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# Billiard with a handle

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We consider an open billiard with two holes, connected by a handle. The central billiard is chosen so that its closed form's islands of stability occupy a significant part of the phase space. Holes destroy these islands, which leads to almost all trajectories of the system being interleaved. We also study the unbalanced flow of billiard particles through the handle, which appears only after a small border site of nonspecular reflection is added to the system. With this site our system is rather a ratchet of a different type, since the site does not produce an explicitly acting force or violate the reversibility of trajectories.

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## I. INTRODUCTION

Mathematical billiards are standard objects of study in the chaos theory; they are the simplest low-dimensional Hamiltonian systems that implement chaotic behavior. The behavior of many physical systems can be reduced to the motion of a particle in a billiard. It is difficult to overestimate the importance of billiards for the basic concepts of statistical physics [1]. In a general case the phase space of a billiard is mixed and contains areas of both regular and chaotic motion. However, there are completely chaotic billiards [2,3] or completely regular ones, such as in the case of an elliptic billiard [4] or a confocal parabolic billiard [5,6]. Big attention is drawn to the study of behavior of trajectories in some generalizations of usual billiards. Such generalizations include soft billiards [7], billiards with a modified law of reflection [8,9], and billiards in external fields [10,11].

An important part of billiards theory comprises open billiards, which are currently intensively studied [12–17]. They are related to such fields of physics as optics, statistical physics, plasma physics, and many others. Detailed description of this interconnection can, for example, be found in the review [18]. Considering open billiards, attention is mainly given to the distribution of escape times from the billiard. In general, exponential distribution law is typical for a strongly chaotic behavior and sedate law in the case of a regular one (see, e.g., [19]). Often, in addition to the main exponential decay, a power-law tail of the distribution is observed. An open billiard may have a border with more than one hole. For example, in the work [20] the Bunimovich billiard with two holes was studied, and it was shown that the existence of a power-law tail depends on the choice of the holes' positions.

In this paper we consider a system consisting of an open billiard with two holes and a handle connecting these holes with each other, so that the whole system is closed. Due to the presence of a handle, the distribution of particles entering the billiard is formed in a natural way. The border of this billiard consists of four parts, two of which are parabolas and other two are circle segments. Parabolas generally are not confocal, but the movement of particles between them largely inherits the properties of motion between confocal parabolas, which is integrable. As a result, there are significant regions of regular motion in the phase space. A family of similar billiards, some periodic orbits, and their stability were studied in [21]. Opening the holes can destroy quasiperiodic motion on the island of stability, since it removes part of the billiard's boundary, which might be hit by the trajectory. The influence of this effect in open billiards is currently incompletely studied. In our billiard it leads to an intermittent character of particle motion.

The presence of a handle raises a natural question about the possibility of unbalanced, directional motion of billiard particles through it. In a pure billiard the occurrence of such flow is impossible. However, such flow may occur after addition to the system of a small border site having almost anything but a specular law of reflection. Billiards with modified reflection laws are also currently under consideration [22,23] and are of interest because such laws of light reflection are doable. For example, the method provided in the paper [24] allows creation of surfaces with structure (metamaterials essentially) which can have almost any preset law of reflection.

We consider both the collisionless case, corresponding to the motion of light rays, and the case of colliding particles. Rare-enough collisions, with a mean free path being an order of magnitude greater than the system's size, have no essential influence on macroscopic flow.

Such a system with a nonspecular border site and stationary flow of particles is far enough from usual billiards. As there is no visible acting force supporting the flow, it may be considered as a ratchet of a new type. Currently, ratchets have been intensively studied both theoretically and experimentally, due to many promising applications in such areas as Brownian and molecular motors [25–27], atomic and optical ratchets [28,29], organic-electric ratchets [30], and many others. The effects associated with the spontaneous appearance of directional motion are experimentally observed in many biological [31] and just micro- and nanodimensional systems.

## II. CENTRAL BILLIARD

Let us consider a billiard with mixed phase space, propitious for maximization of effects related to the islands of stability. The geometry of this billiard is shown at Fig. 1. Its boundary consists of four smooth parts, joined together at the points with coordinates $(\pm a_x, \pm a_y)$. The top and bottom parts are

 

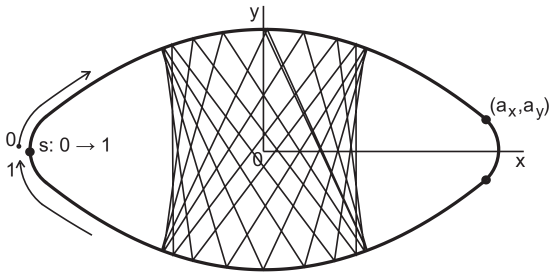


FIG. 1. General view of a billiard with parameters $a_x = 80$, $a_y = 12$, $a = 100$. Parametrization of its boundary and a part of typical trajectory of central island are shown.

parabolic arcs,

$$y = \pm\left(a_y + \frac{a_x^2 - x^2}{2a}\right), \quad x \in [-a_x, a_x], \tag{1}$$

where $a_x, a_y, a$ are the billiard's parameters; parameter $a$ determines the curvature of parabolas. Two lateral parts of the boundary are circle arcs of radius $r = \sqrt{a_y^2 + (\frac{a_x a_y}{a})^2}$ with centers at the points $\pm(a_x - \frac{a_x a_y}{a}, 0)$. They are chosen so that their cross linking with two other border parts was of a smooth $C^1$.

Such a billiard can be regarded as a generalization of Bunimovich billiard [32], since it transfers to it at $a \to \infty$. This billiard will be obtained from a stadium billiard, if the strait border sides between circle arcs are replaced with parabolic segments.

Phase portraits of the considered billiard, built in Birkhoff coordinates $(s_i, \sin\Theta_i)$ for different parameter values, are shown in Fig. 2. The position of a point on the billiard's boundary is determined by the coordinate $s$. It is measured along the boundary of the billiard as shown at Fig. 1 and is normalized by the length of the billiard's perimeter. In other words, $0 \leqslant s < 1$. The coordinate $\Theta_i$ is an angle of incidence for reflection from a boundary point $s_i$.

The phase portrait of a billiard consists of a chaotic sea and islands, among which the central one may be distinguished. The share of phase-space volume, occupied by the system of islands, can vary depending on parameters $a, a_x, a_y$ from zero to almost the entire phase-space volume. Trajectories belonging to the chaotic sea have a positive Lyapunov exponent.

The configuration of the islands of stability depends on the choice of the billiard's parameters. For their certain choice, a significant part of the phase space is occupied by the only one central island of stability, as shown at Fig. 2. Trajectories corresponding to this island of stability collide only with parabolic sites of the billiard's border. A part of the typical trajectory belonging to the central island is shown at the Fig. 1. This simple phase-space organization makes the proposed billiard most convenient for the consideration of effects associated with the presence of islands of stability. A single large-size island makes them well distinguishable.

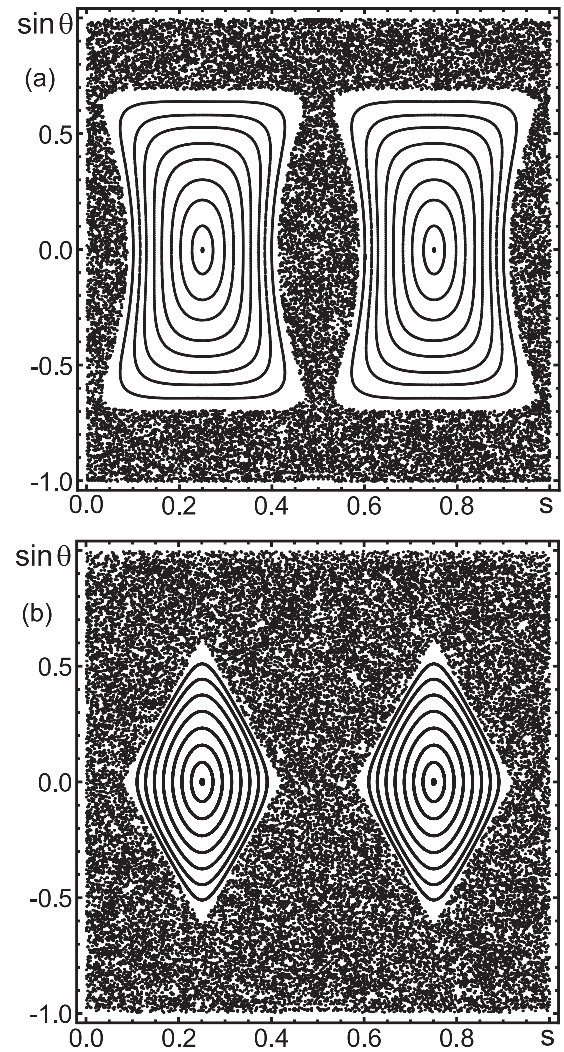


FIG. 2. Phase portraits of billiards with parameters (a) $a_x = 80$, $a_y = 12$, $a = 100$ and (b) $a_x = 80$, $a_y = 34$, $a = 100$ in Birkhoff coordinates.

### III. ADDITION OF A HANDLE TO A BILLIARD

Let us make two holes of sizes $d_1$ and $d_2$ in the boundary of the original billiard so that it becomes open. Through these holes particles can enter and leave the billiard. Let the hole of size $d_1$ be in the upper half of the central billiard and the one of size $d_2$ be in the left half. Now we connect these holes with a handle of two concentric circles, so that the system becomes closed again, as shown at Fig. 3. The hole's sizes will be chosen to be much lesser than the length of the billiard's perimeter. The handle parameters are unequivocally determined by the choice of the holes' sizes. Further, we assume to be not valid a choice where the handle appears to be lying inside the central billiard.

The phase portrait makes it easy to understand the impact of the handle on billiard trajectories. Indeed, the presence of a hole means that trajectories entering the strip $[s_d, s_d + d]$ in the phase space would get into the handle. Here $d$ is the size of a hole and $s_d$ is the coordinate of the beginning of the hole in billiard's boundary. The geometry of the handle guarantees

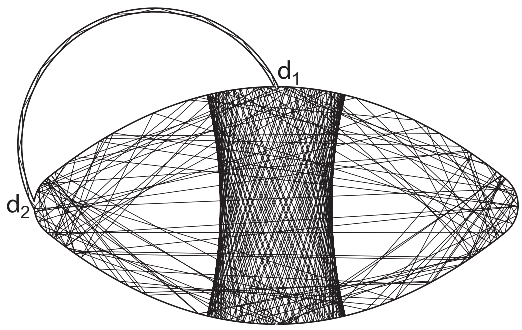


FIG. 3. General view of the considered billiard; it consists of a central billiard and a handle connecting holes. Also shown is a part of typical trajectory, consisting of the motion on the island up to hole $d_1$, passage through the handle, and further motion in the chaotic sea. Billiard parameters are $a = 100$, $a_x = 80$, $a_y = 9.5$, $d_1 = 1.45$, and $d_2 = 2.9$.

that there will be no reversal of the trajectory during motion in it. In fact, this is the main property of a handle, owing to which its specific parameters are of no importance. Trajectory will definitely return to the billiard through the first hole in the case of leaving through the second.

Hole $d_1$ is at the top of the billiard and its opening destroys the central island of stability, because the trajectories of the island were partially falling on the removed part of the border. Now the trajectories of the chaotic sea and island of stability ceased to be separated from each other. Particles from the chaotic sea can now get into the handle and come out on a trajectory that for a closed billiard belongs to the island of stability, and vice versa, as shown in Fig. 3. As a result, a much larger part of phase space is available for a single trajectory. Such a trajectory in the phase portrait fulfills previously inaccessible parts of the phase space so that the result looks like a single chaotic sea. From this point of view, all trajectories appear like chaotic ones. However, the properties of the trajectories still strongly depend on the characteristics of the destroyed island of stability. Thus, before the opening of holes there were only chaotic and regular trajectories. After the holes' opening, all the trajectories passing them change their type to intermittent. Each such trajectory repeatedly transfers from the chaotic sea to the area of regular motion and backwards.

A typical view of emerging intermittent trajectories is shown in Fig. 4. It should be emphasized that the quasiperiod of motion on the destructed island exactly coincides with the quasiperiod of the corresponding trajectory belonging to this island in the closed billiard. The duration of periodicity preservation depends on the characteristics of the island and the size of input window and can be long enough. Thus, with the help of a handle it is possible to connect some islands with the chaotic sea, making their trajectories intermittent.

Intermittent trajectory consists of chaotic and laminar phases of motion. Let us consider what part of the trajectory is chaotic and, accordingly, what is the average share of laminar phases in the intermittent trajectory. As it turned out, the share of chaotic component in the trajectories depends only on the parameters of the billiard and does not depend on the holes sizes $d_1$, $d_2$. However, the sizes of holes determine the average

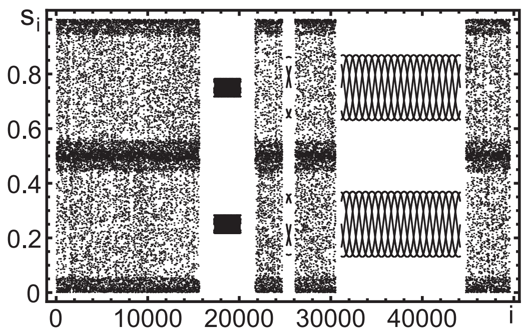


FIG. 4. Typical view of intermittent trajectory $s_i(i)$ for a billiard with parameters $a = 100, a_x = 80, a_y = 3, d_1 = 0.03$, and $d_2 = 0.08$. The gaps correspond to the trajectory being outside the billiard during its motion through the handle.

duration of laminar and chaotic trajectory phases. The smaller are the holes' sizes, the less frequent are transitions through the handle, and the greater are the times a trajectory spends on the island before leaving it or in the chaotic sea before returning to the island. However, the ratio of average durations of chaotic and laminar phases remains constant.

The value of this ratio can be estimated, assuming that average durations of laminar and chaotic phases of motion are proportional to the volumes of corresponding regions in phase space. Generally, it is extremely difficult to calculate analytically the volume of phase space under islands of stability. However, the considered billiard has a region of parameters where the structure of the islands is simplified. In particular, when $a = 100$, $a_x = 80$, and $a_y \in (20,34)$ there is only one island in the phase space, and its size is limited by the periodic "bird" orbit, which is unstable [21]. When $a_y > 34$ or $a_y < 20$ the island's size is limited by the parabola end points and determination of the island's shape becomes complicated. For $a_y \in (20,34)$ in Birkhoff coordinates it has the form of a rhombus [see Fig. 2(b)], whose parameters can be determined from the simple bounding unstable periodic orbit. Appropriate calculation results in the following formula for the island's volume:

$$s_{\rm isl} = x_{\max} \frac{x_{\max}\sqrt{a^2 + x_{\max}^2} + a^2 \operatorname{arcsinh}\left(\frac{x_{\max}}{a}\right)}{a L_{\rm per}\sqrt{a^2 + x_{\max}^2}}, \qquad (2)$$

where

$$x_{\max} = \sqrt{2\left(2aa_y + a_x^2 - a^2\right)},$$

$$L_{\rm per} = 2\sqrt{1 + \frac{a_x^2}{a^2}}\left[a_x + 2a_y \arcsin\left(\frac{a}{\sqrt{a^2 + a_x^2}}\right)\right] + 2a \operatorname{arcsinh}\left(\frac{a_x}{a}\right).$$

The value $x_{\max}$ has the meaning of maximum possible deviation of the island's trajectories from the origin along the $x$ axis, and $L_{\rm per}$ is the length of billiard's perimeter. Comparison of the chaotic sea share in the phase space, calculated according to the formula $1 - s_{\rm isl}$, with a share of chaotic components in the intermittent trajectory, is shown at Fig. 5. It is obvious that they are in good agreement.

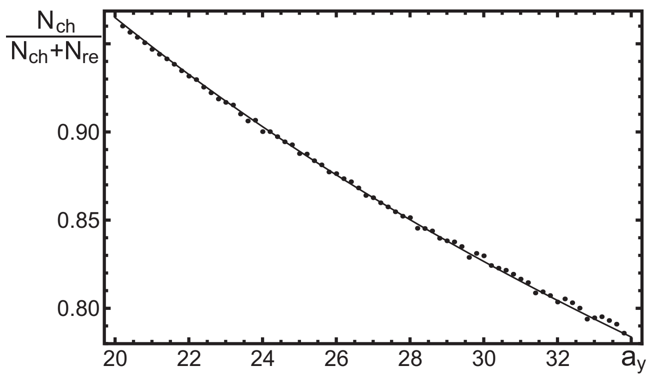


FIG. 5. Dependence of the share of the chaotic component in intermittent trajectory on the billiard's parameter $a_y$ for $a = 100$, $a_x = 80$, and hole sizes $d_1 = 1.45$ and $d_2 = 2.9$. The trajectory was constructed until $N = 0.5 \times 10^5$ chaotic sites was achieved. The solid line is a plot of the share of chaotic sea in the phase space $1 - s_{\rm isl}$.

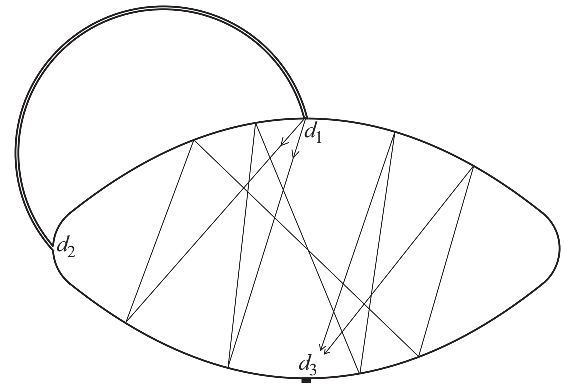


FIG. 6. The position of special site $d_3$ and the property of the central island to focus trajectories on it.

Therefore, the considered billiard allows the construction of intermittent trajectories with predetermined average duration and share of chaotic and laminar phases. The average duration of phases is controlled by the holes' sizes, and their ratio is controlled by the billiard parameters.

## IV. MODIFICATION OF A BILLIARD NONSPECULAR REFLECTION SITE

After the addition of a billiard's handle, a question naturally arises about the possibility of unbalanced flow of particles through the handle. In a certain sense, it could be regarded as a question about the possibility of a modified Maxwell's demon. It is clear that the emergence of a flow is impossible for an ideal gas in the state of equilibrium. However, the question remains for nonequilibrium states. Let us consider in more detail the structure of billiard's trajectories and possibilities to make unbalanced the numbers of passages through the handle in forward and reverse directions.

The positions of holes $d_1$ and $d_2$ were chosen so that through hole $d_1$ the trajectories would mostly get onto the central island and rarely into the chaotic sea, and through the $d_2$ the trajectories would only get into the chaotic sea and never on the island. All the trajectories leaving the island would pass through hole $d_1$. All trajectories that enter through hole $d_2$ are in chaotic sea and may leave the billiard through any hole, mostly through $d_2$. The distribution of leaving-chaotic-sea trajectories between holes $d_1$ and $d_2$ depends on the size of the central island and, accordingly, the parameters of the billiard. For a sufficiently large island, the number of particles leaving through hole $d_2$ is much greater than that through hole $d_1$ of equal size.

For some parameters of the billiard, trajectories of its central island are arranged in such way that if the trajectory starts in the vicinity of hole $d_1$, then after the half period of the island's motion it gets to the border site, opposite hole $d_1$. In other words, as shown in Fig. 6, the island focuses on some border site $d_3$ all passing hole $d_1$ trajectories. This is due to a regular character of motion on the island. We can replace a site $d_3$ with some special border site, which will transfer the trajectories from the island to the chaotic sea. Then most of the trajectories entering the billiard through hole $d_1$ will first get to the island, then through this special site into the chaotic sea, and from there, mainly through hole $d_2$, into the handle and back to hole $d_1$. Thus, there may occur a circular flow of particles entering the billiard through hole $d_1$ and leaving through $d_2$.

For the above-described mechanism to work, it is necessary for the number of particles passing through a special site from an island to a chaotic sea to be greater than the number of particles passing through that site in the opposite direction, from the chaotic sea to the island. At first glance this is expected to be, regardless of the particular arrangement of this site. Because the island due to its regularity rapidly focuses trajectories on this site, it is in a certain sense of a large effective size and is easy to be found, whereas for a chaotic sea that little border site does not differ. However, it appeared that if a special site is organized as some scattering irregularity of a border, in the form of a beak, semicircle, etc., reflection from each point of which is specular, the flow through the handle does not occur. The reason for this is that the billiard's motion preserves the phase volume, and the trajectories in a chaotic sea are evenly distributed over the phase volume. If there is a trajectory that starts in the chaotic sea, passes successively holes $d_1$ and $d_2$, and returns into chaotic sea, there also is a reverse trajectory. The rate of getting a long trajectory in some equal phase-space volumes surrounding such forward and reverse trajectories is the same. Therefore, the appearance of a stationary flow in a pure billiard is impossible.

Let a special border site be a straight site, as shown at Fig. 7 with a modified law of reflection. As a replacement of the specular reflection law we choose for this site the next

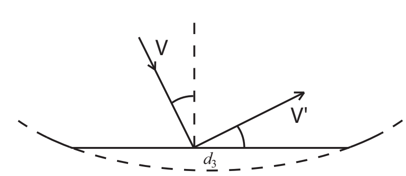


FIG. 7. The replacement of a small part of the border by a special flat site $d_3$ and its law of reflection.

simple law of reflection:

$$v_x' = \text{sgn}(v_x)|v_y|, \\ v_y' = |v_x|. \tag{3}$$

In invariant form it also can be written as

$$V_n' = V_\tau, \quad V_\tau' = \pm V_n. \tag{4}$$

Here the prime denotes the velocity of a particle after reflection, $V_n$ is the velocity component normal to the boundary at a point of incidence, and $V_\tau$ is tangential component. Geometrically, this law corresponds to the reflection at a right angle to the initial direction of particle motion. It is easy to see that this law of reflection preserves some important properties of specular reflection ($V_n = -V_n$, $V_\tau' = V_\tau$). Such reflection does not violate the reversibility of trajectories and energy conservation law. However, every single reflection act does not retain a particle's momentum.

Let us discuss the possibility of implementing such a law of reflection. First, we note that a specular reflection of light, for example, generally speaking, is an abstraction. Upon reflection of a beam of electromagnetic waves from a flat surface, there is a shift of the beam in the plane of incidence, corresponding to the reflection from a certain effective surface lying below the real one. This shift depends on the angle of incidence of the beam. For total reflection of polarized light in addition there is a shift relative to the plane of incidence, called the Fedorov effect [33]. It is therefore of some interest how robust the results obtained for an idealized case of specular reflection.

Apart from some natural nonspecularity, surfaces with a nonmirror law of reflection can be created artificially. For example, the method of creation of surfaces with structure (essentially, a metamaterial), proposed in the paper [24], allows them to have almost any preset law of reflection. In the paper [34] nonspecular reflection of electromagnetic waves from an array of short-circuited coaxial-sector waveguides has been studied, in particular the mode of orthogonal reflection from a two-dimensional periodic lattice, which implements the law of reflection close to our law (3) with complete transformation of energy of incident wave into the energy of the wave reflected in orthogonal direction.

It should be noted that, apart from the law (3), we also consider other laws of reflection, including those violating the reversibility and the conservation of particle's energy. In all cases, a macroscopic flux of particles of greater or lesser intensity also appeared through the handle.

## V. DIRECTED FLOW OF PARTICLES

Let us now consider the behavior of an ideal gas of noncolliding particles placed inside the above-described system with a handle and a special border site. As the particles do not collide with each other, the behavior of the gas can be reduced to the behavior of a single particle provided that its initial data do not belong to the remaining intact islands of stability. Since the system is not ergodic, it is essential that we consider long single trajectories of individual particles, but not uniformly distributed over the phase volume short trajectories. Some time after the beginning of a particle's motion, a stationary state would be established. The establishment of equilibrium in the case of nonergodic gas behavior is considered in detail in the paper [35].

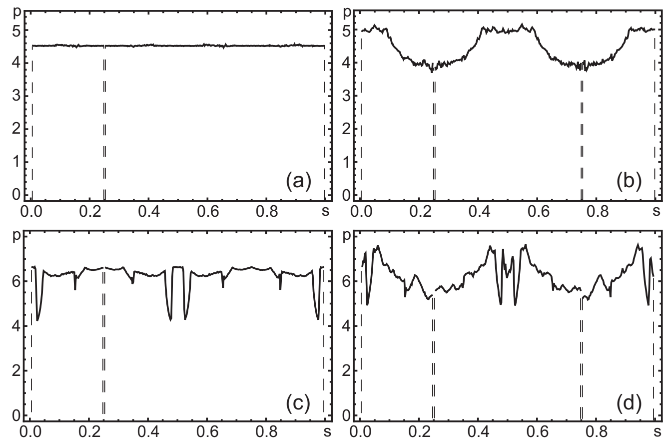


FIG. 8. Pressure on the walls of a central billiard, created by the trajectory of length $N = 10^9$ collisions (a) for the billiard without site $d_3$ and with parameters $a = 100$, $a_x = 80$, $a_y = 12$, $d_1 = 1.45$, $d_2 = 2.9$, and (b) for the billiard with the same parameters and with scattering site $d_3 = 1.45$. Vertical dashed lines indicate the holes and special site positions. (c),(d) Analogous distributions for the same billiard with parameter $a_y = 3$.

Let us consider the distribution of pressure which gas exerts on the walls of billiard. The local pressure sets stationary and does not depend on the initial data of a trajectory, provided that it is not a trajectory belonging to an intact island. Figures 8(a) and 8(c) show the pressure distributions in the billiard with no special site $d_3$. In this case there is no flow of particles through the handle. The case of $a_y = 12$ corresponds to the only one central island in the phase space, and with its destruction the entire phase space become available for any trajectory. In this case the pressure in the system appears practically uniform. If $a_y = 3$, in addition to the central island there are smaller islands of stability in the phase space, part of them remains intact for the open-holes case. As a consequence, the pressure in the system is uneven.

Figures Figs. 8(b) and 8(d) show the same distributions for the billiard with the same parameters, but with site $d_3$ added. It is evident that its introduction radically changes the distribution of pressure in the system. Pressure on those parts of the border where there was a significant contribution of the central island's trajectories had decreased significantly. This is due to the efficient transition of trajectories from the broken island to chaotic sea, where, correspondingly, the pressure increases. In this way there appears a difference of pressure between holes, which causes the appearance of a flow in the handle. It is interesting to note that the pressure inside the handle is uniform, because the particle's motion is regular during any single transition through the handle and the frequency of collisions is constant along the handle's border. Each passage creates additional pressure, the same for all border sites and not depending on the direction in which the handle was passed. Therefore, the pressure gradient inside the handle cannot occur regardless of the presence or direction of a flow.

The direction of motion of a particle in the billiard's handle is constant. If the particle enters the handle, for example, through hole $d_1$, then moving regularly it will, for certain, reach hole $d_2$ and leave the handle through it. This gives the possibility to assign a certain direction to each passage through the handle. We will count as positive a clockwise direction of motion in the handle, from hole $d_2$ to hole $d_1$, and the reverse direction, respectively, as negative. As a value characterizing the emergence of directed flow in the handle w will use the ratio of imbalance in the number of passages $n_+ - n_-$ to the total number of passages in both positive and negative directions $P = \frac{n_+ - n_-}{n_+ + n_-}$.

Before addition of a special site $d_3$ the flow in the system was absent for all possible parameter values. The addition of this site results in the occurrence of a flow of some value $P > 0$, which remains constant over time. Transitions through the handle in the positive direction are more frequent. The reason for this can be understood from the simple considerations. Indeed, without a special site, the numbers of passes through both holes being equal, there is no flow. With the addition of special site, trajectories began to move through this site from the chaotic sea to the island and more frequently in the opposite direction. Transfers from the island to chaotic sea are more frequent as a consequence of the choice of reflection law leading to such special site's property. Therefore, there is an excess of particles in the chaotic sea, and they leave the billiard mostly through hole $d_2$, leading to the excess of the handle's passages in the positive direction. Thus, the direction of emerging flux depends on the choice of a law of reflection from the special site and (in case of law (3)) is positive.

The dependence of a flow intensity on the size of special site $d_3$ has been obtained, having all other system parameters fixed (see Fig. 9). It is visible that with increase in the length of site $d_3$, flux, starting from zero for $d_3 = 0$, increases in a linear manner until the size of site $d_3$ reaches the size of hole $d_1$. With the further increase in $d_3$, the flow intensity also continues to rise approximately linearly, but at a smaller angle of inclination.

There were also built dependencies of a flow magnitude on holes sizes $d_1$ and $d_2$; they are shown in Fig. 9. It is seen that for dependence on $d_1$ the largest value of the difference of number of passages in positive and negative directions is achieved at small $d_1$ sizes. This is due to the fact that for a small hole $d_1$ the trajectories are more effectively focused on the relatively large, in comparison with hole $d_1$, special site $d_3$. With the increase of size $d_1$ having $d_3$ fixed, the magnitude of flow through the handle decreases significantly. The dependence of the flow on hole $d_2$, as can be seen from Fig. 9, is quite insignificant. Probably, it appears because the change of the hole's size $d_2$ leads to the change in the geometry of the handle. This changes the conditions under which the approaching particle would enter the handle. The overall effect of hole $d_2$ on the flow is negligible. The dependence was built starting from the values of $d_2 \approx 1.2$, since for lower values it is impossible to build the handle properly.

It is interesting to note the complexity of influence of the central billiard's geometry. We consider the dependence of the magnitude of flow through the handle on the central billiard's parameter $a_y$ (Fig. 10). It is seen that this dependence is complicated, not monotone, oscillating with considerable

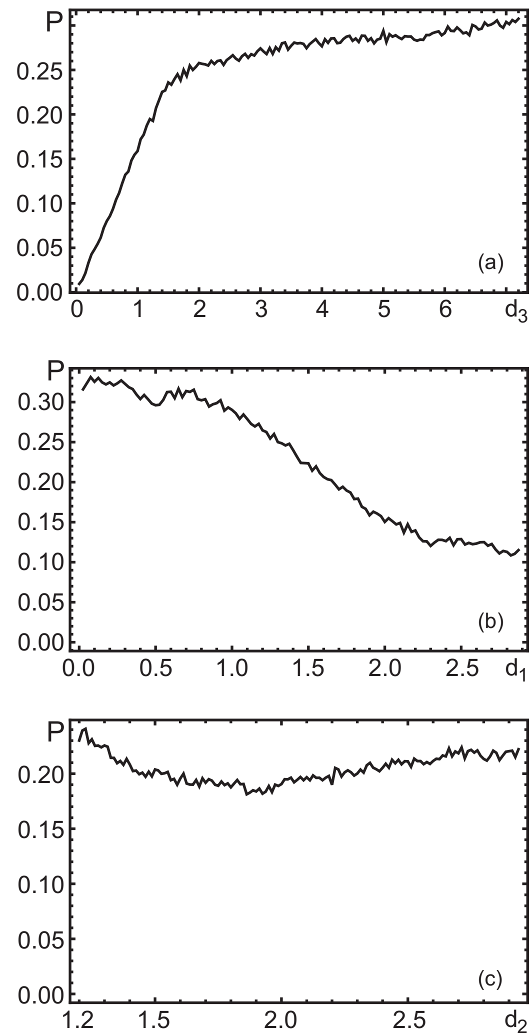


FIG. 9. The dependencies of intensity of a flow through the handle (a) on $d_3$ at $d_1 = 1.45$ and $d_2 = 2.9$, (b) on $d_1$ at $d_2 = 2.9$ and $d_3 = 1.45$, and (c) on $d_2$ at $d_1 = 1.5$ and $d_3 = 1.45$. Parameters of the billiard with handle are $a = 100$, $a_x = 80$, $a_y = 9.5$. The trajectories were built up to the number $n_+ = 0.5 \times 10^5$ of passes through the handle in a positive direction was reached.

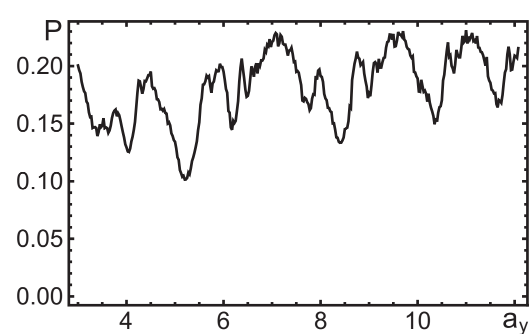


FIG. 10. The dependence of magnitude of flow through the handle on parameter $a_y$. Billiard parameters are $a = 100$, $a_x = 80$, $d_1 = d_3 = 1.45$, $d_2 = 2.9$. The trajectory was constructed up to the achievement of $n_+ = 0.5 \times 10^5$ passages through the handle in a positive direction.

amplitude. The values of the flux magnitude in the vicinity of local minimum and maximum are several times different. This is due to the fact that with the change of parameter $a_y$ there is a reconstruction of the entire phase portrait of the system. In particular, it changes the size of central island of stability and characteristics of its regular motion, including focusing of trajectories on site $d_3$.

## VI. THE INFLUENCE OF COLLISIONS BETWEEN PARTICLES

Let us now consider the influence of collisions on the occurrence of directed flow in the handle. The introduction of collisions between particles qualitatively changes the system; it now cannot be reduced to the motion of one particle in the considered billiard. It should be noted that a reduction to the motion of a single particle in the billiard is still possible, but the dimensions and shape of this billiard will be totally different from those described above (see, for example, [36]). With interparticle collisions turned on, there are two additional parameters of the system: the number of particles in the billiard $N$ and, assuming all particles are circles of equal size, the particle's radius $r$. Of course, the size of the particles will be smaller than the thickness of the handle. The frequency of collisions between particles depends on both these parameters. We assume all collisions to be perfectly elastic. In the case of a billiard with collisions it is necessary to monitor not only the position and direction of the particle's motion, but also its velocity. In an ordinary billiard a trajectory does not depend on it. Collisions require complete knowledge of particle velocities to determine whether there will be a collision.

Let the gas of such particles be placed in the above-described billiard with handle. For unpointlike particles it is convenient to monitor the positions of the centers of the particles, and we assume that they move like those placed in our billiard. This means that for billiard with collisions its actual border is at a distance $r$ from that described above. The main issue to clarify is the dependence of flux value $P$ on the frequency of collisions $p_{\rm coll} = \frac{N_{\rm par}}{N_{\rm par}+N_{\rm bnd}}$ between particles, where $N_{\rm par}$ is the number of collisions between particles and $N_{\rm bnd}$ is the number of collisions of particles with the billiard's boundary. The frequency of collisions may vary in the range $0 < p_{\rm coll} < 1$, where $p_{\rm coll} = \frac{1}{3}$ means that each particle has on average one collision with another particle per one collision with billiard's boundary (two particles collide once with each other and once each with a border). As a result of collisions, a particle may change the direction of its motion in the handle. We then define a quantity characterizing a flow as $P_c = \frac{n_{\rm in}^{(2)} - n_{\rm out}^{(2)} + n_{\rm out}^{(1)} - n_{\rm in}^{(1)}}{n_{\rm in}^{(1)} + n_{\rm out}^{(1)} + n_{\rm in}^{(2)} + n_{\rm out}^{(2)}}$, where $n_{\rm in}^{(1)}$ is the number of particles entering billiard through hole $d_1$, $n_{\rm out}^{(1)}$ is the number of particles leaving billiard through that hole, and $n_{\rm in}^{(2)}$ and $n_{\rm out}^{(2)}$ are the numbers of particles entering and leaving, respectively, hole $d_2$. In the absence of collisions, the value $P_c$ coincides with the value $P$ introduced above.

Using numerical simulation, we calculate the dependence of the flux $P_c$ on the frequency of collisions between particles $p_{\rm coll}$. In the case of a billiard without a special site it appears that a flow still does not occur. This is quite natural, since elastic collisions between particles lead to the establishment of an equilibrium state of gas. The appearance of a directional flow in this case would be in conflict with the laws of thermodynamics. For a billiard with a special site, the obtained dependence is shown at Fig. 11. Except the last two points, the frequency of collisions is increased via increased particle radius without a change in their number. It is seen that at low collision frequencies, when reflections from the border of a billiard dominate, the flow rate is almost the same as in the collisionless case. Collisions between particles make it necessary to consider the system of many particles in a high-dimensional phase space. In fact, collisions open a new channel of additional chaotization in the system. Within the framework of a single-particle approximation it can be expected that each particle during the laminar phase of its motion remains on an island for a long-enough time, so even the small frequency of collisions, in principle, could significantly affect the flow rate. Apparently, the loss of particles is compensated by their arrival on an island from the chaotic sea. With increase in the frequency of collisions between particles, the magnitude of flow through the handle diminishes to complete disappearance. The magnitude of flow decreases quickly enough until the particles start to collide with each other at the same frequency as with billiard's border. The flow at this point is of a small but nonzero magnitude, and with further increase in collisions rate decreases very slowly. To ensure it diminishes to zero, there were built two separate points in the vicinity of $p_{\rm coll} \approx 0.8$.

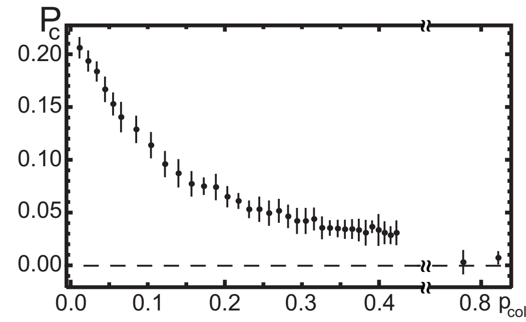


FIG. 11. Dependence of flow intensity on frequency of collisions between particles. The parameters of the billiard are $a = 100$, $a_x = 80$, $a_y = 3$, $d_1 = d_3 = 1.45$, $d_2 = 2.9$; the width of handle corresponding to these parameters is $d = 1.376$. Frequency of collisions between particles is varied from $p_{\rm coll} = 0$ to $p_{\rm coll} = 0.42$ via having the radius of $N = 50$ particles changed from $r = 0$ to $r = 0.64$. The trajectories were constructed up to the achievement of $n_{\rm out}^{(1)} = 0.5 \times 10^5$ passes through hole $d_1$. To verify that the flow completely disappears, two additional points were built for the numbers of particles $N = 300$ and $N = 400$ at $r = 0.5$.

Thus, collisions have little effect on the behavior of the system in the case of mean free path of particles being an order of magnitude higher than a characteristic system size. Unlike many other effects associated with islands of stability, the flow through the handle is resistant to the appearance of rare collisions between particles in the system. If the mean free path becomes of the same order or smaller than the system size, a significant flow of particles through the handle is no longer observed. Thus, the obtained results are relevant for collisionless or thin-enough gas, where particles collide with the boundary an order of magnitude more frequently than with other particles. Such a gas can have density and pressure comparable to atmospheric, but be located, for example, in a gas-filled nanopore in a material.

## VII. CONCLUSIONS

In this paper we consider an open billiard with two holes, connected by a handle. It is shown that the holes destroy islands of stability, which leads to almost all trajectories of the system appearing to be interleaved. The dependence of the share of chaotic component in interleaved trajectories on system parameters was built. It is shown that, by changing system parameters, it is possible to control an average duration and a ratio of chaotic and regular components, varying a level of chaos in a wide range.

It is shown that an unbalanced flow of particles through the handle occurs only after the addition of a special border site to the system. Without this site with a modified law of reflection, such flow cannot occur. Obtained are dependencies of a magnitude of flow through the handle on the parameters of a billiard, the sizes of holes, and the size of a special border site. Distribution of the pressure in the system before and after addition of special site was built. It was shown that the occurrence of a flow through the handle was accompanied by the occurrence of a pressure difference in the vicinity of the holes. Along with this, a pressure gradient inside of the handle for sure does not exist.

The case of colliding particles was considered. It was shown that, for lengths of particles' free path being an order of magnitude higher than the characteristic size of the system, collisions do not affect the system's behavior significantly. For lengths of free path of the same order or less than the system size, there is no considerable flow of particles through the handle.

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