Diffusion in anisotropic fully developed turbulence: Turbulent Prandtl number

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Using the field theoretic renormalization group technique in the leading order of approximation of a perturbation theory the influence of the uniaxial small-scale anisotropy on the turbulent Prandtl number in the framework of the model of a passively advected scalar field by the turbulent velocity field driven by the Navier-Stokes equation is investigated for spatial dimensions $d > 2$. The influence of the presence of the uniaxial small-scale anisotropy in the model on the stability of the Kolmogorov scaling regime is briefly discussed. It is shown that with increasing of the value of the spatial dimension the region of stability of the scaling regime also increases. The regions of stability of the scaling regime are studied as functions of the anisotropy parameters for spatial dimensions $d = 3,4$, and 5. The dependence of the turbulent Prandtl number on the anisotropy parameters is studied in detail for the most interesting three-dimensional case. It is shown that the anisotropy of turbulent systems can have a rather significant impact on the value of the turbulent Prandtl number, i.e., on the rate of the corresponding diffusion processes. In addition, the relevance of the so-called weak anisotropy limit results are briefly discussed, and it is shown that there exists a relatively large region of small absolute values of the anisotropy parameters where the results obtained in the framework of the weak anisotropy approximation are in very good agreement with results obtained in the framework of the model without any approximation. The dependence of the turbulent Prandtl number on the anisotropy parameters is also briefly investigated for spatial dimensions $d = 4$ and 5. It is shown that the dependence of the turbulent Prandtl number on the anisotropy parameters is very similar for all studied cases ($d = 3,4$, and 5), although the numerical values of the corresponding turbulent Prandtl numbers are different.

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I. INTRODUCTION

One of the most fundamental characteristics of diffusion processes in various fluids is the corresponding dimensionless Prandtl number defined as the ratio of the kinematic viscosity of the fluid to the corresponding coefficient of diffusivity of the studied admixture (e.g., the temperature in the problem of thermal diffusion, a weak magnetic field in electrically conductive fluid, or scalar or vector impurities in given environments). It is well known that the values of the Prandtl numbers strongly depend on the microscopic structure of the fluids when the motion of the fluids is characterized by low values of the Reynolds number (Re). However, if the fluids are maintained in the state of fully developed turbulence, i.e., when the values of the Reynolds number are very large (in principle, $\mathrm{Re} \to \infty$), the corresponding Prandtl numbers obtain universal values known as the turbulent Prandtl numbers [1–4].

It is also known that various universal properties of turbulent systems can be effectively and systematically studied by using the renormalization group (RG) technique (see, e.g., Refs. [2,5–7] and references cited therein). By using the RG technique in the framework of the field theoretic formulation of the stochastic problem of a passively advected scalar quantity by the turbulent flow driven by the stochastic Navier-Stokes equation the turbulent Prandtl number was calculated recently in the second-order approximation in the corresponding perturbation expansion (the two-loop approximation in the field theoretic terminology) [8,9]. The most important conclusion obtained in Refs. [8,9] is that the second-order contribution (the two-loop correction) to the turbulent Prandtl number is very small in comparison to its leading one-loop value. Namely, it was shown that the two-loop corrections are less than 2% of the corresponding one-loop result. This rather surprising nontrivial result has very important consequences.

However, before we shall discuss these consequences in detail, let us note that the obtained two-loop theoretical value of the turbulent Prandtl number $\mathrm{Pr_t} = 0.7040$ (see Ref. [9] as well as Ref. [10] where the numerical value of the turbulent Prandtl number was calculated with higher precision) is in very good agreement with its experimental estimations (see, e.g., Refs. [1,11,12]). In addition, let us also note that, as was shown in Ref. [13] where the two-loop RG value of the turbulent magnetic Prandtl number was calculated, there is no difference between the turbulent Prandtl number of the passive scalar quantity advected by the Navier-Stokes turbulence and the corresponding turbulent magnetic Prandtl number of the passive weak magnetic field obtained in the framework of the kinematic magnetohydrodynamic (MHD) turbulence.

The most important consequence of the smallness of the two-loop corrections to the turbulent Prandtl number is the fact that it seems that the turbulent Prandtl number is perturbatively very stable despite the fact that the field theoretic model in the framework of which this result is obtained represents a model with strong coupling constant [7]. It means that, at least in this case, the one-loop result obtained for the turbulent Prandtl number of the passively advected scalar quantity by the Navier-Stokes turbulence is relevant not only qualitatively but also quantitatively with very good precision, at least, in comparison with the two-loop result. Note that the same conclusion is also valid for the turbulent magnetic Prandtl number obtained in the framework of the kinematic MHD turbulence [13] due to its equality with the corresponding turbulent Prandtl number of the passively advected scalar field by the Navier-Stokes turbulence at the one-loop as well as the two-loop level of approximation, of course, only if it is supposed that the corresponding turbulent environments are incompressible and fully symmetric because, e.g., when the turbulent environments are helical, i.e., with the presence of the spatial parity violation, the aforementioned turbulent Prandtl numbers are different [14].

It is worth mentioning that the situation is completely different for the problem of the passive vector advection in turbulent environment (also known as the $A = 0$ model) where the two-loop correction to the one-loop result for the corresponding turbulent vector Prandtl number is rather significant [10,15] (see also Ref. [16] where the two-loop turbulent Prandtl number was studied in the framework of the general A model of passively advected vector field).

The second nontrivial consequence of the smallness of the two-loop corrections to the turbulent Prandtl number is the fact that one can use the one-loop approximation for investigation of the properties of the turbulent Prandtl number even in more complicated situations, e.g., in the cases when some symmetry of the turbulent systems is broken, and he or she can be sure that, with high probability, the obtained results will describe quite adequately the corresponding diffusion processes. This fact is very important especially in the situations when the two-loop calculations are very complicated and therefore can be hardly performed.

In this respect, a typical example is the problem of the influence of the anisotropy of turbulent environments on the processes which take place in them. Here it is known that the complete investigation of the problems related to the presence of even the simplest so-called uniaxial small-scale anisotropy in fully developed turbulence is a highly nontrivial problem even at the one-loop level of approximation [17,18]. This is the reason why, for the time being, the only results in the literature which are related to the theoretical investigation of the influence of the small-scale uniaxial anisotropy on the properties of the models of the passive quantities advected by the Navier-Stokes fully developed turbulence are obtained in the essentially restricted so-called weak anisotropy limit in the framework of which only the linear corrections with respect to the anisotropy parameters are taken into account [19–21] [For completeness, let us note that in Ref. [19] the RG technique based on the Wilson's recursion relations was used; however, the obtained results are not fully correct. On the other hand, in Ref. [20] the problem was analyzed using the field theoretic RG technique, and the results obtained in Ref. [19] were corrected (see also Ref. [6])]. At the same time, it is necessary to stress that, as was shown in the framework of the Gaussian models of turbulent advection (the Kraichnan model, the Kazantsev-Kraichnan model, and their generalizations), the presence of anisotropy has a nontrivial significant impact on the scaling properties of various correlation functions of the corresponding advected fields (see, e.g., the survey papers [22,23] and references cited therein, as well as more recent field theoretical RG investigations, e.g., Refs. [24–32] and references cited therein).

Let us also note that various forms of anisotropy are important in many physical situations related to the turbulence processes: for example, in modeling turbulent processes (see, e.g., Ref. [33]), for investigation of the properties of the Earth's core (see, e.g., Ref. [34]), in the sound turbulence (see, e.g., Ref. [35]), in the problem of mixing layers (see, e.g., Ref. [36]), in the investigation of the dissipation properties of turbulence (see, e.g., Ref. [37]), in the problem of rotating turbulence (see, e.g., Ref. [38]), in the problem of turbulent shearless mixing (see. e.g., Ref. [39]), in the turbulent dynamo problem (see, e.g., Ref. [40]), in the problem of solar wind plasma turbulence (see, e.g., Ref. [41]), in the problem of large-scale structures in anisotropic turbulent flows (see., e.g., Ref. [42]), and many others. Therefore, the investigation and understanding of the fundamental properties of the anisotropic turbulent systems are without doubt desirable and very important.

As was already mentioned, in Refs. [19–21] the model of a passively advected scalar quantity by the stochastic Navier-Stokes velocity field driven by the Gaussian random force with uniaxial small-scale anisotropy was investigated. For example, in Ref. [21] the field theoretic RG analysis of the model was performed in the limit of weak anisotropy within the one-loop approximation, and it was shown that the presence of anisotropy can have significant impact on the properties of this turbulent systems, e.g., on the anomalous scaling of the structure functions of the advected scalar quantity as well as on the value of the turbulent Prandtl number. However, the assumption used in Refs. [19–21], namely, that the anisotropy is weak, i.e., it was supposed that only small deviations from the isotropic case are considered, meant that only linear terms with respect to anisotropy parameters were taken into account in all calculations and all nonlinear corrections were neglected.

Here, however, at least two questions immediately arise. The first question is related to the adequateness of the weak anisotropy assumption, i.e., what restrictions on the absolute values of the anisotropy parameters must be applied to be sure that obtained results in the framework of the weak anisotropy assumption represent good approximation of the unrestricted uniaxial anisotropic model. This is a fundamental question, the answer to which lies only in performing direct calculations in the corresponding anisotropic model without restrictions on the anisotropy parameters. The second question is related to the behavior of a given anisotropic model in the situation when the anisotropy is strong, i.e., when it is impossible to apply the weak anisotropy assumption at all. To find answers on both these questions is the main aim of the present paper. Namely, we will investigate the model of passively advected scalar quantity by the Navier-Stokes turbulence with the presence of strong small-scale uniaxial anisotropy. Here we will concentrate our attention on the detailed analysis of the anisotropic properties of the turbulent Prandtl number. Using the fact that the turbulent Prandtl number is perturbatively stable, as was discussed above, we will find the dependence of the turbulent Prandtl number on the unrestricted anisotropy parameters. It will be shown that the weak small-scale anisotropy approximation is a really well-defined approximation when appropriate restrictions on the absolute values of the anisotropy parameters are applied. On the other hand, it is also shown that the presence of the anisotropy can have a nontrivial impact on the diffusion processes in turbulent environments, in the sense that the value of the turbulent Prandtl number can be significantly influenced by the anisotropy of the system.

The paper is organized as follows. In Sec. II the stochastic model of a passive scalar field advected by the anisotropically driven Navier-Stokes turbulence is described and its field theoretic formulation is given. In Sec. III the ultraviolet (UV) renormalization of the model is discussed and the corresponding RG analysis is performed in the leading order of the perturbation theory. In Sec. IV the dependence of the turbulent Prandtl number on the anisotropy parameters of the model is investigated in detail. Obtained results are briefly reviewed and discussed in Sec. V.

II. MODEL OF PASSIVELY ADVECTED SCALAR FIELD BY AN ANISOTROPIC TURBULENT ENVIRONMENT AND ITS FIELD THEORETIC FORMULATION

Consider a scalar field $\theta \equiv \theta(x)$ $[x \equiv (t,\mathbf{x})]$, described by the stochastic advection-diffusion equation

$$\partial_t\theta + (\mathbf{v}\cdot\partial)\theta = \nu_0 u_0 \Delta\theta + f^\theta, \qquad (1)$$

passively advected by the incompressible turbulent velocity field $\mathbf{v} \equiv \mathbf{v}(x)$ driven by the stochastic Navier-Stokes equation

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\partial)\mathbf{v} = \nu_0\triangle\mathbf{v} - \partial P + \mathbf{f}^{\mathrm{v}}. \qquad (2)$$

The fact that the velocity field is incompressible means that it is divergence-free (solenoidal), i.e., $\partial\cdot\mathbf{v} = 0$.

In Eqs. (1) and (2) we have used the following standard notation: $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, ν_0 is the coefficient of kinematic viscosity, $\nu_0 u_0$ is the coefficient of molecular diffusivity with extracted dimensionless reciprocal Prandtl number u_0, $P \equiv P(x)$ is the pressure, and $f^\theta = f^\theta(x)$ and $\mathbf{f}^{\mathrm{v}} = \mathbf{f}^{\mathrm{v}}(x)$ represent random noises. Note also that, in what follows, each quantity with subscript 0 will always represent the bare value of the corresponding parameter of the unrenormalized theory.

The main role of the scalar random noise f^θ in Eq. (1), which represents the source of fluctuations of the scalar field, is to maintain the steady state of the system. In what follows, it is taken with the Gaussian statistics, i.e., it is supposed that it has zero mean and the correlator in the form

$$D^\theta(x;x') = \langle f^\theta(x) f^\theta(x')\rangle = \delta(t-t')C((\mathbf{x}-\mathbf{x}')/L). \qquad (3)$$

Here L is an integral scale, and the explicit form of the function C is not important in what follows. The only condition which must be satisfied by this function is that it must decrease rapidly for $|\mathbf{x}-\mathbf{x}'| \gg L$.

On the other hand, the explicit form of the transverse random force per unit mass $\mathbf{f}^{\mathrm{v}} = \mathbf{f}^{\mathrm{v}}(x)$ in Eq. (2) is essential, and it must be taken in a form which simulates the energy pumping into the system on large scales. In what follows, we assume that its statistics is also Gaussian with zero mean and we suppose the following explicit form of the pair correlation function:

$$D^v_{ij}(x;x') \equiv \langle f^v_i(x) f^v_j(x')\rangle = \delta(t-t')\int \frac{d^d\mathbf{k}}{(2\pi)^d} D_0 k^{4-d-2\varepsilon} R_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \qquad (4)$$

where d denotes the spatial dimension of the system and $R_{ij}(\mathbf{k})$ is a transverse projector which describes geometric properties of the random force. In the simplest isotropic case, it is standardly defined as ordinary transverse projector $R_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) \equiv \delta_{ij} - k_i k_j/k^2$. It is advisable to write the positive amplitude D_0 in the form $D_0 \equiv g_0\nu_0^3$, where parameter g_0 will play the role of the coupling constant of the present model. It is a formal small parameter of the ordinary perturbation theory and is related to the characteristic ultraviolet (UV) momentum scale Λ (or inner length $l \sim \Lambda^{-1}$) by relation $g_0 \simeq \Lambda^{2\varepsilon}$. Here the physical value of the formally small parameter $0 < \varepsilon \leqslant 2$ is $\varepsilon = 2$. This parameter plays an analogous role as the parameter $\epsilon = 4-d$ in the theory of critical behavior. Note also that in Eq. (4) the needed IR regularization is given by the restriction of the integration from below, namely, $k \geqslant m$, where m corresponds to another integral scale. We shall always suppose that $L \gg 1/m$ in what follows.

Note that the correlation function (4) is chosen in the form which is suitable for description of the real infrared energy pumping to the system because for $\varepsilon \to 2$ the function $D_0 k^{4-d-2\varepsilon}$ is proportional to $\delta(\mathbf{k})$ for an appropriate choice of the amplitude factor D_0, which corresponds to the injection of the energy to the system through interaction with the largest turbulent eddies. In addition, the power-like form of the correlation function (4) allows us to apply the standard RG technique in the analysis of the problem (see, e.g., Refs. [5–7] for all details).

As was mentioned above, the geometric properties of the energy pumping (4) are completely determined by the form of the transverse projector $R_{ij}(\mathbf{k})$. The aim of the present work is to study the model of a passively advected scalar field by the turbulent velocity field driven by the stochastic Navier-Stokes equation when the random force for the velocity field is not isotropic but instead possesses uniaxial anisotropy at all scales. To this end, we take the transverse projector $R_{ij}(\mathbf{k})$ in Eq. (4) in the following form:

$$R_{ij}(\mathbf{k}) = \left(1+\alpha_1\xi_k^2\right)P_{ij}(\mathbf{k}) + \alpha_2 P_{is}(\mathbf{k}) n_s n_t P_{tj}(\mathbf{k}), \qquad (5)$$

where $\xi_k^2 = (\mathbf{n}\cdot\mathbf{k})^2/k^2$. Expression (5) represents the simplest special case of a general uniaxial anisotropic transverse tensor structure (see, e.g., Refs. [17,43,44] for details). In this respect, let us note that Ref. [44] represents the first work where the RG analysis of the problem of developed turbulence based on the anisotropic Navier-Stokes equation was performed in the weak small-scale anisotropy limit. Here n_i is the ith component of the unit vector $\mathbf{n}$, which defines the direction of the axis of the uniaxial anisotropy. At the same time, the necessary condition for positively defined correlator (4) is given by the requirement that the anisotropy parameters α_1 and α_2 satisfy inequalities $\alpha_1 > -1$ and $\alpha_2 > -1$. Note also that the corresponding summations over dummy indices s and t are assumed in Eq. (5).

It is worth mentioning that it is possible to introduce the mixed correlator between the scalar noise f^θ and the vector random force $\mathbf{f}^{\mathrm{v}}$ and to study the corresponding extended stochastic model (see, e.g., Refs. [19,20]). However, the analysis of this extended model in the presence of the strong small-scale uniaxial anisotropy will be studied elsewhere, and here, for simplicity, we concentrate our attention to the model without the mixed correlator [21].

Now, using the general theorem [45], the stochastic model (1)–(2) can be rewritten to the equivalent field theoretic model of the double set of fields $\Phi = \{\mathbf{v},\theta,\mathbf{v}',\theta'\}$ with the action functional

$$S(\Phi) = \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \times \left[v'_i(t_1,\mathbf{x_1}) D^v_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2}) v'_j(t_2,\mathbf{x_2}) + \theta'(t_1,\mathbf{x_1}) D^\theta(t_1,\mathbf{x_1};t_2,\mathbf{x_2})\theta'(t_2,\mathbf{x_2})\right] + \int dt\, d^d\mathbf{x}\{\theta'\{-\partial_t - \mathbf{v}\cdot\partial + \nu_0 u_0[\Delta + \tau_0(\mathbf{n}\cdot\partial)^2]\}\theta + \mathbf{v}'[-\partial_t - \mathbf{v}\cdot\partial + \nu_0(\Delta + \chi_{10}(\mathbf{n}\cdot\partial)^2)]\mathbf{v} + \nu_0\,\mathbf{n}\cdot\mathbf{v}'[\chi_{20}\Delta + \chi_{30}(\mathbf{n}\cdot\partial)^2]\mathbf{n}\cdot\mathbf{v}\}, \tag{6}$$

where $D^\theta(x_1;x_2)$ and $D^v(x_1;x_2)$ are the correlation functions given in Eqs. (3) and (4) for the random forces f^θ and $\mathbf{f^v}$, respectively. The terms with new unrenormalized parameters $\tau_0,\chi_{10},\chi_{20}$, and χ_{30}, which are not present in the original stochastic equations (1) and (2), are related to the presence of the small-scale uniaxial anisotropy in the model, and they must be introduced into the action to obtain multiplicatively renormalizable field theoretic model (see, e.g., Refs. [17,43] for details).

The pressure term $-\partial P$, which is present in Eq. (2), is omitted in the action functional (6) due to the transverse character of the auxiliary field $\mathbf{v}'$.

The needed bare propagators of the model (6) in the framework of the Feynman diagrammatic technique are given by the free part of the action functional and have the following nontrivial form in the frequency-momentum representation [21]:

$$\langle\theta'(\omega,\mathbf{k})\theta(\omega,\mathbf{k})\rangle_0 = \frac{1}{i\omega + \nu_0 u_0 k^2 + \tau_0\nu_0 u_0(\mathbf{n}\cdot\mathbf{k})^2}, \tag{7}$$

$$\langle v'_i(\omega,\mathbf{k}) v_j(\omega,\mathbf{k})\rangle_0 = \frac{P_{ij}}{K_3} - \frac{K_5 P_{is} n_s n_t P_{tj}}{K_3\left[K_3 + K_5\left(1-\xi_k^2\right)\right]}, \tag{8}$$

$$\langle v_i(\omega,\mathbf{k}) v_j(\omega,\mathbf{k})\rangle_0 = -\frac{K_1 P_{ij}}{K_3 K_4} + \frac{P_{is} n_s n_t P_{tj}}{K_3\left[K_4 + K_5\left(1-\xi_k^2\right)\right]} \times \left\{\frac{K_5 K_1}{K_4} + \frac{K_5\left[K_1 + K_2\left(1-\xi_k^2\right)\right]}{K_3 + K_5(1-\xi_k^2)} - K_2\right\}, \tag{9}$$

where

$$K_1 = -g_0\nu_0^3 k^{4-d-2\varepsilon}\left(1+\alpha_1\xi_k^2\right), \tag{10}$$

$$K_2 = -g_0\nu_0^3 k^{4-d-2\varepsilon}\alpha_2, \tag{11}$$

$$K_3 = i\omega + \nu_0 k^2 + \nu_0\chi_{10}(\mathbf{n}\cdot\mathbf{k})^2, \tag{12}$$

$$K_4 = -i\omega + \nu_0 k^2 + \nu_0\chi_{10}(\mathbf{n}\cdot\mathbf{k})^2, \tag{13}$$

$$K_5 = \nu_0\chi_{20} k^2 + \nu_0\chi_{30}(\mathbf{n}\cdot\mathbf{k})^2. \tag{14}$$

Their graphical representation is shown in Fig. 1.

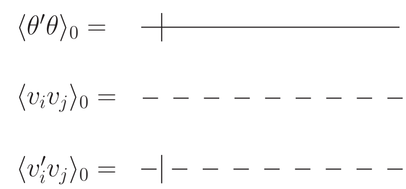

FIG. 1. Graphical representation of needed propagators of the model. The end with a slash in the propagators $\langle\theta'\theta\rangle_0$ and $\langle v'_i v_j\rangle_0$ corresponds to the field θ' and $\mathbf{v}'$, respectively, and the end without a slash corresponds to the field θ and $\mathbf{v}$, respectively.

On the other hand, the model contains two triple interaction vertices $-\theta' v_j\partial_j\theta = \theta' v_j V_j\theta$ and $-v'_i v_j\partial_j v_l = v'_i v_j W_{ijl} v_l$. In the frequency-momentum representation they read $V_j = ik_j$ and $W_{ijl} = i(k_l\delta_{ij} + k_j\delta_{il})$. Their graphical representation is present in Fig. 2.

The formulation of the stochastic problem (1)–(5) in the language of the action functional (6) means that the statistical averages of random quantities are replaced with the functional averages with weight $\exp S(\Phi)$. At the same time, the effective apparatus of the field theoretic RG technique can be used to analyze the universal asymptotic properties of the model.

III. RENORMALIZATION GROUP ANALYSIS OF THE MODEL

The general RG analysis of the model can be found, e.g., in Refs. [21,46]; therefore it is not necessary to discuss it in detail. We restrict our attention only to the basic facts which are important in what follows.

The field theoretic model described by the action functional (6) contains two superficially divergent one-irreducible Green's functions, namely, $\langle v'v\rangle_{1-ir}$ and $\langle\theta'\theta\rangle_{1-ir}$, which can be removed by multiplicative renormalization of the bare parameters g_0,u_0, ν_0, τ_0, and $\chi_{i0}, i=1,2,3$ in the following form:

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon}Z_g, \quad u_0 = uZ_u, \tag{15}$$

$$\tau_0 = \tau Z_\tau, \quad \chi_{i0} = \chi_i Z_{\chi_i}, \quad i = 1,2,3, \tag{16}$$

where parameters g,u,ν,τ, and χ_i, $i=1,2,3$ are dimensionless renormalized counterparts of the bare parameters, μ is the so-called renormalization mass, and $Z_i = Z_i(g,u,\tau,\chi_i;d;\varepsilon)$ are the corresponding renormalization constants which absorb all divergences. In the framework of the minimal substraction (MS) scheme [47], which is used in the present paper, the divergences are realized in the from of poles in ε.

At the same time, the renormalized action functional can be written in the following form: [21]

$$S_R(\Phi) = \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \times \left[v'_i(t_1,\mathbf{x_1}) D^v_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2}) v'_j(t_2,\mathbf{x_2})\right.$$

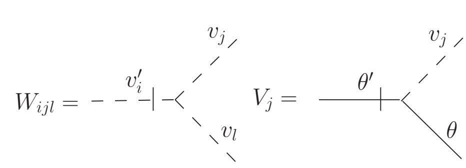

FIG. 2. The interaction vertices of the model. Momentum $\mathbf{k}$ is flowing into the vertexes via the auxiliary fields θ' and v', respectively.

$$+\theta'(t_1,\mathbf{x_1})D^{\theta}(t_1,\mathbf{x_1};t_2,\mathbf{x_2})\theta'(t_2,\mathbf{x_2})\big]$$
$$+\int dt\, d^d\mathbf{x}\{\theta'\{-\partial_t-\mathbf{v}\cdot\partial$$
$$+\nu u[Z_5\Delta+\tau Z_6(\mathbf{n}\cdot\partial)^2]\}\theta$$
$$+\mathbf{v}'\{-\partial_t-\mathbf{v}\cdot\partial+\nu[Z_1\Delta+\chi_1 Z_2(\mathbf{n}\cdot\partial)^2]\}\mathbf{v}$$
$$+\nu\,\mathbf{n}\cdot\mathbf{v}'[\chi_2 Z_3\Delta+\chi_3 Z_4(\mathbf{n}\cdot\partial)^2]\mathbf{n}\cdot\mathbf{v}\}, \tag{17}$$

where Z_i, $i=1,\ldots,6$ represent a new set of independent renormalization constants which are related to the renormalization constants introduced in Eqs. (15) and (16) as

$$Z_\nu = Z_1,\quad Z_g = Z_1^{-3},\quad Z_u = Z_5 Z_1^{-1}, \tag{18}$$

$$Z_\tau = Z_6 Z_5^{-1},\quad Z_{\chi_i} = Z_{i+1}Z_1^{-1},\quad i=1,2,3. \tag{19}$$

It means that the studied model is multiplicatively renormalized through six independent renormalization constants $Z_i, i=1,\ldots,6$. At the same time, in the framework of the MS scheme, each of them has the form of the following infinite series:

$$Z_i = 1+\sum_{n=1}^{\infty} g^n \sum_{j=1}^{n}\frac{z_{nj}^{(i)}}{\varepsilon^j},\quad i=1,\ldots,6. \tag{20}$$

and the ε-independent coefficients $z_{nj}^{(i)}$, $i=1,\ldots,6$ (for given order n of the perturbation expansion) are determined by the requirement that the one-irreducible Green's functions $\langle v_i' v_j\rangle_{1-ir}$ and $\langle\theta'\theta\rangle_{1-ir}$ are free of UV divergences when are written in the renormalized variables, i.e., they have no singularities in the limit $\varepsilon\to 0$.

The one-irreducible Green's functions $\langle v_i' v_j\rangle_{1-ir}$ and $\langle\theta'\theta\rangle_{1-ir}$ are connected to the corresponding self-energy operators $\Sigma^{v'v}$ and $\Sigma^{\theta'\theta}$ through the Dyson equations [21]:

$$\langle v_i' v_j\rangle_{1-ir} = [-i\omega+\nu_0 p^2+\nu_0\chi_{10}(\mathbf{n}\cdot\mathbf{p})^2]\delta_{ij}$$
$$+[\nu_0\chi_{20}p^2+\nu_0\chi_{30}(\mathbf{n}\cdot\mathbf{p})^2]n_i n_j$$
$$-\Sigma_{ij}^{v'v}(\omega,\mathbf{p}), \tag{21}$$

$$\langle\theta'\theta\rangle_{1-ir} = -i\omega+\nu_0 u_0 p^2+\nu_0 u_0\tau_0(\mathbf{n}\cdot\mathbf{p})^2$$
$$-\Sigma_{\theta'\theta}(\omega,\mathbf{p}). \tag{22}$$

On the other hand, the expressions for the self-energy operators $\Sigma^{v'v}$ and $\Sigma^{\theta'\theta}$ are given by the calculations of the corresponding Feynman diagrams. In the one-loop approximation each self-energy operator is determined a by single Feynman diagram as is shown explicitly in Fig. 3.

Thus, in general, when the nth order approximation is considered all ε-independent coefficients $z_{kj}^{(i)}$ for $i=1,\ldots,6$, $k\leqslant n$, and $j\leqslant k$ in Eq. (20) are determined from Dyson equations (21) and (22) by the requirement that all UV divergences vanish when the substitution $e_0 = e\mu^{d_e}Z_e$ for $e=\{g,u,\nu,\tau,\chi_i\}, i=1,2,3$ is performed which are given explicitly in Eqs. (15) and (16). The renormalization constants Z_i can be then found up to the UV finite parts fixed by the renormalization scheme. But, as was already mentioned, we work here in the MS scheme in which the UV finite parts are equal to zero.

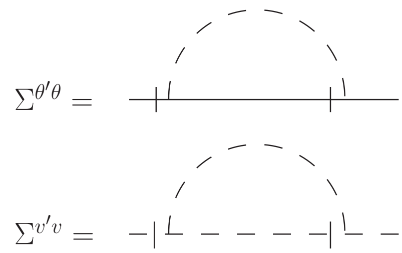

FIG. 3. The graphical representation of the self-energy operators $\Sigma^{v'v}$ and $\Sigma^{\theta'\theta}$ via Feynman diagrams in one-loop approximation.

In the one-loop approximation, which we are interested in, the renormalization constants have the form

$$Z_i = 1+g\frac{z_i}{\varepsilon}+O(g^2),\quad i=1,\ldots,6, \tag{23}$$

where we have introduced notation $z_i\equiv z_{11}^{(i)}$ for further convenience. They are determined by the calculation of the one-loop Feynman diagrams shown in Fig. 3.

The coefficients z_i, for $i=1,\ldots,4$ related to the 1-irreducible Green's function $\langle v_i' v_j\rangle_{1-ir}$ are already known [18]. They have a rather complicated integral form,:

$$z_1 = -\frac{1}{8}\frac{S_{d-1}}{(2\pi)^d(d^2-1)}\int_{-1}^{1}dx\frac{(1-x^2)^{\frac{d-3}{2}}}{(M_1M_2M_3)^3}b_1, \tag{24}$$

$$z_{j+1} = -\frac{1}{8}\frac{S_{d-1}}{(2\pi)^d(d^2-1)}\int_{-1}^{1}dx\frac{(1-x^2)^{\frac{d-3}{2}}}{(M_1M_2M_3)^3}\frac{b_{j+1}}{\chi_j}, \tag{25}$$

for $j=1,2,3$. Here $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface area of the d-dimensional unit sphere, $\Gamma(x)$ represents the Euler's Gamma function, and $M_i, i=1,2,3$ are defined as follows:

$$M_1 = 2(1+\chi_1 x^2)+(\chi_2+\chi_3 x^2)(1-x^2), \tag{26}$$

$$M_2 = 1+\chi_1 x^2+(\chi_2+\chi_3 x^2)(1-x^2), \tag{27}$$

$$M_3 = 1+\chi_1 x^2, \tag{28}$$

and the explicit form of huge coefficients b_i, $i=1,\ldots,4$ can be found in Appendix I in Ref. [18]. Note also that the integration variable x in Eqs. (24) and (25) is the cosine of the angle between the unit vector of the uniaxial anisotropy $\mathbf{n}$ and the momentum $\mathbf{k}$ over which the integration is performed in the framework of the one-loop calculations.

On the other hand, the one-loop coefficients z_5 and z_6 for the renormalization constants Z_5 and Z_6 in Eq. (20), which are related to the 1-irreducible Green's function $\langle\theta'\theta\rangle_{1-ir}$ of the advected scalar field, are unknown and therefore must be calculated.

First of all, it is necessary to find an analytical expression for the self-energy operator $\Sigma^{\theta'\theta}$ defined by the corresponding one-loop Feynman diagram shown in Fig. 3. It can be written in the following form:

$$\Sigma^{\theta'\theta}(p) = -\frac{g\nu}{2\varepsilon}\frac{S_{d-1}}{(2\pi)^d(d-1)}[p^2 a_5+(\mathbf{p}\cdot\mathbf{n})^2 a_6], \tag{29}$$

where the coefficients a_5 and a_6 have the following integral representation:

$$a_i = \int_{-1}^{1} dx \frac{(1-x^2)^{\frac{d-3}{2}}}{2N_1N_2M_1M_2M_3} b_i, \quad i = 5,6, \tag{30}$$

where

$$\begin{aligned} b_5 = {} & (d-1)M_1M_2N_1R_1 \\ & -(1-x^2)\{M_2R_1[M_2(M_2+M_3)+M_1N_3] \\ & +[R_1R_2(M_2+M_3)(M_2+N_2) \\ & +\alpha_2M_3N_2(R_2-2M_2)]x^2-\alpha_2M_3N_2R_2x^4\}, \end{aligned} \tag{31}$$

$$\begin{aligned} b_6 = {} & \alpha_2M_3(1-x^2)[d(1-x^2)-1] \\ & \times[2M_2N_2-R_2(M_3+N_3)(1-x^2)] \\ & +R_1(M_1M_2(1-dx^2)[M_2+N_3-R_2(2-x^2)] \\ & +R_2(M_2+M_3)\{-M_2(d-3+2x^2) \\ & -(M_3+N_3)(1-x^2)\{d(1-x^2)-1]\}), \end{aligned} \tag{32}$$

and

$$N_1 = M_2 + u(1+\tau x^2), \tag{33}$$

$$N_2 = M_3 + u(1+\tau x^2), \tag{34}$$

$$N_3 = u(1+\tau x^2), \tag{35}$$

$$R_1 = 1 + \alpha_1 x^2, \tag{36}$$

$$R_2 = \chi_2 + \chi_3 x^2. \tag{37}$$

At the same time, $M_i, i = 1,2,3$ are defined in Eqs. (26)–(28).

Result (29) for the one-loop self-energy operator $\Sigma^{\theta'\theta}$ together with the Dyson equation (22) gives the final explicit expressions for the one-loop coefficients z_5 and z_6 of the corresponding renormalization constants Z_5 and Z_6:

$$z_5 = -\frac{1}{4u}\frac{S_{d-1}}{(2\pi)^d(d-1)}\int_{-1}^{1} dx \frac{(1-x^2)^{\frac{d-3}{2}}}{M_1M_2M_3N_1N_2} b_5, \tag{38}$$

$$z_6 = -\frac{1}{4u}\frac{S_{d-1}}{(2\pi)^d(d-1)}\int_{-1}^{1} dx \frac{(1-x^2)^{\frac{d-3}{2}}}{M_1M_2M_3N_1N_2} \frac{b_6}{\tau}, \tag{39}$$

When studying fully developed turbulent systems one is usually interested in universal properties of various quantities related to the existence of the corresponding scaling regime. Therefore, to proceed further, the answer on the question about the existence and stability of the Kolmogorov scaling regime within the inertial interval of the present model must be found. It is a nontrivial question in the framework of the models when a uniaxial small-scale anisotropy is present because the stability of the scaling regime can be destroyed when the uniaxial anisotropy defined by the anisotropy parameters has special forms. In this respect, for example, in Ref. [18] a detail mathematical analysis of the influence of the strong small-scale uniaxial anisotropy on the stability of the Kolmogorov regime was performed. It was shown that there is a large region of the values of anisotropy parameters for which the stable Kolmogorov regime exists which includes phenomenologically most interesting region of relatively small absolute values of the anisotropy parameters, i.e., the region close to the isotropic case. The question is if the same can be said in the framework of the enlarged model under consideration. Let us discussed it in detail.

We start with the important conclusion that all fields of the model, namely, $\mathbf{v},\mathbf{v}',\theta$, and θ' are not renormalized in the framework of the present model. This fact means that arbitrary unrenormalized, e.g., connected, correlation function, i.e., written in unrenormalized variables, must be equal to its renormalized counterpart, which is written in renormalized variables. One can therefore write

$$W^R(g,u,\nu,\chi_i,\tau,\mu,\dots) = W(g_0,u_0,\nu_0,\chi_{i0},\tau_0,\dots), \tag{40}$$

where we have denoted as W and W^R the unrenormalized and renormalized counterparts of an arbitrary correlation function. In addition, the dots represent all other variables which are not influenced by renormalization (for example, anisotropy parameters α_1 and α_2, coordinates and times). Now, starting from relation (40), by applying on both sides differential operator $\mu\partial_\mu$ taken at fixed values of the unrenormalized parameters, it is then an easy task to derive the basic differential RG equation which must be satisfied by W^R. One obtains

$$\left(\mu\partial_\mu + \sum_{i=g,u,\chi_j,\tau}\beta_i\partial_i - \gamma_\nu\nu\partial_\nu\right)W^R = 0, \tag{41}$$

where $\beta_i, i = g,u,\chi_j,\tau, j = 1,2,3$, and γ_ν are the corresponding RG functions called, in general, β and γ functions, respectively [7,47]. Using definitions of the renormalization constants in Eqs. (15) and (16) together with relations among them, which are shown in Eqs. (18) and (19), one obtains

$$\beta_g \equiv \mu\partial_\mu g = g(-2\varepsilon + 3\gamma_1), \tag{42}$$

$$\beta_{\chi_i} \equiv \mu\partial_\mu \chi_i = \chi_i(\gamma_1 - \gamma_{i+1}), \quad i = 1,2,3 \tag{43}$$

$$\beta_u \equiv \mu\partial_\mu u = u(\gamma_1 - \gamma_5), \tag{44}$$

$$\beta_\tau \equiv \mu\partial_\mu \tau = \tau(\gamma_5 - \gamma_6), \tag{45}$$

where

$$\gamma_i \equiv \mu\partial_\mu \ln Z_i, \quad i = 1,\dots,6. \tag{46}$$

Note that γ_ν in Eqs. (41) is equal to γ_1 due to relation $Z_\nu = Z_1$ [see the corresponding equation in Eq. (18)]. In the one-loop approximation, from Eq. (20) one immediately comes to the following final compact result:

$$\gamma_i = -2gz_i, \quad i = 1,\dots,6, \tag{47}$$

where $z_i, i = 1,\dots,6$ are given by the integrals in Eqs. (24), (25), (38), and (39). Using this result one can also write all β functions of the model directly in terms of the coefficients z_i of the renormalization constants Z_i:

$$\beta_g \equiv \mu\partial_\mu g = -2g(\varepsilon + gz_1), \tag{48}$$

$$\beta_{\chi_i} \equiv \mu\partial_\mu \chi_i = 2g\chi_i(z_{i+1} - z_1), \quad i = 1,2,3, \tag{49}$$

$$\beta_u \equiv \mu\partial_\mu u = 2gu(z_5 - z_1), \tag{50}$$

$$\beta_\tau \equiv \mu\partial_\mu \tau = 2g\tau(z_6 - z_5). \tag{51}$$

The properties of the β functions are crucial for the existence of a stable IR scaling regime in the model, which is driven by the corresponding IR stable fixed point of the RG equations [7,47]. The coordinates of possible fixed points are given by the requirement of simultaneous vanishing of all β functions of the model:

$$\beta_i(g_*,\chi_{j*},u_*,\tau_*;\alpha_1,\alpha_2,d,\varepsilon) = 0, \quad i = g,\chi_j,u,\tau, \tag{52}$$

where $j = 1,2,3$ and variables with star represent coordinates of the fixed point. On the other hand, the fixed point is IR stable if and only if all eigenvalues of the matrix of the first derivatives of the β functions have positive real parts.

This technique was used in Ref. [21] for finding the IR stable fixed point of the studied model in the limit of weak small-scale anisotropy. It was shown that weak anisotropy, which is characterized by the assumption that the absolute values of the anisotropy parameters are small enough ($|\alpha_i| \ll 1, i = 1,2$) to perform the linear approximation of the problem in respect to the anisotropy parameters α_1 and α_2, has no impact on the stability of the Kolmogorov scaling regime for $d = 3$ and for the physical value $\varepsilon = 2$. As it follows from Ref. [18], the situation can be different in the case when unrestricted anisotropic model is investigated. However, in this case, as was discussed above, the β functions (42)–(45) contain integrals, therefore it is impossible to find a fixed point by direct solving the system of Eqs. (52). Here another possibility exists, namely, to solve directly the system of differential equations (also called the flow equations) for effective (running) variables $\bar{g},\bar{\chi}_i,\bar{u},\bar{\tau}$, $i = 1,2,3$ as functions of the scale parameter $t = k/\Lambda$. In our case the flow equations read

$$t\frac{d\bar{g}}{dt} = \beta_g(\bar{g},\bar{\chi}_j;\alpha_1,\alpha_2,d,\varepsilon), \tag{53}$$

$$t\frac{d\bar{\chi}_i}{dt} = \beta_{\chi_i}(\bar{g},\bar{\chi}_j;\alpha_1,\alpha_2,d,\varepsilon), \quad i = 1,2,3, \tag{54}$$

$$t\frac{d\bar{u}}{dt} = \beta_u(\bar{g},\bar{\chi}_j,\bar{u},\bar{\tau};\alpha_1,\alpha_2,d,\varepsilon), \tag{55}$$

$$t\frac{d\bar{\tau}}{dt} = \beta_\tau(\bar{g},\bar{\chi}_j,\bar{u},\bar{\tau};\alpha_1,\alpha_2,d,\varepsilon), \tag{56}$$

where $j = 1,2,3$, the initial conditions are taken in $t = 1$, and the IR stable fixed point (if exists) is obtained in the limit $t \to 0$, i.e., $\{\bar{g},\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3,\bar{u},\bar{\tau}\}|_{t\to 0} = \{g_*,\chi_{1*},\chi_{2*},\chi_{3*},u_*,\tau_*\}$.

The technique used for solving the system of the six differential equations (53)–(56) is analogous to the technique used for investigation of the pure turbulent problem with the presence of the uniaxial anisotropy in Ref. [18], which is, in fact, described by the closed system of first four equations (53) and (54). In Ref. [18] the technique is discussed in detail, therefore it is not necessary to repeat it here.

From the numerical analysis of the system of differential equations (53)–(56) it follows that the region of the anisotropy parameters α_1 and α_2 where the stable Kolmogorov scaling regime exists is the same as in the case of the pure turbulent system described by the anisotropic stochastic Navier-Stokes equation [18]. It means that the presence of the passively advected scalar field in such turbulent environment does not change the stability of the Kolmogorov regime. In fact, it is an expected result because an arbitrary passive quantity should not have any impact on the fundamental properties of a given turbulent flow by which it is advected. In Fig. 4 the region of the stability of the Kolmogorov scaling regime is shown for the spatial dimension $d = 3$ and the physical value $\varepsilon = 2$. As follows from Fig. 4 the Kolmogorov scaling regime does not exist for simultaneously very small values of the parameters α_1 and α_2, i.e., when they are both very close to -1, and for the large enough values of α_1 and relatively small values of α_2. For our further analysis it is, however, important that there exists rather large region around the isotropic point $\alpha_1 = \alpha_2 = 0$ in all anisotropic directions where the stable IR scaling regime takes place (see Fig. 4).

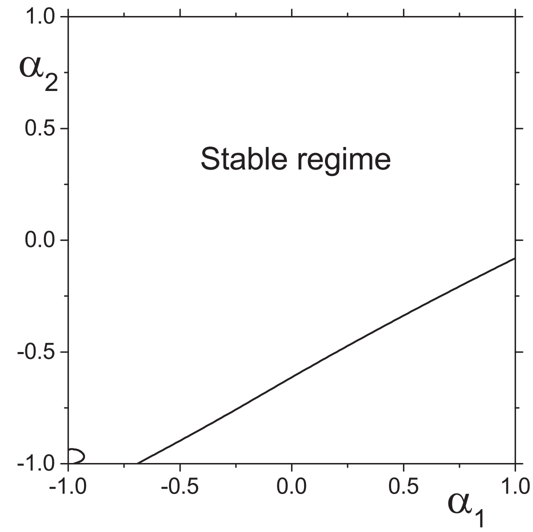

FIG. 4. The region of the stability of the Kolmogorov scaling regime in the plane α_1-α_2 for three-dimensional case $d = 3$ and $\varepsilon = 2$.

In the case, when the Kolmogorov regime exists, i.e., when the anisotropy parameters α_1 and α_2 have values from the region where the stable IR scaling regime takes place, the physics of the model under consideration is driven by the corresponding fixed point values of the running parameters of the model. For example, various correlation functions of the model exhibit well-defined scaling behavior within the inertial interval with appropriate critical dimensions. This is, however, out of scope of the present study.

It is also worth mentioning that the form of all β functions of the model given in Eqs. (48)–(51) is independent of the order of the approximation, i.e., their form is completely determined already at the one-loop level of approximation. It means that every time when the stable scaling regime exists all the γ function of the model, which are usually called the anomalous dimensions, are uniquely defined at the fixed point:

$$\gamma_i^* = \frac{2\varepsilon}{3}, \quad i = 1,\ldots,6. \tag{57}$$

For the physical value $\varepsilon = 2$ they lead to the famous Kolmogorov 2/3 scaling law for the second order correlation function of the velocity field.

Finally, let us note that using the RG technique means that in fact we do not work in the ordinary perturbation theory but rather in the framework of an "improved" perturbation theory with the corresponding RG resummation. In our case it means, e.g., that although we work in the framework of the leading order of the ordinary perturbation theory, i.e., all RG functions of the model (β and γ functions) are found in the one-loop approximation, the subsequent exact solving the RG equations for the correlation functions [see Eq. (41)] will lead to results which take into account the leading contributions given by all orders of the perturbation expansion [7,47].

IV. ANISOTROPY AND TURBULENT PRANDTL NUMBER

A. Anisotropy and turbulent Prandtl number in spatial dimension $d = 3$

The main aim of the present paper is to find the dependence of the turbulent Prandtl number of passively advected scalar field on the uniaxial small-scale anisotropy of the turbulent environment described by the anisotropy parameters α_1 and α_2 discussed in the previous sections. In the framework of the one-loop approximation the turbulent Prandtl number is directly given by the inverse fixed point value of the parameter u:

$$\mathrm{Pr_t}(\alpha_1,\alpha_2,d,\varepsilon) = 1/u_*(\alpha_1,\alpha_2,d,\varepsilon), \tag{58}$$

where we have shown explicitly all parameters of the model which it depends on. Note that it has sense to talk about the well-defined universal turbulent Prandtl number only for the values of the parameters for which the stable scaling regime exists (see the previous section).

However, the definition of the turbulent Prandtl number given in Eq. (58) in the anisotropic case needs some further comment. Strictly speaking, the quantity defined in Eq. (58) represents in fact the well-defined universal "isotropic" (independent of direction) part of the total anisotropic turbulent Prandtl number the value of which must depend on direction. On the other hand, the theoretical definition of the total anisotropic turbulent Ptandtl number is a nontrivial task. For its determination it is necessary to know the total anisotropic turbulent coefficient of diffusion as well as the total anisotropic turbulent viscosity values of which dependent nontrivially on the direction and must be constructed in some well-defined way from the corresponding renormalized diffusion coefficients νu and $\nu u \tau$ and the corresponding renormalized coefficients of viscosity $\nu, \nu\chi_1, \nu\chi_2$, and $\nu\chi_3$ defined, e.g., in action functional (17). Therefore, the question of the total anisotropic turbulent Prandtl number is left for future studies, and here we concentrate our attention to the analysis of the main universal part of the turbulent anisotropic Prandtl number (58) defined by the ratio of the "isotropic" part of the turbulent viscosity ν to the "isotropic" part of the turbulent diffusivity νu, which is uniquely defined and which represents the main contribution to the total anisotropic turbulent Prandtl number, at least, for relatively small absolute values of the anisotropy parameters α_1 and α_2. In what follows we shall call it simply the turbulent Prandtl number.

Let us also note that the turbulent Prandtl number in the isotropic case ($\alpha_1 = \alpha_2 = 0$) was recently calculated up to the second order of the perturbation expansion [8,9] where it was shown that the value of the turbulent Prandtl number is quite surprisingly perturbatively very stable. Namely, it was found that the two-loop isotropic value of the turbulent Prandtl number is $\mathrm{Pr_t}^{(2)} = 0.7040$, which is very close to the corresponding one-loop value $\mathrm{Pr_t}^{(1)} = 0.7179$. Thus, the relative difference between these two values is less than 2%. It means that, as for the investigation of the turbulent Prandtl number of a passively advected scalar quantity, the one-loop approximation seems to be quite accurate. This is a nontrivial conclusion which is crucial for the relevance of the results obtained in the present paper in the framework of the one-loop approximation in the anisotropic case. Here it is also important to stress that the small-scale anisotropy problems in the two-loop approximation are so complicated that, at least at the moment, there does not exist any two-loop calculation even in the weak small-scale anisotropy limit.

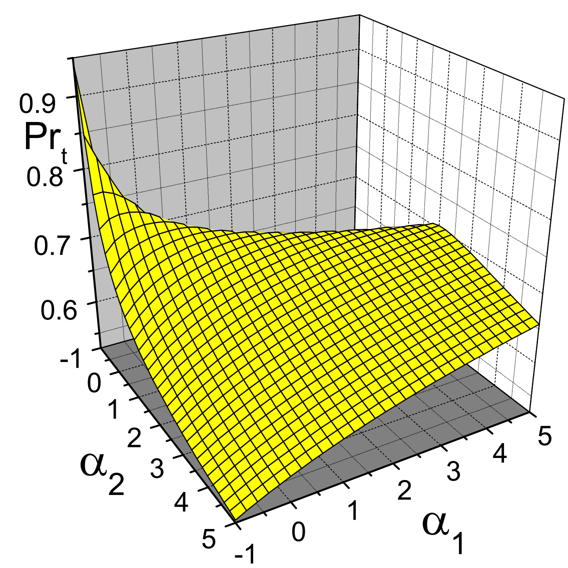

FIG. 5. The turbulent Prandtl number $\mathrm{Pr_t}$ as the function of the anisotropy parameters α_1 and α_2 for $d = 3$ and $\varepsilon = 2$.

In Fig. 5 the dependence of the turbulent Prandtl number on the anisotropy parameters α_1 and α_2 is shown explicitly for the spatial dimension $d = 3$ and physical value $\varepsilon = 2$. It is evident that the turbulent Prandtl number rather strongly depends on the form of the uniaxial anisotropy especially for small values of α_1 when α_2 increases and in the case when both anisotropy parameters are simultaneously very small, i.e., when they both are closed to the value -1. However, while in the first case the turbulent Prandtl number decreases significantly, in the second case it increases deeper into the experimentally estimated interval of its values [11,12].

It is also interesting that when the anisotropy parameters are increasing simultaneously, i.e., when $\alpha_1 = \alpha_2$, the value of the turbulent Prandtl number decreases slowly and it obtains the value $\mathrm{Pr_t} = 0.57848$ in the limit $\alpha_1 = \alpha_2 \to \infty$. On the other hand, when the anisotropy parameters decrease simultaneously, the Prandtl number increases and obtains its maximal value $\mathrm{Pr_t} = 0.93866$ for $\alpha_1 = \alpha_2 = -0.944$. Note that for lower values of $\alpha_1 = \alpha_2 < -0.944$ the stable IR scaling regime does not exist (see Fig. 4). It means that, in this special case, the possible values of the turbulent Prandtl number in the one-loop approximation are restricted to relatively small interval $0.93866 \geqslant \mathrm{Pr_t} \geqslant 0.57848$. The

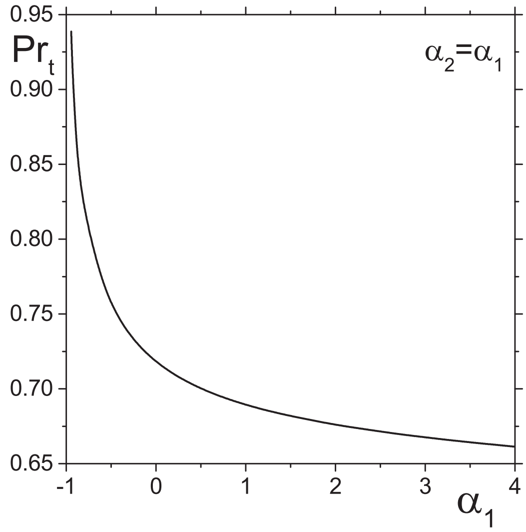

FIG. 6. The turbulent Prandtl number $\mathrm{Pr_t}$ as the function of the anisotropy parameters $\alpha_1 = \alpha_2$ for three-dimensional case $d = 3$ and $\varepsilon = 2$.

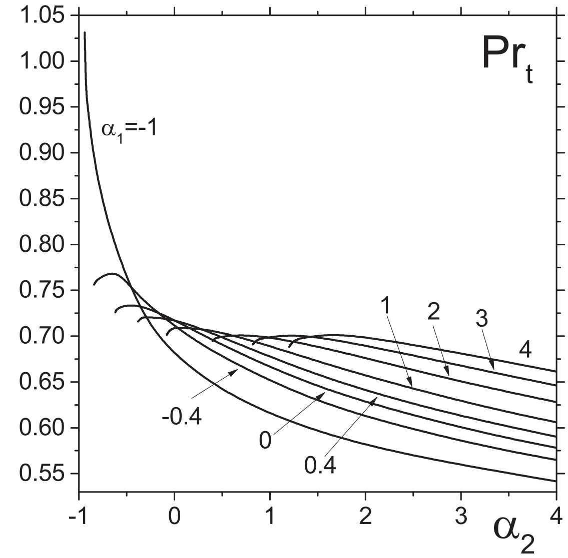

FIG. 8. The turbulent Prandtl number $\mathrm{Pr_t}$ as the function of the anisotropy parameter α_2 for various values of the parameter α_1 for $d = 3$ and $\varepsilon = 2$.

dependence of the turbulent Prandtl number on the parameters $\alpha_1 = \alpha_2$ for $d = 3$ and $\varepsilon = 2$ is shown in Fig. 6.

In addition, in Figs. 7 and 8 the behavior of the turbulent Prandtl number is shown as the function of the parameter α_1 for various values of α_2 and as the function of α_2 for various values of α_1, respectively. The fact that some curves are restricted in these figures is related to the existence of regions where the stable Kolmogorov scaling regime does not exist (see Fig. 4). The values of the turbulent Prandtl number for various values of the anisotropy parameters for $d = 3$ and $\varepsilon = 2$ are shown explicitly in Table I (see the corresponding first values in Table I).

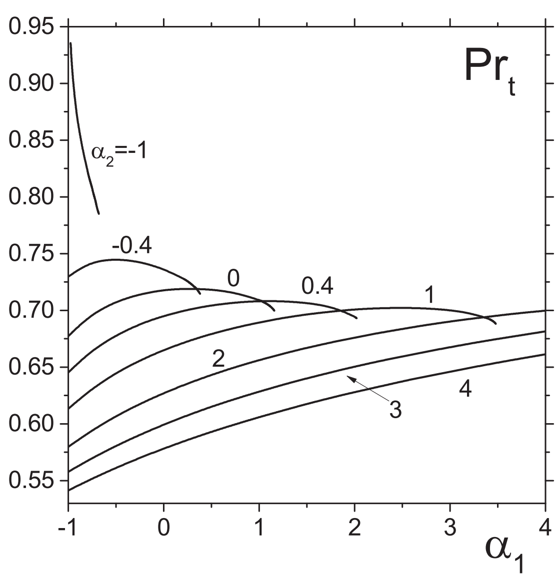

FIG. 7. The turbulent Prandtl number $\mathrm{Pr_t}$ as the function of the anisotropy parameter α_1 for various values of the parameter α_2 for $d = 3$ and $\varepsilon = 2$.

B. Turbulent Prandtl number in the weak anisotropy limit

It is also instructive to compare our results for the turbulent Prandtl number obtained in the framework of unrestricted small-scale uniaxial anisotropy with the corresponding results obtained in the framework of the so-called weak anisotropy limit studied in Ref. [21]. This comparison can tell us important information about the relevance of the weak small-scale anisotropy approximation in the framework of various turbulent problems (see, e.g., Ref. [21] where the influence of the weak small-scale uniaxial anisotropy on the anomalous scaling of the single-time structure functions of the passively advected scalar field by the Navier-Stokes turbulence was studied). For this purpose, we have also calculated the turbulent Prandtl number in the weak anisotropy limit, i.e., in the case when the absolute values of the anisotropy parameters are small enough, namely, we suppose that $|\alpha_i| \leqslant 0.2, i = 1,2$ as was also supposed in Ref. [21]. In Fig. 9 the dependence of the turbulent Prandtl number is shown as the function of the anisotropy parameters α_1 and α_2 in the weak anisotropy limit of the studied model. As follows from this figure the corresponding dependence is exactly linear as it is expected in the approximation when only linear parts of all expressions with respect to anisotropy parameters are taken into account (see Ref. [21] for details). On the other hand, in Fig. 10 the same dependence of the turbulent Prandtl number on the anisotropy parameters is shown in the case when no approximation is performed. It is evident that, in this case, the corresponding dependence of the turbulent Prandtl number on the anisotropy parameters is nonlinear, especially in the direction $\alpha_1 = -\alpha_2$. Nevertheless, by comparing these two figures one can conclude that the weak anisotropy approximation works quite well for small enough anisotropy parameters.

The fact that the weak small-scale anisotropy approximation works with very good accuracy for region of the anisotropy parameters defined by inequalities $|\alpha_i| \leqslant 0.2, i = 1,2$ is also demonstrated in Table II, where the turbulent Prandtl

TABLE I. The turbulent Prandtl number for various values of the anisotropy parameters α_1 and α_2 for $\varepsilon = 2$ and for the spatial dimension $d = 3$ (the first values), $d = 4$ (the second values), and $d = 5$ (the third values).

	$\alpha_2 = -0.8$	$\alpha_2 = -0.4$	$\alpha_2 = 0.0$	$\alpha_2 = 0.4$	$\alpha_2 = 0.8$	$\alpha_2 = 1.2$	$\alpha_2 = 1.6$	$\alpha_2 = 2.0$
$\alpha_1 = -0.8$	0.8183	0.7412	0.6936	0.6613	0.6376	0.6191	0.6041	0.5917
	0.8190	0.7789	0.7557	0.7399	0.7280	0.7187	0.7110	0.7045
	0.8337	0.8089	0.7950	0.7854	0.7782	0.7724	0.7676	0.7636
$\alpha_1 = -0.4$	0.7624	0.7447	0.7107	0.6823	0.6593	0.6405	0.6247	0.6113
	0.8096	0.7834	0.7630	0.7476	0.7355	0.7258	0.7176	0.7107
	0.8321	0.8124	0.7992	0.7895	0.7820	0.7760	0.7709	0.7666
$\alpha_1 = 0.0$	–	0.7365	0.7179	0.6951	0.6744	0.6564	0.6407	0.6271
	0.8023	0.7849	0.7676	0.7533	0.7415	0.7316	0.7233	0.7161
	0.8304	0.8146	0.8023	0.7929	0.7853	0.7791	0.7739	0.7694
$\alpha_1 = 0.4$	–	–	0.7190	0.7030	0.6853	0.6687	0.6537	0.6402
	0.7965	0.7851	0.7706	0.7575	0.7463	0.7366	0.7282	0.7209
	0.8289	0.8159	0.8047	0.7956	0.7882	0.7819	0.7766	0.7720
$\alpha_1 = 0.8$	–	–	0.7144	0.7073	0.6932	0.6784	0.6643	0.6513
	0.7917	0.7845	0.7726	0.7608	0.7502	0.7408	0.7326	0.7252
	0.8275	0.8167	0.8065	0.7979	0.7906	0.7844	0.7790	0.7744
$\alpha_1 = 1.2$	–	–	–	0.7083	0.6988	0.6861	0.6731	0.6608
	0.7878	0.7835	0.7738	0.7633	0.7534	0.7445	0.7364	0.7291
	0.8263	0.8172	0.8080	0.7998	0.7928	0.7866	0.7813	0.7766
$\alpha_1 = 1.6$	–	–	–	0.7055	0.7022	0.6921	0.6806	0.6690
	0.7854	0.7824	0.7746	0.7653	0.7561	0.7476	0.7397	0.7326
	0.8253	0.8175	0.8091	0.8014	0.7946	0.7886	0.7833	0.7786
$\alpha_1 = 2.0$	–	–	–	0.6948	0.7035	0.6967	0.6868	0.6961
	–	0.7811	0.7749	0.7668	0.7584	0.7503	0.7427	0.7357
	0.8244	0.8176	0.8101	0.8028	0.7963	0.7904	0.7852	0.7805

numbers $\mathrm{Pr_t}$ obtained in the framework of the strong uniaxial anisotropy studied in the present paper are compared with the corresponding values of the turbulent Prandtl number $\mathrm{Pr_t^{(w)}}$ obtained in the framework of the weak (linear) anisotropy approximation [21]. In addition, to make this fact more evident, we have also calculated the relative difference ϵ between them, which is defined as

$$\epsilon = \left| \frac{\mathrm{Pr_t^{(w)}} - \mathrm{Pr_t}}{\mathrm{Pr_t}} \right|, \tag{59}$$

and which is also present in Table II. It follows from Table II that the weak anisotropy approximation is in fact very precise for small absolute values of the anisotropy parameters, and, in this case, it can be surely used for qualitative as well as quantitative analysis of problems with the presence of anisotropy. However, it is important to bear in mind that the weak anisotropy approximation loses precision very quickly when the anisotropy parameters get larger and already for $|\alpha_i| \sim 0.5, i = 1,2$ the results obtained in the framework of

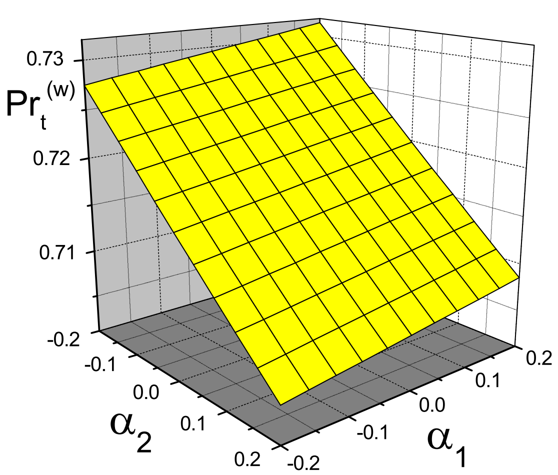

FIG. 9. The turbulent Prandtl number $\mathrm{Pr_t}$ as the function of the anisotropy parameters $|\alpha_i| \leqslant 0.2, i = 1,2$ for $d = 3$ and $\varepsilon = 2$ in the framework of the so-called weak anisotropy approximation [21].

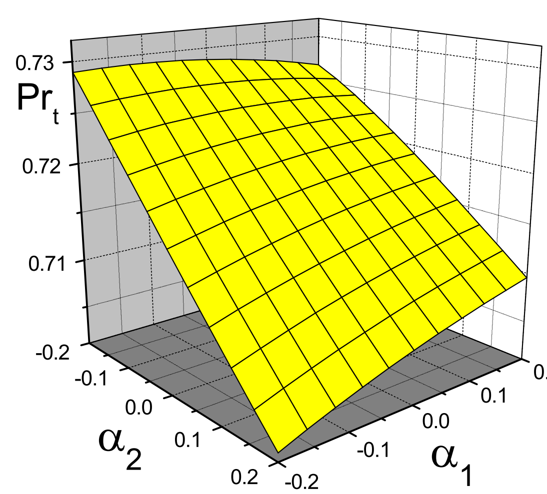

FIG. 10. The turbulent Prandtl number $\mathrm{Pr_t}$ as the function of the anisotropy parameters $|\alpha_i| \leqslant 0.2, i = 1,2$ for $d = 3$ and $\varepsilon = 2$ without any approximation.

TABLE II. The turbulent Prandtl number for various relatively small values of the anisotropy parameters $|\alpha_i| \leqslant 0.2, i = 1,2$ for spatial dimension $d = 3$ and $\varepsilon = 2$. The first values represent the turbulent Prandtl numbers in the framework of the unrestricted model studied in the present paper (see Fig. 10), the second values which are placed in the parentheses are the values of the corresponding turbulent Prandtl numbers in the weak anisotropy approximation (see Fig. 9), and the third numbers correspond to the relative differences $\epsilon \times 10^2[\%]$ between these two values of the turbulent Prandtl numbers, which are defined in Eq. (59).

	$\alpha_2 = -0.2$	$\alpha_2 = -0.1$	$\alpha_2 = 0.0$	$\alpha_2 = 0.1$	$\alpha_2 = 0.2$
$\alpha_1 = -0.2$	0.7290	0.7221	0.7152	0.7084	0.7019
	(0.7275)	(0.7217)	(0.7160)	(0.7103)	(0.7048)
	0.21%	0.06%	0.11%	0.27%	0.41%
$\alpha_1 = -0.1$	0.7291	0.7230	0.7167	0.7105	0.7043
	(0.7285)	(0.7227)	(0.7169)	(0.7113)	(0.7057)
	0.08%	0.04%	0.03%	0.11%	0.20%
$\alpha_1 = 0.0$	0.7288	0.7235	0.7179	0.7121	0.7064
	(0.7295)	(0.7237)	(0.7179)	(0.7122)	(0.7066)
	0.10%	0.03%	0.00%	0.01%	0.03%
$\alpha_1 = 0.1$	0.7280	0.7236	0.7187	0.7135	0.7081
	(0.7305)	(0.7246)	(0.7189)	(0.7132)	(0.7076)
	0.34%	0.14%	0.03%	0.04%	0.07%
$\alpha_1 = 0.2$	0.7268	0.7233	0.7191	0.7145	0.7096
	(0.7315)	(0.7256)	(0.7198)	(0.7141)	(0.7085)
	0.65%	0.32%	0.10%	0.06%	0.16%

the weak anisotropy approximation cannot be considered as useful and valid.

C. Anisotropy and turbulent Prandtl number in higher spatial dimensions: $d = 4$ and $d = 5$

Finally, let us also analyze briefly the influence of the small-scale anisotropy on the turbulent Prandtl number of the studied model in higher spatial dimensions. Although the main purpose for this investigation is purely theoretical, nevertheless it can give us a nontrivial information about universality (or nonuniversality) of the behavior of the turbulent Prandtl number as the function of the anisotropy parameters independent of (or dependent on) the dimensionality of the studied problem. Here let us remind that the technique used in the present paper allows one to study the model in spatial dimensions $d > 2$ only.

In what follows, we concentrate our attention only on the models with spatial dimensions $d = 4$ and $d = 5$. The first question, which we must find answer to, is related to the stability of the corresponding scaling regime in the presence of the small-scale anisotropy. The regions of stability of the Kolmogorov scaling regime in the plane α_1-α_2 for models with spatial dimensions $d = 4$ and $d = 5$ for $\varepsilon = 2$ are shown in Fig. 11, where they are also compared with the corresponding region of stability of the Kolmogorov scaling regime for the case $d = 3$ (see also Fig. 4). From Fig. 11 it is evident that the region of stability of the scaling regime significantly increases with increasing integer value of the spatial dimension.

The explicit dependence of the turbulent Prandtl number on the anisotropy parameters α_1 and α_2 for $\varepsilon = 2$ and for $d = 4$

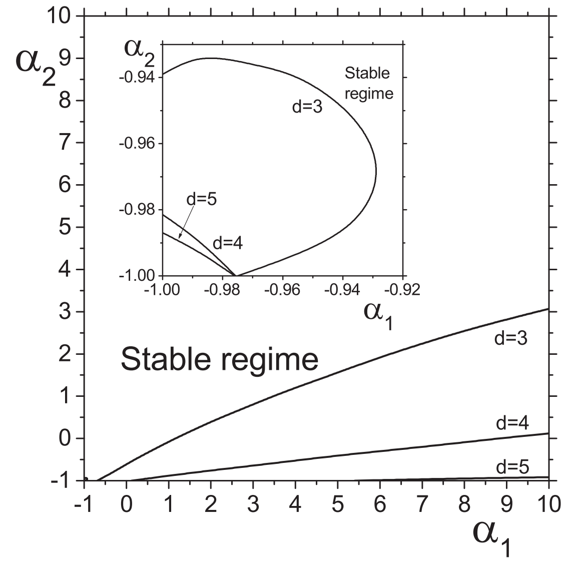

FIG. 11. The region of the stability of the Kolmogorov scaling regime in the plane α_1-α_2 for the model with spatial dimensions $d = 3,4$, and 5 and for $\varepsilon = 2$.

and $d = 5$ is shown in Figs. 12 and 13, respectively. Looking at Figs. 5, 12, and 13 we can conclude that, despite the fact that the numerical values of the turbulent Prandtl number for given values of the anisotropy parameters are different for different values of the spatial dimension, the general properties of its behavior, as the function of the anisotropy parameters, are the same. This behavior can be also seen in detail in Fig. 14, where the turbulent Prandtl numbers for $d = 3,4$, and 5 are shown explicitly as functions of the parameters $\alpha_1 = \alpha_2$ for $\varepsilon = 2$. In addition, the comparison of the numerical values of the turbulent Prandtl number for various values of the anisotropy parameters in spatial dimensions $d = 3,4$, and 5 can be found in Table I. From Table I it is evident that the value of the turbulent Prandtl number for given values of the anisotropy

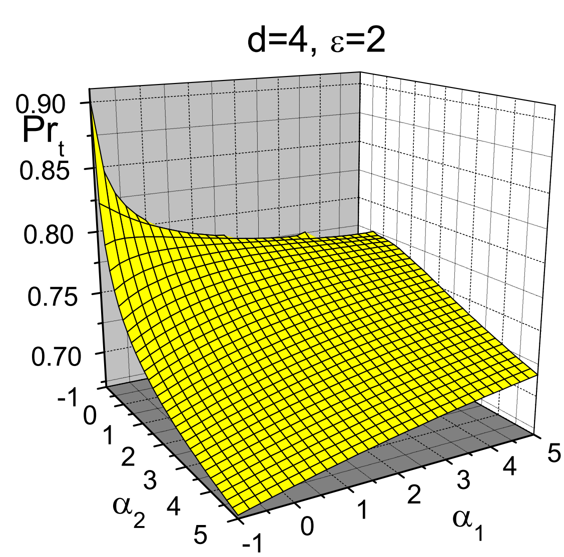

FIG. 12. The turbulent Prandtl number $\mathrm{Pr_t}$ as the function of the anisotropy parameters α_1 and α_2 for $d = 4$ and $\varepsilon = 2$.

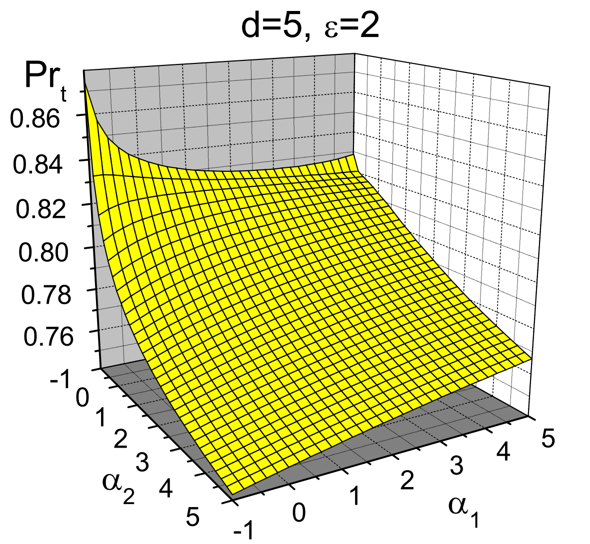

FIG. 13. The turbulent Prandtl number $\mathrm{Pr_t}$ as the function of the anisotropy parameters α_1 and α_2 for $d = 5$ and $\varepsilon = 2$.

parameters increases with increasing the value of the spatial dimension.

Thus, we can conclude that the influence of the anisotropy on the turbulent Prandtl number of the passively advected scalar quantity is known now. However, a few intriguing questions immediately arise here which need further investigations. The first nontrivial question is related to the analogous problem of the passively advected weak magnetic field in the framework of the kinematic magnetohydrodynamic turbulence. It is known that the corresponding turbulent magnetic Prandtl number is equal to the turbulent Prandtl number of a passively advected scalar field if fully symmetric and incompressible turbulent environments are considered. This is true, at least, up to the two-loop approximation [13].

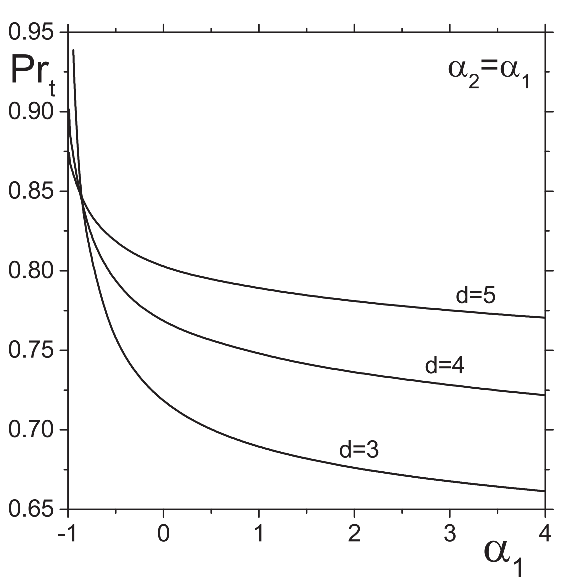

FIG. 14. The dependence of the turbulent Prandtl number $\mathrm{Pr_t}$ on the anisotropy parameters $\alpha_1 = \alpha_2$ for spatial dimensions $d = 3$,4, and 5 and for $\varepsilon = 2$.

On the other hand, it is also known that when spatial parity is violated in these systems, i.e., the systems are helical, then the corresponding turbulent Prandtl numbers become different; i.e., it seems that there is a significant difference between diffusion processes related to the passively advected scalar and vector fields in helical turbulent environments [14]. Thus, the question is whether the same is valid when the corresponding turbulent environments are anisotropic, e.g., when the small-scale uniaxial anisotropy is present in the systems as is studied in the present paper. To answer this question correctly the anisotropically driven kinematic magnetohydrodynamic turbulence must be investigated in the similar way.

Another nontrivial question is related, e.g., to the role of the higher-loop corrections to the anisotropic turbulent Prandtl number. Here it would be quite enough to have leastwise the weak small-scale anisotropy results in the framework of the two-loop approximation. However, even in this significantly simplified case the calculations are too complicated and were not performed yet. These questions and many others are left for future studies.

V. CONCLUSION

In the present paper, we have used the field theoretic RG technique in the leading order of approximation in the framework of the perturbation theory to investigate the turbulent Prandtl number related to the process of a passive advection of a scalar field by the turbulent velocity field described by the anisotropically driven stochastic Navier-Stokes equation. The anisotropy is taken to be uniaxial and small-scale, i.e., it is supposed that it is present at all scales of the studied turbulent problem. At the same time, the field theoretic RG technique is used in the form which allows one to study the problem simultaneously for all spatial dimensions $d > 2$. A detailed analysis of the influence of the presence of the uniaxial anisotropy of the turbulent environment on the turbulent Prandtl number is performed for the most interesting spatial case $d = 3$ as well as for turbulent systems with $d = 4$ and $d = 5$.

First of all, the influence of the uniaxial small-scale anisotropy on the stability of the Kolmogorov scaling regime is studied. It is shown that there exist two separated regions in the plane of the anisotropy parameters in which the stability of the Kolmogorov scaling regime is destroyed. However, an important fact is that there exists a rather large region around the isotropic point in all anisotropic directions where the stable IR scaling regime takes place for all integer spatial dimensions $d > 2$ (see Figs. 4 and 11). At the same time, the region of the stability of the Kolmogorov scaling regime as the function of the anisotropy parameters enlarges sufficiently when the spatial dimension of the problem increases (see Fig. 11).

Further, the influence of the uniaxial small-scale anisotropy on the value of the turbulent Prandtl number is studied in detail in the region where the Kolmogorov scaling regime is stable for spatial dimension $d = 3$ and physical value $\varepsilon = 2$ of the parameter which drives the infrared form of the energy pumping into the system. It follows from our analysis that the anisotropically driven turbulent environments can have a rather significant impact on the rate of the corresponding diffusion processes, i.e., that the value of the turbulent Prandtl number depends quite significantly on the form as well as on the strength of the anisotropy which is represented by the values of the anisotropy parameters (see Figs. 5–8).

In addition, we have also performed the comparison between the obtained results for the turbulent Prandtl number and those obtained in the framework of the so-called weak uniaxial small-scale anisotropy limit [21] where only linear anisotropic corrections are taken into account. It is shown that there exists a relatively large region of small absolute values of the anisotropy parameters where the results obtained in the framework of the weak anisotropy approximation are in very good agreement with results obtained in the framework of the model without any approximation (see Figs. 9 and 10 as well as Table II). It means that the results obtained in the weak anisotropy limit in the framework of investigations of various turbulent problems with the presence of anisotropy can be considered as relevant for the quite large region of the anisotropy parameters.

Finally, we have also analyzed the dependence of the turbulent Prandtl number of the model on the anisotropy parameters for larger spatial dimensions, for $d = 4$ and $d = 5$. It is shown that the form of the dependence of the turbulent Prandtl number on the anisotropy parameters is very similar for all studied cases, i.e., for $d = 3,4$ and 5, as can be seen in Figs. 5, 12, 13, and 14. At the same time, however, the numerical values of the corresponding turbulent Prandtl numbers are quite different. A comparison of the values of the turbulent Prandtl numbers for cases $d = 3,4$, and 5 for various values of the anisotropy parameters is performed in Table I.

Note also that the relevance of our one-loop results is based on the fact that in the fully isotropic case the relative difference between the one-loop and the two-loop values of the turbulent Prandtl number is less than 2% [8,9]. Of course, it would be quite interesting to have also the two-loop anisotropic results, at least, in the weak small-scale anisotropy limit to discuss the higher-loop tendencies in the anisotropic case. However, the corresponding calculations are left for the future.

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