# Algebraic study of drifting spiral waves

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This two-dimensional study is motivated by cardiac electrophysiology, and focuses on rotating spiral waves in reaction-diffusion (RD) models. Here we deal with a spiral's translational drift under a constant externally imposed gradient **G**. A long-standing problem may be stated as follows: Given the dimensionless drift velocity $\mathbf{V}/G$, find its nontrivial direction angle $\Gamma$ relative to **G**. A deductive algebraic treatment yields a solution, $\cos\Gamma = -V/G$. Three features are worth noting: the combination of algebraic and RD contexts; a somewhat extensive derivation contrasting with a compact result; and the generality due to the absence of reaction details in the formula. Agreement with a computational database is good to fair, if spirals of very low density are excluded.

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## I. INTRODUCTION

Over the last few decades, electric excitation waves in the heart muscle have become the object of ever more elaborate and realistic models [1–3]. Much of the mathematics makes use of reaction-diffusion (RD) theories. The present work, which assumes an RD system, is algebraically based, and deals with uniformly rotating nonmeandering spiral waves. If a weak constant external vector field **G** is applied to the propagation medium, a translational drifting motion of the spiral is observed. The weakness of **G** preserves the spiral's rotation, and one may use a perturbation method to analyze the drift.

The physical nature of **G** is unspecified. It could be an electric field, or the concentration of a chemical substance, etc. For our purposes its important effect is to break the rotational symmetry of the medium and induce a drift with constant velocity **V**.

The drift problem of concern in this study asks for the unique angle between **V** and **G**. The main result consists of Eq. (30) further on. For definiteness we assume that **G** points in the $+x$ direction. To begin, the equations are assumed to be unperturbed, $\mathbf{G} = 0$. The mathematical assumptions are as follows.

Two variables, $u(x, y, t)$ and $v(x, y, t)$, are assumed to propagate in a medium that is uniform and independent of time; it is also isotropic in the absence of an external gradient. Under appropriate conditions the above-mentioned waves are produced. We start with the modified FitzHugh-Nagumo equations [4]

$$\partial_t u - \nabla^2 u + \Phi(u, v) = 0, \tag{1}$$

$$\partial_t v + \Psi(u, v) = 0. \tag{2}$$

In these equations, $u$ stands for the (macroscopically measured) transmembrane potential of an electrically excitable heart muscle cell; $v$ represents a local ionic current (also macroscopically measured) across the membrane. In a more realistic model, there may be well over a dozen such currents.

The coefficient of $-\nabla^2 u$ (the diffusion constant) is set equal to 1 for simplicity. The reaction functions $\Phi$ and $\Psi$ are usually nonlinear and can be quite complicated; they are assumed given, and have no dependence on space or time except implicitly by way of the variables $u$ and $v$. Uniformity of the medium is expressed by that feature, and applies also to the other terms $\partial_t u$, $\partial_t v$, and $\nabla^2 u$. This fact lets us define the combination

$$-\nabla^2 u\ (u, v) + \Phi(u, v) \equiv \Xi(u, v), \tag{3}$$

leading for Eq. (1) to a form similar to (2), namely

$$\partial_t u + \Xi(u, v) = 0. \tag{4}$$

## II. ZERO-GRADIENT CASE

The unperturbed spiral is assumed here to undergo rigid rotation about a fixed center (the origin of coordinates); the literature of the subject mostly assumes a clockwise rotation. We accordingly have

$$\partial_t \partial_x = \omega \partial_y, \tag{5}$$

$$\partial_t \partial_y = -\omega \partial_x, \tag{6}$$

where $\omega$ is the angular frequency. Equations (5) and (6) are valid to first order in $x$ or $y$ from the rotation center. [In order to confirm these equations and their signs one may apply the operators in question to a clockwise rotating distribution such as $(x \sin\omega t + y \cos\omega t)$ in the $(x, y)$ plane; a bit further from the origin, and with added terms in $x^2$, etc., the equations are seen to break down.]

Applying $\partial_x$ and $\partial_y$ to Eqs. (4) and (2), we have

$$\omega \partial_y u + (\partial_x u)\partial_u \Xi + (\partial_x v)\partial_v \Xi = 0, \tag{7}$$

$$-\omega \partial_x u + (\partial_y u)\partial_u \Xi + (\partial_y v)\partial_v \Xi = 0, \tag{8}$$

and similarly for the $\partial_t v$ equations using $\Psi(u, v)$. Thus we obtain four simultaneous linear equations for the four quantities $\partial_x u$, $\partial_y u$, $\partial_x v$, $\partial_y v$. Setting for short

$$\partial_u \Xi = a,\ \partial_v \Xi = b,\ \partial_u \Psi = c,\ \partial_v \Psi = d, \tag{9}$$

we find for the above-mentioned four equations the coefficient matrix

$$M = \begin{pmatrix} a & \omega & b & 0 \\ -\omega & a & 0 & b \\ c & 0 & d & \omega \\ 0 & c & -\omega & d \end{pmatrix}. \tag{10}$$

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A nontrivial solution requires

$$\det M = 0, \tag{11}$$

or, after calculation,

$$(\omega^2 - ad + bc)^2 + (a + d)^2 = 0. \tag{12}$$

Thus, remarkably, condition (11) splits into a pair,

$$\omega^2 - ad + bc = 0 \tag{13}$$

and

$$a + d = 0. \tag{14}$$

All cofactors of $\det M$ can be shown to vanish.

### III. COMOVING $(X, Y)$ UNDER NONZERO G

Application of $\mathbf{G}$ is particularly simple in its effect on $M$: namely, in the perturbation expansion, $M$ has a zero first-order term in $G$. This is readily seen from the fact that, while $\mathbf{G}$ is a vector, every element of $M$, including $\omega$, is a scalar. Since the calculations that follow are limited to first order in $G$, all elements of $M$ will be treated as independent of $G$.

The perturbed Eq. (4) reads

$$\partial_t u - G\partial_x u + \Xi(u, v) = 0. \tag{15}$$

We implement the unknown drift by transforming the $(x, y, t)$ coordinates to comoving $(X, Y, T)$:

$$x = V_x T + X, \quad y = V_y T + Y, \quad t = T, \tag{16}$$

so that

$$\partial_x = \partial_X, \quad \partial_y = \partial_Y, \quad \partial_t = -V_x\partial_X - V_y\partial_Y + \partial_T. \tag{17}$$

In the above, $\partial_T$ is taken at constant $X, Y$. Equation (4) now reads

$$(-V_x\partial_X - V_y\partial_Y + \partial_T)u - G\partial_X u + \Xi(u,v) = 0. \tag{18}$$

It will be useful to evaluate this equation at the center of rotation $\mathcal{C}$ where $X = Y = T = 0$ (the origins coincide) and where

$$\partial_T u = \partial_T v = 0; \tag{19}$$

it should be noted that the constancy of $u$ and $v$ at $\mathcal{C}$ is the essential marker for that point, whether stationary or comoving. We focus on the vicinity of $\mathcal{C}$, where, using (16) in Eq. (18) we have

$$u(x, y, t) = u(X + V_x T\ , Y + V_y T\ , T) \tag{20}$$

and similarly for $v$. We are assuming $t = T = 0$ for the time when moving and laboratory origins coincide. Therefore, at point $\mathcal{C}$, Eq. (20) just assigns to $u$ its $G = 0$ value, and the same is true for $v$. According to Eqs. (4) and (2), that also implies $\Xi = \Psi = 0$ in Eq. (18). Taking (19) into account, Eq. (18) now reduces to

$$(V_x + G)\partial_X u + V_y\partial_Y u = 0. \tag{21}$$

In an entirely similar way we find

$$V_x\partial_X v + V_y\partial_Y v = 0. \tag{22}$$

Further information can be obtained from (21) and (22) by applying $\partial_T$ to them.

In addition, $\partial_T$ may be applied to Eq. (18) *before* taking $T = 0$. With use of Eq. (20) as well as $\partial_T u = (\partial_T)^2 u = 0$ and similarly for $v$, the result is

$$\begin{aligned} &-\omega(V_x + G)(\partial_Y u) + \omega V_y(\partial_X u) \\ &\quad + V_x[(\partial_u \Xi)(\partial_X u) + (\partial_v \Xi)(\partial_X v)] \\ &\quad + V_y[(\partial_u \Xi)(\partial_Y u) + (\partial_v \Xi)(\partial_Y v)] = 0. \end{aligned} \tag{23}$$

Correspondingly, from the $\Psi$ equation analogous to (18),

$$\begin{aligned} &\omega V_x(\partial_Y v) - \omega V_y(\partial_X v) + V_x[(\partial_u \Psi)(\partial_X u) + (\partial_v \Psi)(\partial_X v)] \\ &\quad + V_y[(\partial_u \Psi)(\partial_Y u) + (\partial_v \Psi)(\partial_Y v)] = 0. \end{aligned} \tag{24}$$

### IV. $V$-DEPENDENT MATRICES

The coefficients of $\partial_X u, \partial_Y u, \partial_X v, \partial_Y v$, in that order, can now be listed in a four-by four matrix $N$ whose rows represent Eqs. (21)–(24):

$$N = \begin{pmatrix} V_x + G & V_y & 0 & 0 \\ 0 & 0 & V_x & V_y \\ -\omega V_y + aV_x & \omega(V_x + G) + aV_y & bV_x & bV_y \\ cV_x & cV_y & -\omega V_y + dV_x & \omega V_x + dV_y \end{pmatrix}. \tag{25}$$

Concluding the algebraic information from the RD equations, yet another set of four rows, the matrix $P$ below, is obtained by again applying $\partial_T$ to (21)–(24):

$$P = \begin{pmatrix} V_y & -V_x - G & 0 & 0 \\ 0 & 0 & V_y & -V_x \\ \omega(V_x + G) + aV_y & \omega V_y - aV_x & bV_y & -bV_x \\ cV_y & -cV_x & \omega V_x + dV_y & \omega V_y - dV_x \end{pmatrix}. \tag{26}$$

Further use of $\partial_T$ brings nothing new; that operator simply toggles between $N$ and $P$. In particular, we see by inspection that $\det N = \det P$. Together, those two matrices represent a set of eight simultaneous linear homogeneous equations that have a nontrivial solution in common. Thus, every subset of four equations has a nontrivial solution and must have a

zero determinant. Calculating any such determinant generally gives a redundant result. However, two determinants can be sufficient to yield the values of $V_x$ and $V_y$ as functions of $G$. [A simplification occurs as follows: in the $4 \times 8$ matrix consisting of $N$ and $P$, multiplication of rows or columns by $c$ or $c^{-1}$ makes all resultant $4 \times 4$ determinants dependent on the product $bc$ rather the individual $b$ or $c$. Then, owing to Eqs. (13) and (14), the surviving parameters are only $\omega$ and $\delta$.]

## V. SOLVING FOR Γ VERSUS V

As a first example we require $\det N = 0$. After calculation, and omitting an overall factor $V_x^2 + V_y^2$, that condition reads

$$G^2 + G\left(2V_x - \frac{d}{\omega}V_y\right) + \left(V_x^2 + V_y^2\right) = 0. \tag{27}$$

Our second chosen constraint uses a composite matrix $Q$, described as follows. The eight equations in $N$ and $P$ will be labeled $(N1, \ldots, N4)$ and $(P1, \ldots, P4)$; the matrix $Q$ is defined with rows $(N1, N4, P2, P3)$, and its $\det Q = 0$ condition reads, with omission of a $d^2(V_x^2 + V_y^2)$ factor,

$$GV_x + V_x^2 + V_y^2 = 0. \tag{28}$$

For comparison with simulations, we define the drift speed $V$ and drift angle $\Gamma$ by

$$V_x = V\cos\Gamma, \quad V_y = V\sin\Gamma. \tag{29}$$

Equation (28) then states that

$$\cos\Gamma = -\frac{V}{G}, \tag{30}$$

and with its help, (27) reduces to

$$\left(\frac{V}{G}\right)^2 + \frac{d}{\omega}\frac{V}{G}\sin\Gamma - 1 = 0, \tag{31}$$

and therefore

$$\tan\Gamma = -\frac{d}{\omega}. \tag{32}$$

## VI. DISCUSSION

Equation (30), which follows from a purely algebraic analysis, is the main result presented here. It makes no reference to the reaction functions and accordingly, it possesses unexpected universality. Running the unperturbed spiral simulation is not required. A particlelike characterization of the rotation center, leading to Eqs. (5) and (6), is the only approximation that needed to be made.

Formula (32), however, requires knowledge of $\omega$. That parameter can be viewed as incorporating the details of reaction functions $\Phi$ and $\Psi$ in all space-time. In order to obtain $\omega$, we need to run the unperturbed spiral. Thus, compared to (30), Eq. (32) seems less useful; it also requires a good estimate of $d$. Such an estimate may require us to know the location of the rotation center in the $(u, v)$ plane. If steep changes exist in $u$ or $v$, considerable errors in $d$ can be expected. In the presently used database, $d$ happens to be a discontinuous function of $u$. With some care a reasonable prediction for $\Gamma$ can still be obtained; the analysis lies beyond the present scope, however.

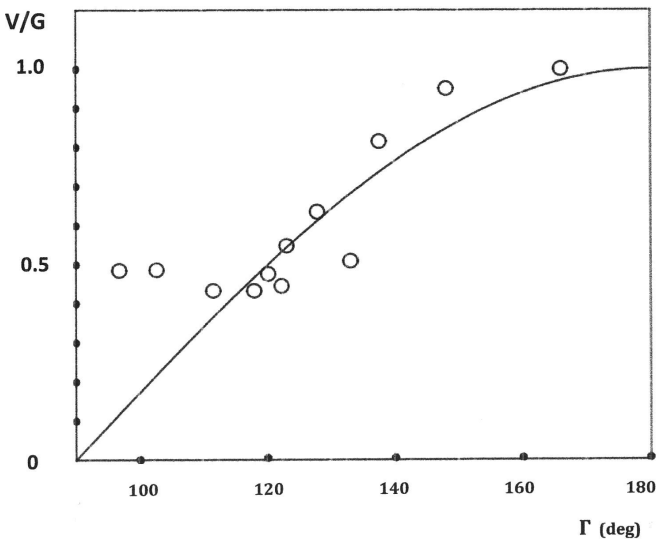


FIG. 1. Theory vs simulation data: the dimensionless drift speed $V/G$ is plotted against the drift angle $\Gamma$. The curve follows Eq. (30); the points are taken from the database of Ref. [4]. Comments are to be found in the text.

In verifying the results presented above, the limits of applicability are as follows. According to (30), **V** must point into the $x < 0$ half plane. This is in fact the case for a majority of database simulations in Ref. [4]. According to (32), **V** must then point into the $y > 0$ half plane, as it in fact does in simulations, a result of clockwise rotation and the positivity of $d = \partial_v \Psi$. In short, theoretical predictions limit **V** to the second quadrant.

The result of simulations is compared with theory in Fig. 1. The two leftmost points are quite close to the 90° limit of validity of (30) and should not be expected to satisfy that equation. The other points are reasonably close to the theory; the worst case, with coordinates (133.0, 0.509), when compared to the closest curve point, is 5.2° off in angle, and 13% off in terms of the unit speed amplitude.

Our particlelike treatment of the rotation center is what ultimately yields a formula as compact and general as Eq. (30). The particle approximation, however, does produce a chain of consequences that can influence the fit to some simulations. First, Eqs. (5) and (6), must break down at some distance from the center. Therefore, as the data indicate, large-size spiral cores are affected more than small ones. Second, the effect becomes more visible close to the 90° validity cutoff. The reason lies in the fact, observed years ago [5], that sparse spirals (large cores) tend to drift against the gradient **G**, oppositely to dense spirals (small cores). The rule is confirmed in the database.

The remaining points' comparison to the theoretical curve ranges from good to fair. The reasonable conclusion appears to be that formula (30) and the wave simulations themselves are mutual quality tests. Higher-resolution work should decide the question one way or the other.

Future improvements of the validity limits might conceivably be achieved with the help of eikonal results, involving wave speed as related to wave front curvature [6–9].

An alternative treatment of drifting spirals, due to Biktashev and collaborators [10], needs mentioning. That work considers a nonzero area around the rotation center and makes use of a numerically precalculated response function. The result is a very accurate determination of drift parameters. In that method, a gradient that causes the drift can be applied to a reaction parameter. In contrast, the present treatment applies specifically to the diffusion term, the situation that prevails in many electrocardiographic cases. Very little overlap is found between Ref. [10] and the present method. In the present case, no response function is calculated. Thus the result is in some sense universal. It is also unexpectedly simple—formula (30) is readily memorized. However, as discussed earlier in the section, some data points are only in rough agreement with the theoretical curve. The method and data may converge better after some future work.

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