Cascading failures in interdependent networks with finite functional components

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(Received 22 March 2016; published 5 October 2016; corrected 29 November 2016)

We present a cascading failure model of two interdependent networks in which functional nodes belong to components of size greater than or equal to s. We find theoretically and via simulation that in complex networks with random dependency links the transition is first order for $s \geqslant 3$ and continuous for $s = 2$. We also study interdependent lattices with a distance constraint r in the dependency links and find that increasing r moves the system from a regime without a phase transition to one with a second-order transition. As r continues to increase, the system collapses in a first-order transition. Each regime is associated with a different structure of domain formation of functional nodes.

DOI: 10.1103/PhysRevE.94.042304

I. INTRODUCTION

Modern real-world infrastructures can be modeled as a system of several interdependent networks [1–4]. For example, a power grid and the communication network that executes control over its power stations constitute a system of two interdependent networks. Power stations depend on communication networks to function, and communication networks cannot function without electricity. There have been several recent attempts to model these systems [5–20]. One of these is based on a model of mutual percolation (MOMP) in which a node in each network can function only if (1) it receives a crucial commodity from support nodes in other networks and (2) it belongs to the giant component (GC) formed by other functional nodes in its own network.

If the nodes within each network of the system are randomly connected, and the support links connecting the nodes in different networks are also random, then the MOMP for an arbitrary network of networks (NON) can be solved analytically using the framework of generating functions, which allows to map the stochastic model into node percolation.

It turns out that a NON is significantly more vulnerable than a single network with the same degree distribution. In regular percolation of a single network, the size of the GC gradually approaches zero when the fraction p of nodes that survived the initial failure, approaches the critical value p_c. In contrast, in the MOMP, the fraction of nodes in the mutual GC, $\mu(p)$ undergoes a discontinuous first-order phase transition at $p = p_\tau > p_c$, dropping from a positive value, μ_τ, for $p \geqslant p_\tau$ to zero for $p < p_\tau$.

The authors of Ref. [12] extended MOMP to Euclidian lattices by studying the process of cascading failures in two lattices A and B of the same size L in which the dependency links are limited by a distance constraint r. In this case there is a particular value of r denoted by $r_{\max}$ below which there is a second-order transition and above which the system collapses in a first-order transition. This process is characterized by the formation of spatial holes that burn the entire system when $r \geqslant r_{\max}$ [21].

The first rule of MOMP is quite general and can be easily verified from an engineering standpoint, but the second rule is not easy to verify. Although it seems that a functioning node must belong to the giant component in order to receive sufficient power, information, or fuel from its own network, this condition can be relaxed, i.e., the second rule in the MOMP can be replaced by a more general rule (2′) in which a node in order to be functional must belong to a connected component of size greater than or equal to s, formed by other functional nodes of this network. This rule is significantly more general and realistic than rule (2) because the nodes in finite components are still able to receive sufficient commodities to continue functioning. Note that the original rule (2) is actually a particular case of rule (2′) for $s = \infty$. In this paper, we will show how the replacement of condition (2) by the more general condition (2′) with $s < \infty$ affects the results in complex networks and Euclidean lattices [5,12].

II. THEORETICAL FORMALISM FOR COMPLEX NETWORKS

The most important role of the MOMP of a NON is played by the function $g_i(y_i)$ [5] such that $y_i g_i(y_i)$ is the fraction of nodes in the *giant component* of network i of the NON after a random failure of a fraction $1 - y_i$ of its nodes. The generating function of the degree distribution of network i is given by [5,11].

$$G_i(x) = \sum_{k=0}^{\infty} P_{k,i} x^k, \tag{1}$$

where $P_{k,i}$ is the degree distribution of network i and the generating function of the excess degree distribution is

$$H_i(x) = \frac{d}{dx}\frac{G_i(x)}{\langle k_i \rangle} = \sum_{k=0}^{\infty} P_{k+1,i}(k+1)x^k / \langle k_i \rangle, \tag{2}$$

where

$$\langle k_i \rangle = \sum_{k=1}^{\infty} k \; P_{k,i} = G_i^{'}(x)|_{x=1} \tag{3}$$

is the average degree of network i.

The fraction of nodes in the giant component relative to the fraction y of surviving nodes is given by

$$g_i(y) = 1 - G_i[f_i(y)y + 1 - y], \tag{4}$$

where $f_i(y)$ is the probability that the branches do not reach the GC, which satisfies the recursive equation [22]

$$f_i(y) = H_i[f_i(y)y + 1 - y]. \tag{5}$$

We also compute the generating function of the component size distribution [23]

$$C_i(x,y) = \sum_{s=1}^{\infty} \pi_{i,s}(y)x^s = xG_i[B_i(x,y)y + 1 - y], \tag{6}$$

where $\pi_{i,s}(y)$ is the fraction of nodes belonging to components of size s in network i relative to the fraction y of surviving nodes, and $B_i(x,y)$ satisfies the recursive equation

$$B_i(x,y) = xH_i[B_i(x,y)y + 1 - y]. \tag{7}$$

Note that when $x = 1$, Eqs. (6) and (7) are equivalent to Eqs. (4) and (5), respectively, and hence

$$C_i(1,y) = \sum_{s=1}^{\infty} \pi_{i,s}(y) = 1 - g_i(y). \tag{8}$$

To move from rule (2) to rule (2′) we replace function $g_i(y_i)$ with function $g_{i,s}(y_i)$, defined the same as $g_i(y_i)$ but replacing the words *giant component* with *components of size larger than or equal to s*. Thus,

$$g_{i,s}(y) = 1 - \sum_{r=1}^{s-1} \pi_{i,r}(y). \tag{9}$$

III. ANALYTIC SOLUTION IN RANDOM REGULAR AND ERDÖS RÉNY NETWORKS

In this section we present the analytic solution for two random regular (RR) and two Erdös Rény (ER) interdependent networks. From Eq. (9), using the Lagrange inversion formula [23], we obtain the coefficients $\pi_{i,s}(y)$ for $s > 1$

$$\pi_{i,s}(y) = \frac{y\langle k_i\rangle}{(s-1)!}\frac{d^{s-2}}{dx^{s-2}}[H_i(x\ y + 1 - y)]^s|_{x=0} \tag{10}$$

and

$$\pi_{i,1}(y) = G_i(1 - y). \tag{11}$$

For ER graphs with a Poisson degree distribution and an average degree $\langle k\rangle$ and for RR graphs with degree z, we can obtain an analytical solution for Eq. (10) for $\pi_{i,s}(y)$. For ER networks $\pi_{\mathrm{ER},s}(y)$ is given by

$$\pi_{\mathrm{ER},s}(y) = \frac{(s\ y\ \langle k\rangle)^{s-1}\ \exp(-s\ y\ \langle k\rangle)}{s!}, \tag{12}$$

and for RR graphs, with degree z, for $s = 1$, $\pi_{\mathrm{RR},1}(y)$ is given by

$$\pi_{\mathrm{RR},1}(y) = (1 - y)^z, \tag{13}$$

and when $s > 1$, $\pi_{\mathrm{RR},s}(y)$ is

$$\pi_{\mathrm{RR},s}(y) = z\ p^{s-1}(1 - y)^{s\ (z-2)+2}\frac{[s\ (z-1)]!}{(s-1)![s\ (z-2)+2]!}. \tag{14}$$

IV. MODEL IN COMPLEX NETWORKS

To illustrate our model, we consider two networks A and B with degree distributions in which bidirectional interdependency links establish a one-to-one correspondence between their nodes as in Ref. [5]. The initial random failure of a fraction $1 - p$ of nodes in one network at $t = 0$ produces a failure cascade in both networks.

A. Theory

At step t of the failure cascade, the effective fraction of surviving nodes $\tilde{\mu}_{A,t}(p)$ and $\tilde{\mu}_{B,t}(p)$ of networks A and B, respectively, satisfies the recursive equations

$$\begin{aligned}\tilde{\mu}_{A,t}(p) &= p\ g_{B,s}(\tilde{\mu}_{B,t-1}(p)),\\ \tilde{\mu}_{B,t}(p) &= p\ g_{A,s}(\tilde{\mu}_{A,t}(p)),\end{aligned} \tag{15}$$

and the fractions of nodes belonging to components of size greater than or equal to s, $\mu_{A,t}(p)$ and $\mu_{B,t}(p)$, are given by

$$\begin{aligned}\mu_{A,t}(p) &= \tilde{\mu}_{B,t}(p)\ g_{B,s}(\tilde{\mu}_{B,t}(p)),\\ \mu_{B,t}(p) &= \tilde{\mu}_{A,t}(p)\ g_{A,s}(\tilde{\mu}_{A,t}(p)),\end{aligned} \tag{16}$$

where $\tilde{\mu}_{A,0}(p) = p$ and $\mu_{A,0}(p) = p\ g_{A,s}(p)$. The process is iterated until the steady state is reached, where

$$\begin{aligned}\tilde{\mu}_A(p) &= p\ g_{B,s}[\tilde{\mu}_B(p)],\\ \tilde{\mu}_B(p) &= p\ g_{A,s}[\tilde{\mu}_A(p)],\end{aligned} \tag{17}$$

and

$$\mu\ (p) \equiv \mu_A(p) = \mu_B(p) = \tilde{\mu}_A(p)\tilde{\mu}_B(p)/p.$$

When $p = p_\tau$, the order parameter of our model, $\mu(p)$, transitions from $\mu(p) > 0$ when $p > p_\tau$ to $\mu(p) = 0$ when $p \leqslant p_\tau$. In the most simple case when the networks have identical degree distributions, $g_{A,s}(x) = g_{B,s}(x) \equiv g_s(x)$. At the threshold, $p = p_\tau$ and $\tilde{\mu}(p_\tau)$ satisfy

$$\begin{aligned}\tilde{\mu}(p_\tau) &= p_\tau\ g_s[\tilde{\mu}(p_\tau)]\\ 1 &= p_\tau\ g_s'[\tilde{\mu}(p_\tau)],\end{aligned} \tag{18}$$

where $g_s'(y) = dg_s(y)/dy$. Because $g_2(y) = 1 - G(1 - y)$, the second derivative of $g_2(y)$ is always negative, and thus Eq. (18) has a trivial solution at $\tilde{\mu}(p_\tau) = 0$ from which $p_\tau = 1/G'(1) = 1/\langle k\rangle$, where $G'(1) = dG(y)/dy|_{y=1}$, and, as a consequence, the system undergoes a continuous phase transition. For networks with a nondivergent second moment of the degree distribution the transition is third order; but, for networks with a divergent second moment the transition is of a higher order. However, when $s \geqslant 3$, $g_s(y)$ changes the sign of its second derivative from positive at $y = 0$ to negative at $y = 1$, and hence Eq. (18) has a nontrivial solution in the interval $0 < p < 1$ at which $\tilde{\mu}(p)$ abruptly changes from a positive value above p_τ to zero below p_τ. Thus for $s \geqslant 3$ we always have a first-order transition, which was previously

found [5] but only for $s = \infty$. The different kinds of transitions that we find in our model are reminiscent of the ones found in k-core percolation [24–27]. A k core of a graph is a maximal connected subgraph of the original graph in which all vertices have degree at least k, formed by repeatedly deleting all vertices of degree less than k. In particular, in 2-core there is a continuous transition, while for $k \geqslant 3$ the transition is first order, as in our model for $s = 2$ and $s \geqslant 3$, respectively. The key difference between the k-core transition and our model is that in our model the functionality of a node is not based on its degree but rather on the size of the finite components to which it belongs. The similarity between the phase transitions in our model and the ones in k-core is due to a resemblance between the pruning rules of both processes. For example, in our model with $s = 2$, the final state is constituted of nodes with at least one active link in their own network and one dependency link and, hence, all nodes have two active links as in the final state of 2-core. Next we will see that the similarities of the phase transitions arise due to the similarities in the leading terms of the Taylor expansions of the equations that govern k-core and our model. However, we will also demonstrate that both models do not belong to the same universality class.

1. Scaling behavior of the fraction of active nodes for $s = 2$ in our model

From Eq. (18) for $s = 2$, at the steady state, the effective fraction of remaining nodes $\tilde{\mu}(p) \equiv \tilde{\mu}$ is given by

$$\tilde{\mu} = p[1 - G(1 - \tilde{\mu})], \tag{19}$$

where p is the fraction of nodes that survived the initial damage, and $G(x)$ is the generating function of the degree distribution. For RR, ER, and scale-free networks with nondivergent second moment ($\lambda > 3$), close to the threshold p_τ at which $\tilde{\mu}(p_\tau) = 0$, expanding Eq. (19) around $\tilde{\mu} = 0$ gives

$$\tilde{\mu} = p[G'(1)\tilde{\mu} - G''(1)\tilde{\mu}^2/2 + O(\tilde{\mu}^3)], \tag{20}$$

and solving this equation for $\tilde{\mu}$ leads to

$$\tilde{\mu} = 2\frac{pG'(1) - 1}{p\ G''(1)} + O(\tilde{\mu}^2). \tag{21}$$

Equation (21) shows that $\tilde{\mu} \to 0$, when $p \to 1/G'(1)$; thus there is a continuous phase transition at $p = p_\tau \equiv 1/G'(1)$. Recalling that for any degree distribution with converging first and second moments, $G'(1) = \langle k \rangle$, $G''(1) = \langle k^2 \rangle - \langle k \rangle$, we can rewrite Eq. (21) as

$$\tilde{\mu} = 2\frac{\delta p \langle k \rangle}{(p_\tau + \delta p)(\langle k^2 \rangle - \langle k \rangle)} + O[(\delta p)^2], \tag{22}$$

where $p = p_\tau + \delta p$, with $\delta p \to 0$. Since the denominator does not diverge, then $\tilde{\mu} \sim (p - p_\tau)^{\beta'}$, with $\beta' = 1$.

For two interdependent networks with the same degree distribution, the order parameter is given by

$$\mu = \tilde{\mu}^2/p, \tag{23}$$

and thus $\mu \sim (p - p_\tau)^\beta$ with $\beta = 2$.

For $2 < \lambda < 3$, the second moment diverges, thus using the Tauberian theorem [28] the expansion of $\tilde{\mu}$ is given by

$$\tilde{\mu} = p[G'(1)\tilde{\mu} - A\tilde{\mu}^{\lambda-1} + O(\tilde{\mu}^{\lambda-2})], \tag{24}$$

in which, for $p = p_\tau + \delta p$,

$$\tilde{\mu} = \left[\frac{\delta p G'(1)}{pA}\right]^{1/(\lambda-2)} + O(\delta p)^{1/(\lambda-2)} \tag{25}$$

$$\sim (p - p_\tau)^{1/(\lambda-2)}, \tag{26}$$

so $\beta' = 1/(\lambda - 2)$ and, as a consequence [see Eq. (23)], $\beta = 2/(\lambda - 2)$. Thus there is a fourth-order phase transition for $8/3 < \lambda < 3$. In general, for scale-free (SF) networks with $2 < \lambda < 3$, the transition is of mth order for $2 + 2/(m - 1) < \lambda < 2 + 2/(m - 2)$.

2. Scaling behavior of the fraction of active nodes in 2-core

In contrast with Eq. (19), for 2-core percolation, the fraction of active nodes q obeys the equation

$$q = p[1 - G(1 - f) - fG'(1 - f)], \tag{27}$$

where p is the fraction of nodes that survived the initial damage and f is the effective fraction of survived links obeying a self-consistent equation

$$f = p\left[1 - \frac{G'(1 - f)}{G'(1)}\right]. \tag{28}$$

For homogeneous networks, such as RR and ER, after expanding Eq. (28) around $f = 0$, we obtain

$$f = \frac{p}{G'(1)}[G''(1)f - G'''(1)f^2/2 + O(f^3)]. \tag{29}$$

If $G'''(1) < \infty$, then $p_\tau = G'(1)/G''(1) = \langle k \rangle/(\langle k^2 \rangle - \langle k \rangle)$ as in regular percolation, and

$$f = \frac{2[\delta p G''(1)]}{p\ G'''(1)}. \tag{30}$$

Finally expanding Eq. (27) around $f = 0$ leads to $q = pf^2G''(1)/2 + O(f^3) \sim (p - p_\tau)^2$, which indicates a third-order phase transition.

For SF networks, if $3 < \lambda \leqslant 4$, from the Tauberian theorem [28]

$$f = p\left[\frac{G''(1)}{G'(1)}f - Af^{\lambda-2} + O(f^{\lambda-2})\right], \tag{31}$$

from where $f \sim (p - p_\tau)^{1/(\lambda-3)}$ with $p_\tau = G'(1)/G''(1)$ and $q \sim (p - p_\tau)^{2/(\lambda-3)}$ and the transition becomes of the order m if $3 + 2/m < \lambda \leqslant 3 + 2/(m - 1)$. If $2 < \lambda < 3$, $G''(1) = \infty$, then $p_\tau = 0$, $f \sim p^{1/(3-\lambda)}$ and $q \sim p^{1/(3-\lambda)}$. Thus for $2 < \lambda < 3$ there is a phase transition but at $p_\tau = 0$, and the order parameter of this transition changes in reverse order from infinity for $\lambda = 3$ to 3 for $\lambda = 2 + \epsilon$ with $\epsilon \to 0$.

Thus we have a close analogy between the model of functional finite component interdependent networks with $s = 2$ and 2-core percolation in terms of the order of the phase transition. This analogy stems from the similarities in the Taylor expansion of the equations describing these two models, but the physical basis on which these equations are constructed totally differs. In addition, the order of the transitions differs for SF networks with $2 < \lambda < 4$, and thus the two models do not belong to the same universality class.

B. Simulations in complex networks

We test our theoretical arguments with stochastic simulations in which we use the Molloy-Reed algorithm [29] to construct networks with a given degree distribution. The procedure is as follows:

(1) At $t = 0$ we remove a random fraction of nodes $1 - p$ in network A, remove all the nodes in the components of network A smaller than s, and remove all the dependent nodes in network B.

(2) At $t \geqslant 1$ we remove all the nodes in the components of network B smaller than s and remove all the nodes in network A dependent on dead nodes in B.

(3) We repeat (2) until no more nodes can be removed.

We perform simulations for a system of two ER graphs, two RR graphs in which all nodes have the same degree z, each of $N = 10^6$ nodes, and two SF graphs with $N = 5 \times 10^6$ (see Fig. 1). The SF networks have a degree distribution $P_k \propto k^{-\lambda}$ with $k_{\min} \leqslant k \leqslant k_{\max}$, where λ is the exponent of the SF network. We set $k_{\min} = 2$ and $k_{\max} = \sqrt{N}$. To compare our simulations with the theoretical results [Eq. (18)] we use analytical expressions for $\pi_{i,s}(p)$ given in the case of ER and RR networks by Eq. (10). For SF networks we compute $\pi_{i,s}(p)$ numerically. The details of the analytical solution for ER and RR networks are presented in Sec. III.

Figures 1(a), 1(b), and 1(c) show perfect agreement between the theoretical results and the simulations. Figure 1(d) shows a plot of p_τ as a function of s for two RR networks with degree $z = 3$, two ER networks with $\langle k \rangle = 3$, and two SF networks with $\lambda = 3$, $k_{\min} = 2$ and an average degree $\langle k \rangle = 3.18$. As predicted, $p_\tau = 1/\langle k \rangle$ for $s = 2$ and increases as s increases. For $s \to \infty$ we recover the mutual percolation threshold of Ref. [5] shown as dashed lines in Fig. 1(d).

V. MODEL IN INTERDEPENDENT EUCLIDEAN LATTICES

We also study the same model for square lattices, generalizing Refs. [12,14]. When there are random interdependency links, i.e., when there is no geometric constraint on the interdependencies, we use the exact results for the perimeter polynomials of the finite components to compute $g_s(p)$,

$$g_s(p) = 1 - \sum_{n=1}^{s-1} n\, p^{n-1}\, D_n(1-p), \tag{32}$$

where $D_n(1-p)$ are the perimeter polynomials for small components on a square lattice [30].

Here the system undergoes a first-order phase transition when $s \geqslant 3$ at the predicted values of $p_\tau = 0.485$ for $s = 3$ and $p_\tau = 0.5506$ for $s = 4$, obtained by solving Eq. (18). When the interdependency links satisfy distance restrictions, we define the distance between the two interdependent nodes in lattices A and B as the shortest path between the nodes along the bonds of the lattices, i.e., $|x_A - x_B| + |y_A - y_B| \leqslant r$, where (x_A, y_A) and (x_B, y_B) are the coordinates of the interdependent nodes in lattices A and B, respectively. Using simulations we see a first-order phase transition emerging at a certain value of

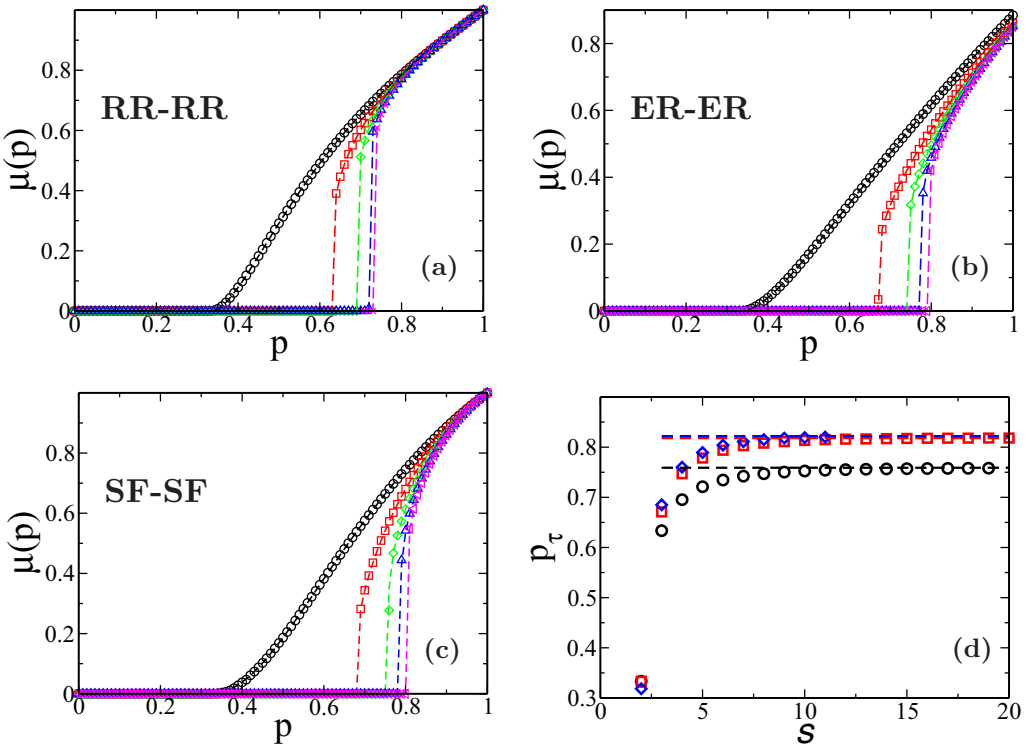

FIG. 1. $\mu(p)$ as a function of p for different values of s for two (a) RR networks with degree $z = 3$ for $N = 10^6$, (b) ER networks with average degree $\langle k \rangle = 3$ for $N = 10^6$, and (c) SF networks with $\lambda = 3$ for $N = 5 \times 10^6$ with 1000 networks realizations for different values of s, from $s = 2$ to $s = 6$, (the left most curves, indicated by circles). The symbols are the simulations and the dashed lines are the theory. (d) The threshold p_τ as a function of s obtained from the theory for the same RR (black circles), ER (red squares), and SF (blue diamonds) networks presented in (a), (b), and (c). The dashed lines are used as a guide to show p_τ for $s \to \infty$.

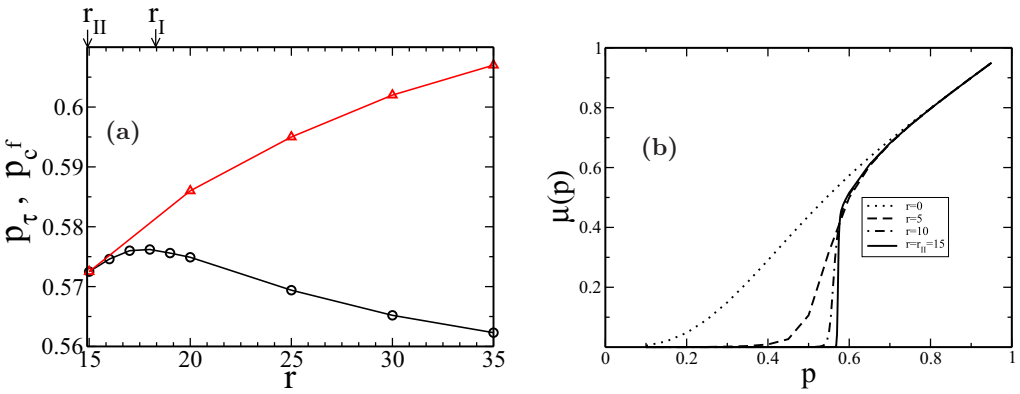

FIG. 2. For interdependent lattices with interdependent distance r and survival component size $s = 4$, for $L = 512$ (a) $p_\tau(r)$ (◯) vs. r for the first-order $r \geqslant r_{\rm I} = 18$ and the continuous phase transition $r_{\rm II} = 15 \leqslant r < r_{\rm I} = 18$ and $p_c^f(r)$ (△). The lines are used as a guide for the eyes. (b) $\mu(p)$ vs. p for different values of r. For $r < r_{\rm II} = 15$ the results do not depend on the lattice size L. The system size dependence emerges only at $r = r_{\rm II} = 15$.

$r = r_I$ in qualitative agreement with the case $s = \infty$ studied by Li *et al.* [12]. At this value of r the system reaches maximum vulnerability, indicated by a maximum of $p_\tau(r)$ as a function of r [see Fig. 2(a)].

The $r_{\rm I}$ value is much greater than the value obtained for the MOMP ($s = \infty$). For r close to $r_{\rm I}$, the cascading failures propagate via node destruction on the domain perimeters composed of surviving node components, and this creates moving interfaces when the size of the void separating the domains is greater than r. These moving interfaces belong to the class of depinning transitions characterized by a threshold $p = p_c^f(r)$ that increases with r (see Fig. 2). Here $p = p_c^f(r)$ is the critical fraction of nodes remaining after the initial failure, such that for $p > p_c^f(r)$ the interface of an infinitely large void will be eventually pinned and stop to propagate. In contrast, when $p < p_c^f$, the interface of the voids propagates freely without pinning and eventually burns the entire system. Near $p_c^f(r)$, the velocity of the domain interfaces approaches zero with a power-law behavior $v \sim (p_c^f - p)^\theta$, where $\theta > 0$ is a critical exponent [31]. In order to compute p_c^f, we compute the velocity v of the growing interface as a function $p^f - p$ until we get a straight line in a log-log plot, which corresponds to the value of the critical threshold p_c^f. The value of the slope of $v \sim p_c^f - p$ is the critical exponent θ. We find $\theta = 0.53$, suggesting that the interface belongs to the universality class of a Kardar-Parisi-Zhang (KPZ) equation [32] with quenched noise. As $p = p_c^f(r)$ increases, the probability that large voids with a diameter greater than r will spontaneously form, decreases, and becomes vanishingly small in a system of a finite size. Thus in a finite system we must decrease p below $p_c^f(r)$ in order to create these voids. When $p < p_c^f(r)$, the interface of the voids begins to freely propagate without pinning and eventually, like a forest fire, burns through the entire system. Thus the emergence of a first-order transition in a finite system depends on the system size, i.e., the larger the system, the larger the $r_{\rm I}$ value

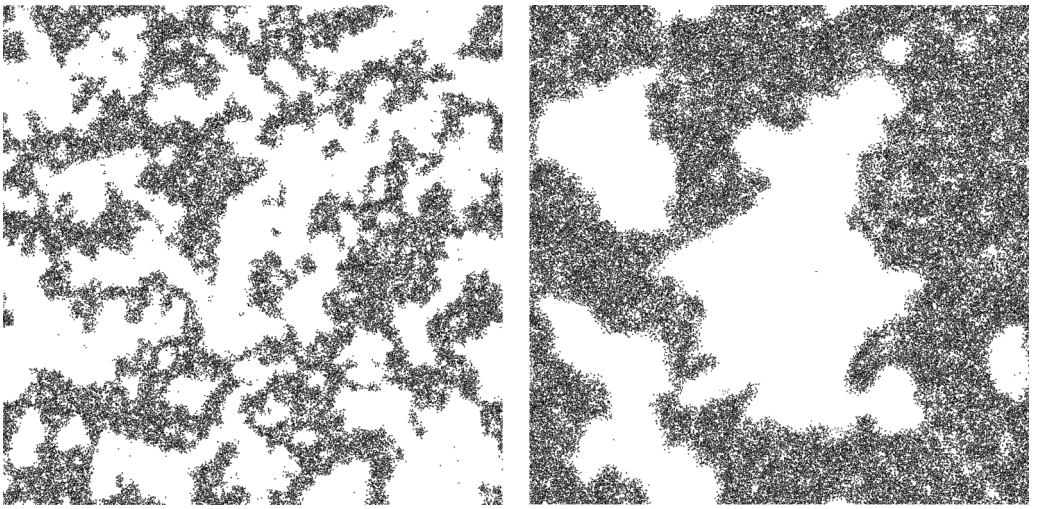

FIG. 3. Snapshots of the model of interdependent lattices for $s = 4$, $L = 1024$ and different values of $r = 10 < r_{\rm II}$, $p = 0.56$ (left) and $r = r_{\rm II} = 15$, $p = p_m = 0.572$ (right) at the end of the cascade of failures. It can be seen that for small $r < r_{\rm II}$ the system is divided into many independent domains, while for $r = r_{\rm II}$ the domains coalesce, and the cascades are driven by the propagation of the interface near the depinning transition.

at which the effective first-order (all-or-nothing) transition is observable.

Figure 2(a) shows that as r continues to increase, p_τ begins to decrease and slowly approaches the p_τ value for random interdependence as $r \to \infty$. There is no second-order percolation transition for finite s and small r that governs the size of the voids, in contrast to what was found by Li *et al.* [12] for $s \to \infty$. For finite s, a second-order transition emerges when the r value is large, $r = r_{\mathrm{II}} < r_{\mathrm{I}}$, but when $r < r_{\mathrm{II}}$ there is no transition, the fraction of survived nodes $\mu(p)$ is zero only at $p = 0$, and it continues to be differentiable and independent of the system size for any positive value of p. Note, however, that as r approaches r_{II} the derivative of $\mu(p)$ develops a sharp peak at a certain value of $p \equiv p_m(r)$ below which $\mu(p)$ is very small but finite. At $r = r_{\mathrm{II}}$ we see a second-order transition because the height of the peak of the derivative of $\mu(p)$ now increases with the lattice size L, which is typical of a second-order transition. This behavior is associated with different regimes of domain formation. For small values, $r < r_{\mathrm{II}}$, the first stages of the cascading failure fragment the system into small independent regions, each of which has its own pinned interface (see Fig. 3). In this regime, after the first stages of the cascade of failures the system practically does not change. After the first stages, the interfaces propagate very slow and can stop at any point leaving the resulting snapshots indistinguishable from the one obtained in the steady state. A single interface emerges only when these regions coalesce at $r = r_{\mathrm{II}}$, and a second-order phase

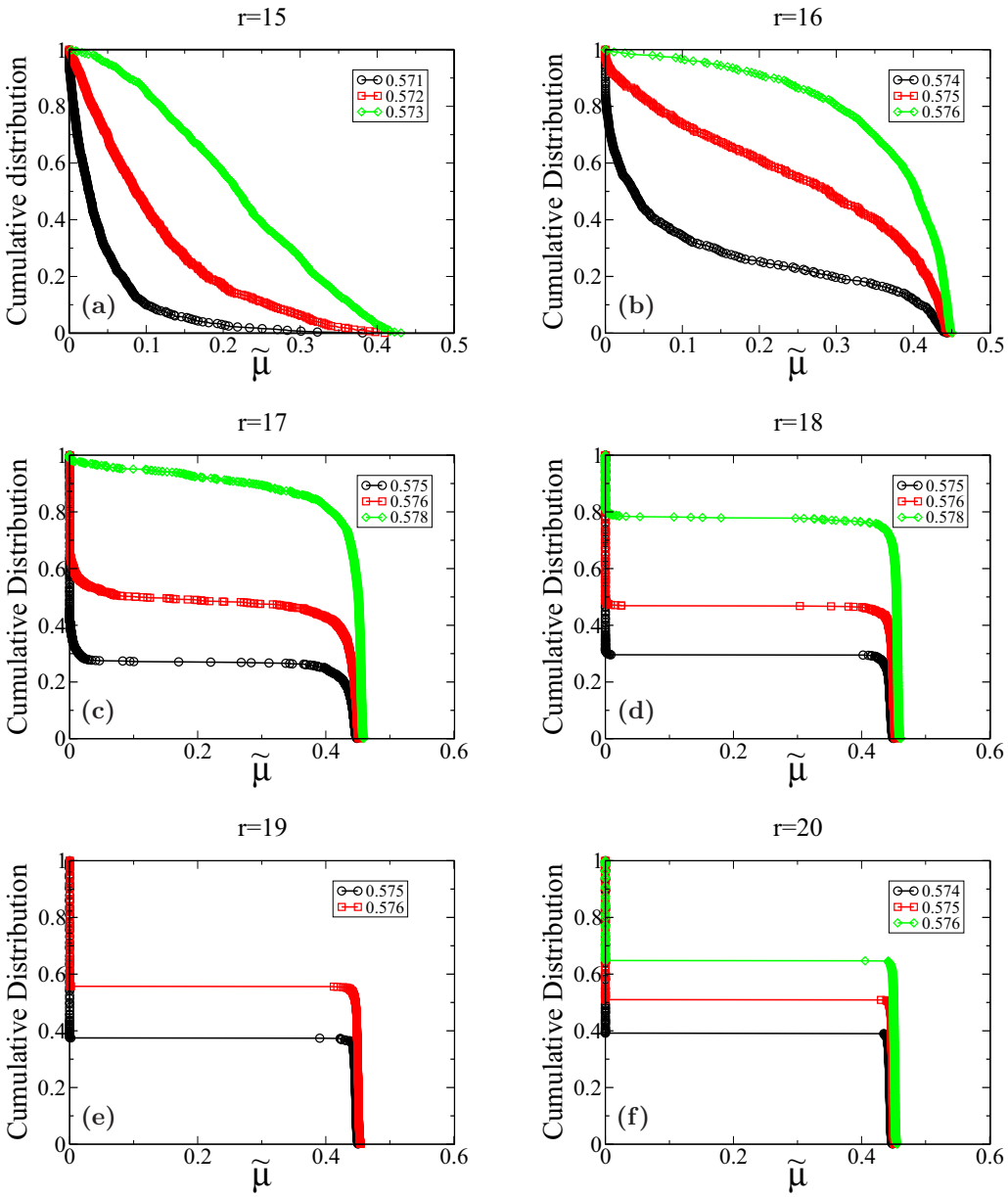

FIG. 4. Cumulative distribution of the fraction of survived nodes, $\tilde{\mu}$, for different values of r. As we can see from the plots, as r increases above r_{II}, a plateau develops in the cumulative distribution for $p \approx p_\tau$, which means that the distribution of the values of $\tilde{\mu}$ is bimodal and the system will eventually reach a first-order transition at $r \geqslant r_{\mathrm{I}}$. In this regime, there is a large gap between the values of $\tilde{\mu}$, indicating that for the same value of p, either a large fraction of the system can stay functional or the system can completely collapse.

transition related to the propagation of this interface through the entire system emerges. This second-order phase transition observed for $r = r_{\rm II}$ has a unimodal distribution of the fraction of surviving nodes $\tilde{\mu}(p)$, and we use the maximum slope of the graph $\tilde{\mu}(p)$ to compute the critical point $p_\tau = p_m(r_{\rm II})$. As r increases between $r_{\rm II}$ and $r_{\rm I}$, the distribution of $\tilde{\mu}(p)$ becomes bimodal, and we compute the transition point p_τ using the condition of equal probability of both modes. Note that p_τ reaches a maximum at $r = r_{\rm I}$ where the two peaks of the distribution of $\tilde{\mu}(p)$ separate completely, as indicated by a wide plateau in the cumulative distribution of $\tilde{\mu}(p)$ [33]. The cumulative distribution of $\tilde{\mu}(p) \equiv \tilde{\mu}$ for square lattices is presented in Fig. 4.

The emergence of the first-order phase transition above $r_{\rm II}$ is related to the decrease of the correlation length as we move away from $r_{\rm II}$. We thus find that when s is small, $r_{\rm I}$ is significantly larger than $r_{\rm I}(\infty)$. For the shortest path metric $r_{\rm I}(\infty) = 11$, and $r_{\rm II}(4) = 15$ and $r_{\rm I}(4) = 18$ for $L = 1024$. As s increases, $r_{\rm I}$ gradually decreases and coincides with $r_{\rm I}(\infty)$ for $s \to \infty$.

VI. CONCLUSION

In summary, we find that in complex networks with $s > 2$, our model has a first-order transition as for the previously studied case of MOMP with $s \to \infty$. For $s = 2$, our model has a higher-than-second-order transition similar to that found in k core, but the order of the transitions in SF networks differs depending on the exponent of the degree distribution. However, the finite component generalization of MOMP in spatially embedded networks has a totally different behavior, which is not related to k core. In this case, the transitions, when they exist, are dominated by the behavior of the pinning transition of void's interfaces. Our model in spatially embedded networks is a rich and interesting phenomenon, which has many practical applications for studying the cascade of failures in real-world infrastructures embedded in space. Our work can be extended to any NON model incorporating MOMP, but our finite component model is significantly more general and realistic. We can generalize our model to derive equations for a partially interdependent NON. Here the second-order transition will also appear when $s > 2$ if the fraction of interdependent nodes is small. The value of s can differ in different networks of the NON and can be a stochastic variable, such that a component of size s survives with probability $p(s)$, as in the heterogeneous k core [27,34].

ACKNOWLEDGMENTS

The Boston University and the Yeshiva University work was supported by DTRA Grant No. HDTRA1-14-1-0017; by DOE Contract No. DE-AC07-05Id14517; and by NSF Grants No. CMMI-1125290, No. PHY-1505000, and No. CHE-1213217. S.V.B. acknowledges the partial support of this research through the B. W. Gamson Computational Science Center at Yeshiva University. L.A.B. and M.A.D.M. thank UNMdP and FONCyT, Pict 0429/13, for financial support.

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