireland
(Received 21 June 2016; published 21 October 2016)

We investigate the stability of thin liquid curtains with respect to two-dimensional perturbations. The dynamics of perturbations with wavelengths exceeding (or comparable to) the curtain's thickness are examined using the lubrication approximation (or a kind of geometric optics). It is shown that, contrary to the previous theoretical results, but in agreement with the experimental ones, all curtains are stable with respect to small perturbations. Large perturbations can still be unstable, however, but only if they propagate upstream and, thus, disrupt the curtain at its outlet. This circumstance enables us to obtain an effective stability criterion by deriving an existence condition for upstream propagating perturbations.

DOI: 10.1103/PhysRevE.94.043110

I. INTRODUCTION

The process of curtain coating is traditionally used for manufacturing photographic materials and, more recently, paper. Typically, a liquid curtain is formed using a reservoir with a slot (outlet) in its bottom; having emerged from the outlet, the curtain falls under gravity until it hits the substrate to be coated. Clearly, for this to work as an industrial process, the falling curtain should be hydrodynamically stable.

The stability of liquid curtains with respect to small perturbations was first considered in Ref. [1] using a somewhat intuitive approach (not based on a formal asymptotic expansion). A stability criterion was obtained, predicting that all curtains with a sufficiently small Weber number We < 1 are unstable. It was also argued that the instability is caused by sinuous perturbations, traveling upstream. The conclusions obtained appeared to agree with the experimental results reported in Refs. [2,3]. The effect of the surrounding air on the curtain's stability was examined in Refs. [4,5] using the same intuitive approach as that of Ref. [1].

Note, however, that, in experiments [2,3], the curtain was perturbed by a solid object inserted in the flow so that the resulting perturbations could hardly be assumed small. Subtler experiments were carried out in Refs. [6,7] with perturbations created by either fluctuations of air pressure or a thin needle, respectively. On the basis of these experiments the authors of Ref. [6] concluded that "despite the fact that previous work . . . shows that curtains . . . where We < 1 are unstable to small disturbances, our experiments show that these curtains can exist over a wide range of flow conditions." A similar conclusion was drawn in Ref. [7] where it was also claimed that a curtain can disintegrate only due to a "hole" (i.e., a large perturbation).

In the present paper, we examine the stability of liquid curtains through a formal asymptotic expansion and thus show that all curtains are linearly stable with respect to sinuous perturbations. We also calculate an expression for the perturbations' speed (which is similar to the corresponding result of Refs. [1,4,5] but with an extra term). Then, adopting the hypothesis of Ref. [8]—that sufficiently strong upstream traveling perturbations can destabilize the curtain—we obtain a stability criterion for large perturbations.

This paper has the following structure. In Sec. II, we formulate the problem. In Secs. III and IV, we derive an asymptotic model for long-wave perturbations and use it to examine their stability. In Sec. V, we examine perturbations whose wavelengths are comparable to the curtain's thickness. Large-amplitude perturbations are discussed in Sec. VI where we also compare our results to those of Ref. [1].

II. FORMULATION

Let the horizontal and vertical axes x and z be directed to the right and downwards, respectively, and consider a two-dimensional liquid sheet emerging from an outlet located at $z = 0$ (see Fig. 1). The free boundaries of the sheet are described by the equation $x = x_\pm(t,z)$, where t is the time. The liquid is assumed incompressible with density ρ, kinematic viscosity ν, and surface tension σ; and the flow is characterized by its velocity (u,w) and pressure p.

A. The governing equations

We employ the Navier-Stokes equations,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \quad (1)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = g + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (3)$$

where g is the acceleration due to gravity. The kinematic conditions at the free boundaries are

$$\frac{\partial x_\pm}{\partial t} - u + w\frac{\partial x_\pm}{\partial z} = 0, \quad \text{at} \quad x = x_\pm, \quad (4)$$

and the standard dynamic conditions are rearranged into a more convenient form

$$2\frac{\partial u}{\partial x} - \frac{p - \sigma c_\pm}{\rho\nu} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\partial x_\pm}{\partial z}, \quad \text{at} \quad x = x_\pm, \quad (5)$$

*eugene.benilov@ul.ie; http://www.staff.ul.ie/eugenebenilov/hpage/
†ricardo.barros@ul.ie; https://www.researchgate.net/profile/Ricardo_Barros5
‡Stephen.OBrien@ul.ie; http://www.staff.ul.ie/obriens

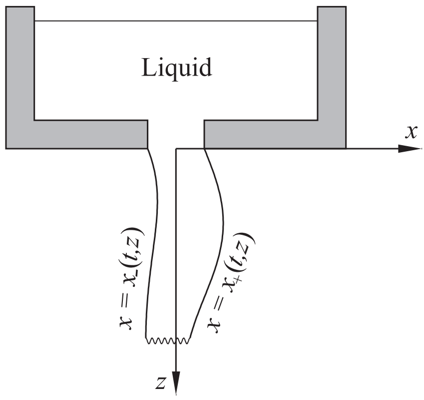

FIG. 1. The setting: a free-falling liquid curtain.

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\left(2\frac{\partial w}{\partial z}-\frac{p-\sigma c_\pm}{\rho\nu}\right)\frac{\partial x_\pm}{\partial z},\quad \text{at}\quad x=x_\pm, \tag{6}$$

where the curvatures of the free boundaries are

$$c_\pm=\mp\frac{\partial^2 x_\pm}{\partial z^2}\left[1+\left(\frac{\partial x_\pm}{\partial z}\right)^2\right]^{-3/2}. \tag{7}$$

We will use the following nondimensional variables:

$$\begin{gathered}t_*=\frac{t}{T},\quad x_*=\frac{x}{X},\quad z_*=\frac{z}{Z},\quad u_*=\frac{u}{U},\\ w_*=\frac{w}{W},\quad p_*=\frac{p}{P},\quad x_{\pm *}=\frac{x_\pm}{X},\quad c_{\pm *}=\frac{Z^2}{X}c_\pm,\end{gathered} \tag{8}$$

where the dimensional scales are constrained by the standard hydrodynamic balance,

$$\frac{1}{T}=\frac{U}{X}=\frac{W}{Z}. \tag{9}$$

Let the vertical acceleration of liquid particles be comparable to g, the horizontal pressure gradient be balanced by viscosity, and X and W are linked to the volumetric flow rate Q per unit width of the curtain, i.e.,

$$\frac{W}{T}=g,\quad \frac{P}{\rho X}=\frac{\nu U}{X^2},\quad WX=Q. \tag{10}$$

Conditions (9) and (10) imply that

$$T=\frac{Q}{gX},\quad Z=\frac{Q^2}{gX^2}, \tag{11}$$

$$U=\frac{gX^2}{Q},\quad W=\frac{Q}{X},\quad P=\frac{\rho g\nu X}{Q}. \tag{12}$$

Rewriting the governing set (1)–(7) in terms of the nondimensional variables (8), taking into account conditions (11) and (12), and omitting asterisks, we obtain

$$\begin{aligned}&\delta\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)+\frac{\partial p}{\partial x}\\ &=\frac{\partial^2 u}{\partial x^2}+\varepsilon\frac{\partial^2 u}{\partial z^2}\delta\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)+\varepsilon\frac{\partial p}{\partial z}\\ &=\delta+\frac{\partial^2 w}{\partial x^2}+\varepsilon\frac{\partial^2 w}{\partial z^2},\end{aligned} \tag{13}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{14}$$

$$\frac{\partial x_\pm}{\partial t}-u+w\frac{\partial x_\pm}{\partial z}=0,\quad \text{at}\quad x=x_\pm, \tag{15}$$

$$\begin{gathered}2\frac{\partial u}{\partial x}-p+\delta\gamma c_\pm=\left(\varepsilon\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{\partial x_\pm}{\partial z},\\ \varepsilon\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\varepsilon\left(2\frac{\partial w}{\partial z}-p+\delta\gamma c_\pm\right)\frac{\partial x_\pm}{\partial z},\\ \text{at}\quad x=x_\pm,\end{gathered} \tag{16}$$

$$c_\pm=\mp\frac{\partial^2 x_\pm}{\partial z^2}\left[1+\varepsilon\left(\frac{\partial x_\pm}{\partial z}\right)^2\right]^{-3/2}, \tag{17}$$

where

$$\delta=\frac{gX^3}{\nu Q},\quad \varepsilon=\left(\frac{gX^3}{Q^2}\right)^2,\quad \gamma=\frac{\sigma X}{\rho Q^2}. \tag{18}$$

Observe that δ multiplies the convective derivatives in the Navier-Stokes equations (13)—hence, it should be interpreted as the Reynolds number. ε, in turn, is the curtain's aspect ratio (it can be shown that $\varepsilon = X^2/Z^2$), and γ characterizes surface tension.

B. The governing parameters

Note that the horizontal scale X characterizing the curtain's thickness has not been fixed so far. Physically, the best choice for X is the outlet's width X_{outlet}.

Assuming, thus, $X = X_{\text{outlet}}$, we formulate our main assumptions in the form

$$\delta \ll 1,\quad \varepsilon \ll 1. \tag{19}$$

These conditions make the flow "slow" and the curtain thin—together, they amount to the lubrication approximation. We do not make any assumptions about γ, so it will be treated as an order-one parameter.

III. LONG-WAVE PERTURBATIONS: THE ASYMPTOTIC EQUATIONS

When the curtain emerges from the outlet, its velocity profile is parabolic (as that of a Poiseuille flow). However, since the side walls supporting this shape are no longer in place, the flow has to adjust to the new boundary conditions—and it does so in a certain adjustment region, whose size is sometimes referred to as the "exit length." Unfortunately, this parameter has been calculated only for large Reynolds numbers [9,10], whereas we are interested in the opposite limit $\delta \ll 1$. One can assume, however, that, for small δ, the "exit problem" is symmetric to the "entrance problem" (where a plug flow enters a channel and eventually assumes the Poiseuille profile). The latter setting has been examined in more detail; in particular, the entrance length has been computed for a wide range of δ in Ref. [11]. It has turned out to be surprisingly small: about half of the channel's width.

In this paper, we are not concerned with the flow behavior in the small adjustment region. We only assume that the curtain's thickness does not change much over the exit length and, thus, remains approximately equal to the outlet's width.

A. The analysis

There are two approaches to the asymptotic analysis of problems with multiple small parameters. In most cases, one assumes a certain relationship between these—say, $\delta = c\varepsilon$ with $c = O(1)$—and then expands the problem in ε only. Alternatively, one keeps the small parameters unrelated and works with expansions in $\varepsilon^n\delta^m$, which yields more general results but may involve more algebra.

In what follows, we employ the latter approach. The price to pay for the generality is not too high in the problem at hand as we do not need to go beyond the first-order perturbations, i.e., linear in ε and δ.

Thus, let the solution of Eqs. (13)–(17) have the form

$$(u,w,p,x_\pm) = (u^{(0)},w^{(0)},p^{(0)}) + (u^{(1)},w^{(1)},p^{(1)}) + \cdots,$$

where

$$(u^{(0)},w^{(0)},p^{(0)}) = O(1), \quad (u^{(1)},w^{(1)},p^{(1)}) = O(\varepsilon,\delta).$$

Note that the unknowns characterizing the free-surface $x_\pm$ are not expanded (as one often does when using the lubrication approximation).

To leading order, (13)–(17) yield

$$w^{(0)} = w_0, \quad u^{(0)} = -x\frac{\partial w_0}{\partial z} + u_0, \tag{20}$$

$$p^{(0)} = -2\frac{\partial w_0}{\partial z}, \tag{21}$$

$$\frac{\partial x_\pm}{\partial t} + x_\pm\frac{\partial w_0}{\partial z} - u_0 + w_0\frac{\partial x_\pm}{\partial z} = 0, \tag{22}$$

where the undetermined functions $w_0(z,t)$ and $u_0(z,t)$ are independent of x.

In the next order, we only need Eqs. (13), (14), and (16). Taking into account (20) and (21), we obtain

$$\begin{aligned}\frac{\partial p^{(1)}}{\partial x} = {}& \frac{\partial^2 u^{(1)}}{\partial x^2} + \varepsilon\left(\frac{\partial^2 u_0}{\partial z^2} - x\frac{\partial^3 w_0}{\partial z^3}\right)\\ &+\delta\left\{x\left[\frac{\partial^2 w_0}{\partial t\,\partial z} - \left(\frac{\partial w_0}{\partial z}\right)^2 + w_0\frac{\partial^2 w_0}{\partial z^2}\right]\right.\\ &\left.-\delta\frac{\partial u_0}{\partial t} + u_0\frac{\partial w_0}{\partial z} - w_0\frac{\partial u_0}{\partial z}\right\},\end{aligned} \tag{23}$$

$$\frac{\partial^2 w^{(1)}}{\partial x^2} = -3\varepsilon\frac{\partial^2 w_0}{\partial z^2} + \delta\left(\frac{\partial w_0}{\partial t} + w_0\frac{\partial w_0}{\partial z} - 1\right), \tag{24}$$

$$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial w^{(1)}}{\partial z} = 0, \tag{25}$$

$$2\frac{\partial u^{(1)}}{\partial x} - p^{(1)} = 4\varepsilon\frac{\partial w_0}{\partial z}\left(\frac{\partial x_\pm}{\partial z}\right)^2 \pm \delta\gamma\frac{\partial^2 x_\pm}{\partial z^2}, \quad \text{at} \quad x = x_\pm, \tag{26}$$

$$\frac{\partial w^{(1)}}{\partial x} = \varepsilon\left(4\frac{\partial w_0}{\partial z}\frac{\partial x_\pm}{\partial z} + x_\pm\frac{\partial^2 w_0}{\partial z^2}\frac{\partial^2 w_0}{\partial z^2}\frac{\partial u_0}{\partial z}\right), \quad \text{at} \quad x = x_\pm. \tag{27}$$

It can readily be verified that Eq. (24) and the boundary conditions (27) have a solution for $w^{(1)}$ only if

$$\begin{aligned}&\delta\left(\frac{\partial w_0}{\partial t} + w_0\frac{\partial w_0}{\partial z} - 1\right)\\ &\quad = \varepsilon\left[4\frac{\partial^2 w_0}{\partial z^2} + \frac{4}{x_+ - x_-}\frac{\partial w_0}{\partial z}\frac{\partial(x_+ - x_-)}{\partial z}\right].\end{aligned} \tag{28}$$

If condition (28) holds, one can find $w^{(1)}$—substitute it in Eq. (26) and, thus, find $u^{(1)}$—then substitute $u^{(1)}$ and $w^{(1)}$ into (23) and (26) and solve these for $p^{(1)}$. It turns out that the solution for $p^{(1)}$ exists only if

$$\begin{aligned}&\delta\left\{\frac{\partial u_0}{\partial t} + w_0\frac{\partial u_0}{\partial z} - u_0\frac{\partial w_0}{\partial z} - \frac{x_+ + x_-}{2}\right.\\ &\quad\left.\times\left[\frac{\partial^2 w_0}{\partial t\partial z} + w_0\frac{\partial^2 w_0}{\partial z^2} - \left(\frac{\partial w_0}{\partial z}\right)^2\right] - \frac{4\gamma}{\Delta}\frac{\partial^2(x_+ + x_-)}{\partial z^2}\right\}\\ &= \frac{4\varepsilon}{x_+ - x_-}\frac{\partial}{\partial z}\left[(x_+ - x_-)\frac{\partial w_0}{\partial z}\frac{\partial(x_+ + x_-)}{\partial z}\right].\end{aligned} \tag{29}$$

Note that this equation has been simplified using (28). Equations (22), (28), and (29) form a closed set for $x_\pm$, w_0, and u_0. They take a much simpler form in terms of

$$C = \frac{x_+ + x_-}{2}, \quad h = x_+ - x_-,$$

$$U = u_0 - \frac{x_+ + x_-}{2}\frac{\partial w_0}{\partial z}, \quad w = w_0,$$

where $C(z,t)$ and $U(z,t)$ are the position and horizontal velocity of the curtain's center line and $h(z,t)$ is the curtain's thickness. One can now rewrite (22), (28), and (29) in the form

$$\frac{\partial h}{\partial t} + \frac{\partial(wh)}{\partial z} = 0, \quad \delta\left(\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} - 1\right) = \frac{4\varepsilon}{h}\frac{\partial}{\partial z}\left(h\frac{\partial w}{\partial z}\right), \tag{30}$$

$$\frac{\partial C}{\partial t} + w\frac{\partial C}{\partial z} - U = 0,$$

$$\delta\left(\frac{\partial U}{\partial t} + w\frac{\partial U}{\partial z} - \frac{2\gamma}{h}\frac{\partial^2 C}{\partial z^2}\right) = \frac{4\varepsilon}{h}\frac{\partial}{\partial z}\left(h\frac{\partial w}{\partial z}\frac{\partial C}{\partial z}\right). \tag{31}$$

B. Discussion

(1) Observe that Eq. (29) governs the varicose (symmetric) part of the curtain's evolution, whereas (30) governs the sinuous part. Interestingly, the former do not involve C and U and, thus, decouple from the latter, but the latter do involve w and h and, thus, depend on the solution of the former.

Thus, physically, sinuous motions are affected by the varicose ones but not vice versa.

(2) Since ε and δ have already played their roles of "indicators" of small terms in the governing equations, we

can scale them out of Eqs. (29) and (30). To do so, let

$$t=\left(\frac{\varepsilon}{\delta}\right)^{1/3}t_{\text{new}},\quad z=\left(\frac{\varepsilon}{\delta}\right)^{2/3}z_{\text{new}},$$

$$w=\left(\frac{\varepsilon}{\delta}\right)^{1/3}w_{\text{new}},\quad h=\left(\frac{\varepsilon}{\delta}\right)^{-2/3}h_{\text{new}},$$

$$C=C_{\text{new}},\quad U=\left(\frac{\varepsilon}{\delta}\right)^{-1/3}U_{\text{new}}.$$

We emphasize that this rescaling does not imply extra physical assumptions as it will not be used for omitting any terms in the equations derived. Note also that the new variables are still related to the dimensional ones by (8)–(12) but only if the spatial scale X is given by

$$X=\frac{Q}{(g\nu)^{1/3}}.$$

Note, however, that definitions (18) of ε and δ still imply $X=X_{\text{outlet}}$.

Rewriting (29) and (30) in terms of the new variables and omitting the subscripts $_{\text{new}}$, we obtain

$$\frac{\partial h}{\partial t}+\frac{\partial(wh)}{\partial z}=0,\quad \frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}-1=\frac{4}{h}\frac{\partial}{\partial z}\left(h\frac{\partial w}{\partial z}\right), \tag{32}$$

$$\frac{\partial C}{\partial t}+w\frac{\partial C}{\partial z}-U=0,$$
$$\frac{\partial U}{\partial t}+w\frac{\partial U}{\partial z}-\frac{2\gamma}{h}\frac{\partial^2 C}{\partial z^2}=\frac{4}{h}\frac{\partial}{\partial z}\left(h\frac{\partial w}{\partial z}\frac{\partial C}{\partial z}\right). \tag{33}$$

IV. LONG-WAVE PERTURBATIONS: STABILITY OF A STEADY CURTAIN

In terms of Eqs. (32) and (33), steady vertical curtains are described by

$$h=\bar{h}(z),\quad w=\bar{w}(z),\quad C=0,\quad U=0.$$

Since our nondimensionalization implies that the nondimensional flux equals unity, Eq. (33) reduces to

$$\bar{w}\bar{h}=1, \tag{34}$$

$$\left(\bar{w}-\frac{4\bar{w}'}{\bar{w}}\right)'=\frac{1}{\bar{w}}, \tag{35}$$

where the prime denotes differentiation with respect to z. Equation (35) (first presented in Ref. [2]) admits an analytic solution in terms of the Airy function [12], which can be used to show that all physically meaningful solutions of (35) are such that

$$\bar{w}=\tfrac{1}{8}(z-\bar{z})^2+O[(z-\bar{z})^3],\quad \text{as}\quad z\to\bar{z}+0, \tag{36}$$

$$\bar{w}=(2z)^{1/2}+O(z^{-1/2}),\quad \text{as}\quad z\to+\infty, \tag{37}$$

where $\bar{z}$ is arbitrary [due to the translational invariance of (35)]. As $\bar{w}\to 0$ implies $\bar{h}\to\infty$, $\bar{z}$ should be smaller than the coordinate of the curtain's outlet; the latter is assumed to be at $z=0$—hence, $\bar{z}<0$ [see Fig. 2(a)]. Note also that the limit of small X_{outlet} is described by the limit of large negative $\bar{z}$ in which case all of the curtain is described by asymptotics (37).

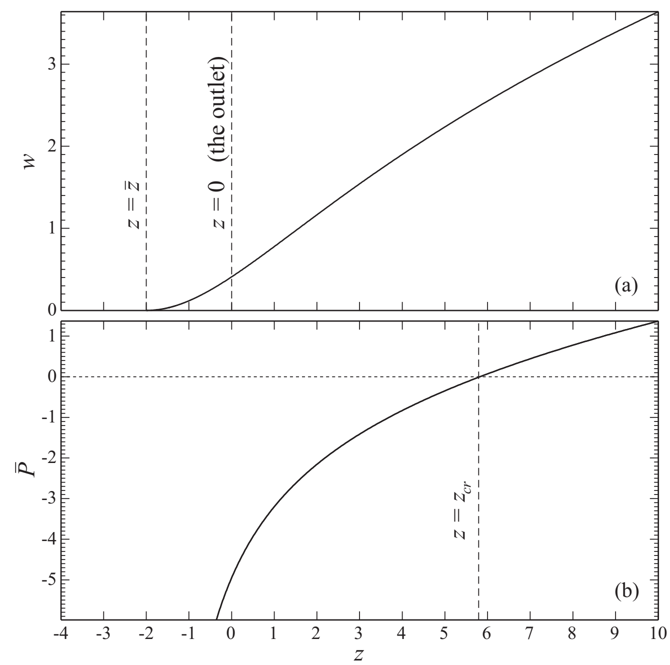

FIG. 2. The solution of the boundary-value problem (35)–(37) for $\bar{z}=-2$. The curtain's outlet is located at $z=0$. (a) $\bar{w}$ vs z; (b) $\bar{P}$ [defined by (41)] vs z. z_{cr} is the coordinate of the critical point such that $\bar{P}(z_{\text{cr}})=0$.

Note that all available experiments [2,3,6,7] suggest that varicose perturbations are stable—therefore, we only examine sinuous perturbations. To do so, we let $w=\bar{w}$, $h=\bar{h}$ in Eq. (33) and use (34) to eliminate $\bar{h}$, which yields

$$\frac{\partial C}{\partial t}+\bar{w}\frac{\partial C}{\partial z}-U=0, \tag{38}$$

$$\frac{\partial U}{\partial t}+\bar{w}\frac{\partial U}{\partial z}=\bar{w}\frac{\partial}{\partial z}\left(\frac{4\bar{w}'}{\bar{w}}\frac{\partial C}{\partial z}\right)+2\gamma\bar{w}\frac{\partial^2 C}{\partial z^2}. \tag{39}$$

Note that these equations were not linearized with respect to C and U as the original Eq. (33) is already linear.

We confine our study to perturbations with exponential dependence on t,

$$C=\hat{C}(z)e^{\lambda t},\quad U=\hat{U}(z)e^{\lambda t},$$

where $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ are the growth rate and frequency. Reducing then (38) and (39) to a single equation for $\hat{C}$, we obtain

$$(\bar{P}\hat{C}')'+2\lambda\hat{C}'+\frac{\lambda^2}{\bar{w}}\hat{C}=0, \tag{40}$$

where

$$\bar{P}(z)=\bar{w}-\frac{4\bar{w}'}{\bar{w}}-2\gamma. \tag{41}$$

Before setting boundary conditions for (40), note that the original partial differential Eqs. (38) and (39) are of hyperbolic type and their characteristics $z(t)$ satisfy the following differential equations:

$$\frac{dz}{dt} = \bar{w}(z) \pm \sqrt{2\gamma\bar{w}(z) + 4\bar{w}'(z)}. \tag{42}$$

Evidently, if

$$\bar{P}(z) \geqslant 0 \quad \text{for all} \quad z \in (0,\infty), \tag{43}$$

where $\bar{P}(z)$ is defined by (41), then (42) implies $dz/dt \geqslant 0$. Keeping in mind that, physically, characteristics are associated with waves, we conclude that these cannot travel upstream. If, however, a region of negative $\bar{P}(z)$ exists, some of the waves there do travel upstream. Solving of the steady-curtain problems (35)–(37) numerically, one can show that this region, if it exists, is adjacent to the outlet, i.e., occupies an interval $[0,z_{cr})$ with z_{cr} such that

$$\bar{P}(z_{cr}) = 0.$$

An example of $\bar{P}(z)$ allowing for waves traveling upstream is plotted in Fig. 2(b).

Next, the number of boundary conditions at $z = 0$ should equal the number of characteristics emerging from the outlet. Thus,

(1) if $\bar{P} \geqslant 0$ for $z \in [0,\infty)$, (40) requires two boundary conditions at $z = 0$;

(2) if $\bar{P} < 0$ for $z \in [0,z_{cr})$, (40) requires one boundary condition at $z = 0$.

In Case 1, we require

$$\text{Case 1: } \hat{C} = 0, \quad \hat{C}' = 0, \quad \text{at} \quad z = 0, \tag{44}$$

i.e., the position of the outlet is fixed, and the curtain is vertical there.

It can readily be shown that (40)–(44) admit only the trivial solution $\hat{C} = 0$. Thus, no unstable solutions exist, and we conclude that, in Case 1, the curtain is stable.

In Case 2, the waves coming from below still cannot change the outlet's position—hence,

$$\text{Case 2: } \hat{C} = 0, \quad \text{at} \quad z = 0. \tag{45}$$

The upstream traveling waves, however, can alter the angle at which the curtain emerges from the outlet. The condition for $\hat{C}'$ should, thus, be discarded, leaving us one boundary condition short.

Observe, however, that the coefficient of the highest derivative of Eq. (40) vanishes at $z = z_{cr}$, making it a singular point. Then, using the Frobenius method, one can show that

$$\hat{C} = a_1\hat{C}_1 + a_2\hat{C}_2, \tag{46}$$

where $a_{1,2}$ are constants and

$$\begin{aligned}\hat{C}_1 &= 1 + O(z - z_{cr}),\\ \hat{C}_2 &= (z - z_0)^{-2\lambda/\bar{P}'(z_{cr})}[1 + O(z - z_{cr})], \quad \text{as} \quad z \to z_{cr}.\end{aligned} \tag{47}$$

We are only interested in unstable solutions for which $\operatorname{Re}\lambda > 0$. In this case $\hat{C}_2$ is unbounded as $z \to z_{cr}$ and, therefore, should be eliminated by setting $a_2 = 0$. Assuming without loss of generality that $a_1 = 1$, we reduce (46) and (47) to

$$\hat{C} = 1, \quad \text{as} \quad z = z_{cr}. \tag{48}$$

It turns out, however, that problems (40), (45), and (48) do not have solutions with $\operatorname{Re}\lambda > 0$. To prove this, multiply (40) by $\hat{C}^*$ and integrate from $z = 0$ to $z = z_{cr}$. Integrating by parts, using the boundary conditions (45) and (48), and separating the real and imaginary parts, we obtain

$$\lambda_i + \lambda_r \int_0^{z_{cr}} \operatorname{Im}\left(\hat{C}^*\frac{d\hat{C}}{dz} - \frac{d\hat{C}^*}{dz}\hat{C}\right)dz + 2\lambda_r\lambda_i \int_0^{z_{cr}} \frac{|\hat{C}|^2}{\bar{w}}dz = 0, \tag{49}$$

$$\lambda_r - \lambda_i \int_0^{z_{cr}} \operatorname{Im}\left(\hat{C}^*\frac{d\hat{C}}{dz} - \frac{d\hat{C}^*}{dz}\hat{C}\right)dz - \int_0^{z_{cr}} \bar{P}\left|\frac{d\hat{C}}{dz}\right|^2 dz + \left(\lambda_r^2 - \lambda_i^2\right)\int_0^{z_{cr}} \frac{|\hat{C}|^2}{\bar{w}}dz = 0, \tag{50}$$

where $\lambda_r = \operatorname{Re}\lambda$ and $\lambda_i = \operatorname{Im}\lambda$. Equality (49) can be used to eliminate the integral involving $\operatorname{Im}(\cdots)$ from (50), which yields

$$\int_0^{z_{cr}} \bar{P}\left|\frac{d\hat{C}}{dz}\right|^2 dz = |\lambda|^2\left(\int_0^{z_{cr}} \frac{|\hat{C}|^2}{\bar{w}}dz + \frac{1}{\lambda_r}\right). \tag{51}$$

Since in Case 2 $\bar{P} < 0$ for $z \in [0,z_{cr})$, equality (51) cannot hold for $\lambda_r > 0$. We conclude that the eigenvalue problems (40), (45), and (48) do not have unstable solutions.

Thus, no unstable solutions exist in either Case 1 or Case 2, which should be interpreted as stability.

V. SHORT-WAVE PERTURBATIONS

We will now consider short-wave perturbations with wavelengths comparable to the curtain's thickness. Note that, although the lubrication approximation does not apply to these, it still applies to the curtain—which circumstance will considerably simplify our task.

Seek a solution of Eqs. (13)–(17) (with $\delta = \varepsilon$) in the form

$$\begin{aligned}u &= u_s(x,z) + \tilde{u}(t,x,z), \quad w = w_s(x,z) + \tilde{w}(t,x,z),\\ p &= p_s(x,z) + \tilde{p}(t,x,z),\\ x_\pm &= x_{s\pm}(z) + \tilde{x}_\pm(t,z), \quad c_\pm = c_{s\pm}(z) + \tilde{c}_\pm(t,z),\end{aligned}$$

where the variables with the subscript $_s$ and tildes describe the curtain and a small perturbation, respectively. The latter is governed by the linearized version of (13)–(17) (again, with $\delta = \varepsilon$),

$$\varepsilon\left(\frac{\partial\tilde{u}}{\partial t} + u_s\frac{\partial\tilde{u}}{\partial x} + \tilde{u}\frac{\partial u_s}{\partial x} + w_s\frac{\partial\tilde{u}}{\partial z} + \tilde{w}\frac{\partial u_s}{\partial z}\right) + \frac{\partial\tilde{p}}{\partial x} = \frac{\partial^2\tilde{u}}{\partial x^2} + \varepsilon\frac{\partial^2\tilde{u}}{\partial z^2}, \tag{52}$$

$$\varepsilon\left(\frac{\partial\tilde{w}}{\partial t} + u_s\frac{\partial\tilde{w}}{\partial x} + \tilde{u}\frac{\partial w_s}{\partial x} + w_s\frac{\partial\tilde{w}}{\partial z} + \tilde{w}\frac{\partial w_s}{\partial z} + \frac{\partial\tilde{p}}{\partial z}\right) = \frac{\partial^2\tilde{w}}{\partial x^2} + \varepsilon\frac{\partial^2\tilde{w}}{\partial z^2}, \tag{53}$$

$$\frac{\partial\tilde{u}}{\partial x} + \frac{\partial\tilde{w}}{\partial z} = 0, \tag{54}$$

$$\frac{\partial \tilde{x}_\pm}{\partial t} - \tilde{u} - \frac{\partial u_s}{\partial x}\tilde{x}_\pm + w_s\frac{\partial \tilde{x}_\pm}{\partial z} + \left(\tilde{w} + \frac{\partial w_s}{\partial x}\tilde{x}_\pm\right)\frac{\partial x_{s\pm}}{\partial z} = 0, \quad \text{at} \quad x = x_{s\pm}, \tag{55}$$

$$2\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial^2 u_s}{\partial x^2}\tilde{x}_\pm\right) - \tilde{p} - \frac{\partial p_s}{\partial x}\tilde{x}_\pm + \varepsilon\gamma\tilde{c}_\pm = \left(\varepsilon\frac{\partial u_s}{\partial z} + \frac{\partial w_s}{\partial x}\right)\frac{\partial \tilde{x}_\pm}{\partial z} + \left[\varepsilon\left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial^2 u_s}{\partial x\,\partial z}\tilde{x}_\pm\right) + \frac{\partial \tilde{w}}{\partial x} + \frac{\partial^2 w_s}{\partial x^2}\tilde{x}_\pm\right]\frac{\partial x_{s\pm}}{\partial z} \quad \text{at} \quad x = x_{s\pm}, \tag{56}$$

$$\varepsilon\left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial^2 u_s}{\partial x\,\partial z}\tilde{x}_\pm\right) + \frac{\partial \tilde{w}}{\partial x} + \frac{\partial^2 w_s}{\partial x^2}\tilde{x}_\pm = \varepsilon\left\{\left(2\frac{\partial w_s}{\partial z} - p_s + \varepsilon\gamma c_{s\pm}\right)\frac{\partial \tilde{x}_\pm}{\partial z} + \left[2\left(\frac{\partial \tilde{w}}{\partial z} + \frac{\partial^2 w_s}{\partial x\partial z}\tilde{x}_\pm\right) - \tilde{p} - \frac{\partial p_s}{\partial x}\tilde{x}_\pm + \varepsilon\gamma\tilde{c}_\pm\right]\frac{\partial x_{s\pm}}{\partial z}\right\} \quad \text{at} \quad x = x_{s\pm}, \tag{57}$$

$$\tilde{c}_\pm = \mp\frac{\partial^2 \tilde{x}_\pm}{\partial z^2}\left[1 + \varepsilon\left(\frac{\partial x_{s\pm}}{\partial z}\right)^2\right]^{-3/2} \pm 3\varepsilon\frac{\partial^2 x_{s\pm}}{\partial z^2}\frac{\partial x_{s\pm}}{\partial z}\frac{\partial \tilde{x}_\pm}{\partial z}\left[1 + \varepsilon\left(\frac{\partial x_{s\pm}}{\partial z}\right)^2\right]^{-5/2}. \tag{58}$$

Since the perturbation is no longer assumed to be long wave, these equations need to be rescaled. To understand how, recall our original nondimensionalization where the x-to-z aspect ratio was $O(\varepsilon^{1/2})$. Accordingly, the assumption that the dimensional wavelength is comparable to the curtain's thickness amounts to making the nondimensional wavelength $\sim\varepsilon^{1/2}$.

With regard to the characteristic period τ of the perturbation, two different modes can be distinguished

$$\text{Mode 1:}\, \tau \sim \varepsilon^{1/2},$$
$$\text{Mode 2:}\, \tau \sim \varepsilon.$$

It turns out that the two modes also differ by the amplitude of the free-surface displacement,

$$\text{Mode 1:}\, \tilde{x} \sim \tilde{u},$$
$$\text{Mode 2:}\, \tilde{x} \sim \varepsilon\tilde{u}.$$

As for the curtain itself, it is still described by the steady version of Eqs. (13)–(17) with no rescaling necessary. This circumstance allows us to "reuse" the lubrication results of Sec. III, i.e., set

$$w_s = \bar{w} + O(\varepsilon), \quad u_s = -x\bar{w}' + O(\varepsilon), \tag{59}$$

$$p_s = -2\bar{w}' + O(\varepsilon), \quad x_{s\pm} = \pm\frac{1}{2\bar{w}} + O(\varepsilon), \tag{60}$$

where $\bar{w}(z)$ satisfies the boundary-value problems (35)–(37).

Perturbations of Modes 1 and 2 are examined in Secs. V A and V B, respectively—in both cases using an asymptotic approach similar to the geometric optics.

A. Mode 1

Seek a solution describing a wave packet traveling in a slowly changing medium,

$$\tilde{u} = \hat{u}(x,z,t)\exp\left\{i\varepsilon^{-1/2}\left[\int k(z)dz - \omega t\right]\right\}, \quad \tilde{w} = \varepsilon^{1/2}\hat{w}(x,z,t)\exp\left\{i\varepsilon^{-1/2}\left[\int k(z)dz - \omega t\right]\right\}, \tag{61}$$

$$\tilde{p} = \hat{p}(x,z,t)\exp\left\{i\varepsilon^{-1/2}\left[\int k(z)dz - \omega t\right]\right\}, \quad \tilde{x}_\pm = \hat{x}_\pm(z,t)\exp\left\{i\varepsilon^{-1/2}\left[\int k(z)dz - \omega t\right]\right\}, \tag{62}$$

where the variables with hats describe the so-called envelope of the wave packet and ω and k are the frequency and wave number of the carrier wave. The former is a constant, whereas the latter depends on the local properties of the curtain where the perturbation is currently traveling. In the case under consideration, $k(z)$ can immediately be determined from Eq. (55), which admits a nontrivial leading-order solution only if

$$k = \frac{\omega}{\bar{w}(z)}. \tag{63}$$

Thus, the perturbation speed locally coincides with that of the flow.

Substituting (61) and (62) into (52)–(58) and keeping the leading-order terms only, we obtain

$$\frac{\partial \hat{p}}{\partial x} = \frac{\partial^2 \hat{u}}{\partial x^2} - \frac{\omega^2}{\bar{w}^2}\hat{u}, \quad \frac{i\omega}{\bar{w}}\hat{p} = \frac{\partial^2 \hat{w}}{\partial x^2} - \frac{\omega^2}{\bar{w}^2}\hat{w}, \tag{64}$$

$$\frac{\partial \hat{u}}{\partial x} + \frac{i\omega}{\bar{w}}\hat{w} = 0, \tag{65}$$

$$\frac{\partial \hat{x}_\pm}{\partial t} - \hat{u} + \bar{w}'\hat{x}_\pm + \bar{w}\frac{\partial \hat{x}_\pm}{\partial z} = 0, \quad \text{at} \quad x = \pm\frac{1}{2\bar{w}}, \tag{66}$$

$$2\frac{\partial \hat{u}}{\partial x} - \hat{p} \pm \frac{\gamma\omega^2}{\bar{w}^2}\hat{x}_\pm = 0, \quad \text{at} \quad x = \pm\frac{1}{2\bar{w}}. \tag{67}$$

$$\frac{i\omega}{\bar{w}}\hat{u} + \frac{\partial \hat{w}}{\partial x} = \frac{4i\omega}{\bar{w}}\bar{w}'\hat{x}_\pm, \quad \text{at} \quad x = \pm\frac{1}{2\bar{w}}. \tag{68}$$

Equations (64) and (65) form a set of ordinary differential equations in x, which can be readily solved

$$\hat{u} = (A + Bx)\exp\frac{\omega x}{\bar{w}} + (F + Gx)\exp\left(-\frac{\omega x}{\bar{w}}\right),$$

$$\hat{w} = i\left[\left(A + \frac{\bar{w}}{\omega}B + Bx\right)\exp\frac{\omega x}{\bar{w}} - \left(F - \frac{\bar{w}}{\omega}G + Gx\right)\exp\left(-\frac{\omega x}{\bar{w}}\right)\right],$$

$$\hat{p} = 2B\,\exp\frac{\omega x}{\bar{w}} + 2G\,\exp\left(-\frac{\omega x}{\bar{w}}\right),$$

where A, B F, and G are undetermined functions of z and t. Substituting the above expressions into the boundary conditions (66)–(68) and eliminating A, B, F, and G, we obtain (after straightforward but tedious algebra)

$$\frac{\partial \hat{X}}{\partial t} + \bar{w}\frac{\partial \hat{X}}{\partial z} + D_X\hat{X} = 0, \quad \frac{\partial \hat{h}}{\partial t} + \bar{w}\frac{\partial \hat{h}}{\partial z} + D_h\hat{h} = 0, \quad (69)$$

where

$$\hat{X} = \tfrac{1}{2}(\hat{x}_+ + \hat{x}_-), \quad \hat{h} = \hat{x}_+ - \hat{x}_-$$

represent the sinuous and varicose perturbations, and

$$D_X = \frac{\bar{w}'\left(\sinh\frac{\omega}{\bar{w}^2} + \frac{\omega}{\bar{w}^2}\right) + \frac{\gamma\omega}{2\bar{w}}\left(\cosh\frac{\omega}{\bar{w}^2} + 1\right)}{\sinh\frac{\omega}{\bar{w}^2} - \frac{\omega}{\bar{w}^2}},$$

$$D_h = \frac{\bar{w}'\left(\sinh\frac{\omega}{\bar{w}^2} - \frac{\omega}{\bar{w}^2}\right) + \frac{\gamma\omega}{2\bar{w}}\left(\cosh\frac{\omega}{\bar{w}^2} - 1\right)}{\sinh\frac{\omega}{\bar{w}^2} + \frac{\omega}{\bar{w}^2}}.$$

Equation (69) can readily be solved by characteristics, and it can be shown that, since $D_X, D_h > 0$ for $\omega \neq 0$, the solution decays as $t \to \infty$, which means that Mode-1 perturbations are stable.

B. Mode 2

Seek a solution in the form

$$\tilde{u} = \hat{u}(x,z,t)\exp\left\{i\left[\varepsilon^{-1/2}\int k(z)dz - \varepsilon^{-1}\omega t\right]\right\},$$

$$\tilde{w} = \varepsilon^{1/2}\hat{w}(x,z,t)\exp\left\{i\left[\varepsilon^{-1/2}\int k(z)dz - \varepsilon^{-1}\omega t\right]\right\},$$

$$\tilde{p} = \hat{p}(x,z,t)\exp\left\{i\left[\varepsilon^{-1/2}\int k(z)dz - \varepsilon^{-1}\omega t\right]\right\},$$

$$\tilde{x}_\pm = \varepsilon\hat{x}_\pm(z,t)\exp\left\{i\left[\varepsilon^{-1/2}\int k(z)dz - \varepsilon^{-1}\omega t\right]\right\}.$$

Substituting these expressions into (52)–(58) and keeping the leading-order terms only, we obtain

$$-i\omega\hat{u} + \frac{\partial \hat{p}}{\partial x} = \frac{\partial^2 \hat{u}}{\partial x^2} - k^2\hat{u}, \quad -i\omega\hat{w} + ik\hat{p} = \frac{\partial^2 \hat{w}}{\partial x^2} - k^2\hat{w}, \quad (70)$$

$$\frac{\partial \hat{u}}{\partial x} + ik\hat{w} = 0, \quad (71)$$

$$-i\omega\hat{x}_\pm - \hat{u} = 0, \quad \text{at} \quad x = \pm\frac{1}{2\bar{w}}, \quad (72)$$

$$2\frac{\partial \hat{u}}{\partial x} - \hat{p} = 0, \quad ik\hat{u} + \frac{\partial \hat{w}}{\partial x} = 0, \quad \text{at} \quad x = \pm\frac{1}{2\bar{w}}. \quad (73)$$

Observe that, unlike their Mode-1 counterparts, Eqs. (70)–(73) do not involve the derivatives of $\hat{x}_\pm$ with respect to z and t (they appear in the next order of the expansion). We will still be able, however, to draw a conclusion about stability as the relationship between ω and k described by (70)–(73) turns out to be complex and corresponding to wave decay. This kind of decay is even faster than the one for the Mode-1 perturbation.

Omitting technical details, we just state that (70)–(73) yield the following dispersion relation for sinuous perturbations:

$$4\sqrt{1 - \frac{i\omega}{k^2}}\tanh\left(\sqrt{1 - \frac{i\omega}{k^2}}\frac{k}{2\bar{w}}\right) = \left(2 - \frac{i\omega}{k^2}\right)^2\tanh\frac{k}{2\bar{w}}. \quad (74)$$

To illustrate how this equality can be used, we solve it for ω for the following limits:

$$\omega \sim -\frac{ik^4}{3\bar{w}^2}, \quad \text{as } k \to 0,$$

$$\omega \sim -iak^2, \quad \text{as} \quad k \to \infty,$$

where $a \approx 0.91262$. These dispersion relations correspond to the linear fourth- and second-order diffusion equations, whose solutions rapidly decay. Generally, if a dispersion relation is resolved with respect to ω and yields

$$\operatorname{Im}\omega < 0 \quad \text{for real} \quad k, \quad (75)$$

one can safely assume stability.

Unfortunately, (74) does not have an obvious solution for ω, so condition (75) has been verified numerically. This is not a difficult task as $\bar{w}$ can be scaled out from (74), so it has to be solved only for a range of k.

Thus, Mode-2 perturbations are stable.

VI. LARGE PERTURBATIONS

Even though liquid curtains are stable with respect to small perturbation, experiments [2,3,6,7] show that they can be unstable with respect to large ones. According to a hypothesis put forward in Ref. [8], curtains are unstable if (large) perturbations can travel upstream. Such perturbations can disrupt the flow near the outlet, whereas downstream traveling perturbations are simply swept away.

Adopting this hypothesis and using expression (42) for the wave speed, one can readily show that instability occurs if

$$\bar{w} < 2\gamma + \frac{4\bar{w}'}{\bar{w}}. \quad (76)$$

The second term on the right-hand side of criterion (76) makes it different from its counterpart obtained in Ref. [1], which gives rise to two questions. First, why the two criteria are different, and, second—if we insist that ours is correct—how can we explain the agreement of the results of Ref. [1] with the experimental ones [2,3,6,7]?

The first question is a mathematical one and, thus, can be answered with certainty. Reference [1] as well as the follow-up papers [4,5] assumed that the stability properties of a curtain depend only on its local characteristics and, thus, neglected its streamwise variability. We, on the other hand, use a formal asymptotic procedure not based on any *ad hoc* assumptions and obtained an expression (42) for the wave speed involving local terms as well as a nonlocal one (associated with the streamwise gradient of the curtain's velocity). Most importantly, *the local and nonlocal terms are on the same order*, which leaves us with no other option but to conclude that the locality hypothesis was unjustified.

Before we address the second question, we emphasize that it has nothing to do with the first one. The issue of which of the two criteria is correct can only be resolved by mathematical means. It is still puzzling though that measurements agree with a theoretical result missing a term.

The most likely explanation of the paradox is that, in the above-mentioned experiments,

$$\varepsilon \ll \delta \quad (77)$$

[for the definitions of these parameters, see (18)]. Indeed, recalling the nonrescaled equations (30) and (31), one can see that the nonlocal term [the one on the right-hand side of the second equation in (31)] involves ε and, thus, disappears subject to (77).

To illustrate this argument, we quantified estimate (77) for the experiments described in Fig. 7 of Ref. [6] for which

$$\varepsilon \approx 4.5 \times 10^{-4}, \quad \delta \approx 0.13.$$

Thus, nonlocal effects are indeed weak in this case, so the speed of sinuous perturbations calculated in Refs. [1,4,5] should apply.

VII. CONCLUDING REMARKS

The main result of this paper is condition (76) guaranteeing the existence of upstream traveling perturbations in a liquid curtain—and, hypothetically, instability of the curtain itself. Recalling how our variables were nondimensionalized [see (8), (11)–(12)], one can rewrite the instability criterion (76) in the dimensional form

$$\mathrm{We} < 1 + \frac{2\rho\nu Q w'}{\sigma w}, \tag{78}$$

where ρ, ν, and σ are the liquid's density, kinematic viscosity, and surface tension, Q is the volumetric flow rate per unit width of the curtain, w is the dimensional velocity, w' is its streamwise gradient, and

$$\mathrm{We} = \frac{\rho Q w}{2\sigma}$$

is the Weber number.

Criterion (78) was derived for *thin* curtains with a *slow* flow, i.e., such that satisfy restrictions (19). Substituting definitions (18) of the Reynolds number δ and the aspect ratio ε into (19) and letting $X = X_{\mathrm{outlet}}$, one obtains

$$\frac{g X_{\mathrm{outlet}}^3}{\nu Q} \ll 1, \quad \left(\frac{g X_{\mathrm{outlet}}^3}{Q^2}\right)^2 \ll 1. \tag{79}$$

Note that, even though these restrictions have never been stated explicitly in the previous work on liquid curtains, they underlie implicitly all analytical results obtained so far. If either of conditions (79) is violated, one simply does not have a small parameter to use when calculating the characteristics of a liquid curtain.

Finally, we list possible extensions of our results. First, using the same lubrication-style expansion, one can derive *three*-dimensional asymptotic equations for liquid curtains, which can describe the transverse variability of the liquid curtains reported in Ref. [7]. Second, one can examine how curtains are affected by the surrounding air—which can be important as the air-curtain interaction gives rise to instability of perturbations propagating downstream [13]. Interestingly, these perturbations are varicose, and the amplitude required for triggering them off is much smaller than that for exciting the instability due to upstream-propagating sinuous perturbations.

ACKNOWLEDGMENTS

The authors acknowledge support from the Science Foundation Ireland under Grants No. 11/RFP.1/MTH3281 and No. 12/IA/1683.

[1] S. P. Lin, J. Fluid Mech. **104**, 111 (1981).
[2] D. R. Brown, J. Fluid Mech. **10**, 297 (1961).
[3] S. P. Lin and G. Roberts, J. Fluid Mech. **112**, 443 (1981).
[4] X. Li, Chem. Eng. Sci. **48**, 2973 (1993).
[5] X. Li, Acta Mech. **106**, 137 (1994).
[6] D. S. Finnicum, S. J. Weinstein, and K. J. Ruschak, J. Fluid Mech. **255**, 647 (1993).
[7] J. S. Roche, N. Le Grand, P. Brunet, L. Lebon, and L. Limat, Phys. Fluids **18**, 082101 (2006).
[8] R. J. Dyson, J. Brander, C. J. W. Breward, and P. D. Howell, J. Eng. Math. **64**, 237 (2009).
[9] J. P. K. Tillett, J. Fluid Mech. **32**, 273 (1968).
[10] Y. Miyake, E. Yukai, and Y. Iemoto, Bull. JSME **22**, 1382 (1979).
[11] R. K. Shah, J. Fluids Eng. **100**, 177 (1978).
[12] N. S. Clarke, Mathematica **12**, 51 (1966).
[13] H. Lhuissier, P. Brunet, and S. Dorbolo, J. Fluid Mech. **795**, 784 (2016).