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Inverse Gaussian and its inverse process as the subordinators of fractional Brownian motion

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In this paper we study the fractional Brownian motion (FBM) time changed by the inverse Gaussian (IG) process and its inverse, called the inverse to the inverse Gaussian (IIG) process. Some properties of the time-changed processes are similar to those of the classical FBM, such as long-range dependence. However, one can also observe different characteristics that are not satisfied by the FBM. We study the distributional properties of both subordinators, namely, IG and IIG processes, and also that of the FBM time changed by these subordinators. We establish also the connections between the subordinated processes considered and the continuous-time random-walk model. For the application part, we introduce the simulation procedures for both processes and discuss the estimation schemes for their parameters. The effectiveness of these methods is checked for simulated trajectories.

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I. INTRODUCTION

Fractional Brownian motion (FBM) is a Gaussian stochastic process that is non-Markovian. It belongs to the class of long-range-dependent (LRD) systems with self-similarity property. It is also considered as one of the classical second-order processes with so-called anomalous diffusion behavior. The process is closely related to fractional Langevin equation motion [1] and can be considered as a generalization of the classical Brownian motion (BM). One can find in the literature many research papers devoted to different theoretical aspects related to FBM [2–6]. The FBM has been widely used in several applications such as polymer translocation through a pore [7] and the dynamics of a tagged monomer [8] and is also a commonly used probabilistic model in areas such as finance [9] and hydrology [10].

However, for many real-life data with long-range dependence, the classical FBM cannot be considered an appropriate model. One of the possible solutions is the time-changed FBM. In general, the time-changed processes are constructed as the superposition of two stochastic systems. The first one is called an external process, while the second one, called an internal process or a subordinator, is generally a nondecreasing process with stationary independent increments with right continuous left limits sample paths. In the literature, one can find different papers related to processes delayed by subordinators or even inverse subordinators that are constructed as the inverse processes to subordinators [11]. One can also find applications of subordinated processes to different fields such as financial time series [12,13], indoor air quality data [14], biology [15], physics [11], and many other disciplines; see [16] and the references therein. It is worth mentioning that the idea of subordination was introduced by Bochner [17].

In this paper we analyze the time-changed FBM with two different subordinators: the inverse Gaussian (IG) and its inverse [inverse to the inverse Gaussian (IIG)]. We mention that the normal inverse Gaussian (NIG) process was introduced and studied by Barndorff-Nielsen [18]. It is obtained by subordinating BM having drift with the IG process to model the financial data. It is a process with stationary and independent increments with sample paths having jumps. The NIG density exhibits the so-called semiheavy tails, typically observable in finance and geophysics [19]. The NIG distribution has also found interesting applications in a multiscale entropy description [20]. In this paper we use both IG and IIG processes as subordinators and indicate their main characteristics.

The time-changed FBM with IG processes in the literature is known as the fractional normal inverse Gaussian (FNIG) process [19]. In this paper we extend the results of [19] and present more interesting properties, especially related to the long-range-dependence property, infinite divisibility, simulation of sample paths, and parameter estimation of the FNIG process. Moreover, as it was mentioned, we analyze also the FBM time changed by the IIG process. We compare the main properties of both subordinated processes, such as distributional characteristics and covariance structures. We show that both time-changed processes, like FBM, exhibit the long-range-dependence property. We also present the simulation procedures for both subordinated processes considered. The procedures are based on the fact that the analyzed systems are

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superpositions of two independent mechanisms. Further, we also propose parameter estimation schemes for both processes. It is noted that the estimation techniques for both time-changed systems are different. For the FBM delayed by the IG process, the estimation method is based on the asymptotic behavior of the right tail and the fact that the mean square displacement of the process has the same asymptotic behavior as the classical FBM. For the FBM delayed by the IIG process, the constant time periods observable in the trajectory constitute a sample of independent data from the IG distribution. In order to estimate the parameters of this distribution, we use here a method based on the rescaled modified cumulative distribution function. In order to check the effectiveness of estimation techniques, we analyze the simulated trajectories of the processes considered.

The rest of the paper is organized as follows. In Sec. II we introduce the IG and IIG processes and indicate their main characteristics. In Sec. III we study the FBM time changed by the IG process and show its properties related to the asymptotic behavior of distribution characteristics and indicate the relationship between this process and the continuous-time random-walk (CTRW) model. In Sec. IV we analyze the equivalent characteristics of the FBM time changed by the IIG process. In Sec. V we show how to simulate the subordinated processes and discuss the methods of estimation of their parameters. We summarize in Sec. VI.

II. INVERSE GAUSSIAN AND ITS INVERSE PROCESS

A. The IG process

In this section the main properties of the IG process are given. Some of these results are taken from the literature, while other results such as the asymptotic behavior of the probability density function (PDF), tail probabilities, and asymptotic behavior of moments are presented and used in the present paper.

Let $B(t)$ be the standard BM and $B_\gamma(t)$ be the standard Brownian motion with positive drift such that $B_\gamma(t) = B(t) + \gamma t$, where $\gamma > 0$. The inverse Gaussian process $G(t)$ with the parameters δ and γ, both positive, is defined by [21]

$$G(t) = \inf\{s > 0 : B_\gamma(s) > \delta t\}. \tag{2.1}$$

As one can see, the $G(t)$ process can be considered as the first time the BM with drift γ hits the barrier δt. The increments $G(t+s) - G(s)$ follow the inverse Gaussian $\mathcal{G}(\delta t,\gamma)$ distribution having the PDF [22]

$$g(x,t) = \frac{\delta t}{\sqrt{2\pi x^3}} e^{\delta\gamma t - (\delta^2 t^2/x + \gamma^2 x)/2}, \quad x > 0. \tag{2.2}$$

The IG process $G(t)$ is a nondecreasing Lévy process (i.e., process with stationary independent increments) with Lévy measure π given by [22]

$$\pi(dx) = \frac{\delta}{\sqrt{2\pi x^3}} e^{-(\gamma^2/2)x} dx, \quad x > 0. \tag{2.3}$$

Let $\mathcal{L}_{x\to u}(g(x,t)) = \tilde{g}(u,t)$ be the Laplace transform (LT) of g with respect to the space variable x. Then

$$\tilde{g}(u,t) = \mathbb{E}(e^{-uG(t)}) = e^{-t\delta(\sqrt{2u+\gamma^2}-\gamma)}. \tag{2.4}$$

Therefore, the Laplace exponent $\Psi_G(u) = \delta(\sqrt{2u+\gamma^2} - \gamma)$. Note that almost all sample paths of $G(t)$ are strictly increasing with jumps, since the sample paths of $B_\gamma(t)$ are continuous and have intervals where paths are decreasing. It is easy to see that

$$g(x,t) \sim \frac{\delta t}{\sqrt{2\pi}} e^{\delta\gamma t} x^{-3/2} e^{-(\gamma^2/2)x} \quad \text{as } x \to \infty,$$

where $f(x) \sim g(x)$ as $x \to x_0$ means $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$. Thus, the tail probability for the IG process satisfies

$$\mathbb{P}(G(t) > x) \sim \sqrt{\frac{2}{\pi}} \frac{\delta t}{\gamma^2} e^{\delta\gamma t} x^{-3/2} e^{-(\gamma^2/2)x} \quad \text{as } x \to \infty, \tag{2.5}$$

which is exponentially decaying and hence all moments of $G(t)$ process are finite. Let $K_\nu(\omega)$ be the modified Bessel function of the third kind with index ν, defined by

$$K_\nu(\omega) = \frac{1}{2}\int_0^\infty x^{\nu-1} e^{-(x+x^{-1})/2} dx, \quad \omega > 0. \tag{2.6}$$

An asymptotic expansion for $K_\nu(\omega)$ for large ω [see Eq. (A.9) in [23]] is

$$K_\nu(\omega) = \sqrt{\frac{\pi}{2}} \omega^{-1/2} e^{-\omega} \left(1 + \frac{\mu-1}{8\omega} + \frac{(\mu-1)(\mu-9)}{2!(8\omega)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8\omega)^3} + \cdots\right), \tag{2.7}$$

where $\mu = 4\nu^2$. The qth moment of the $\mathcal{G}(\delta t,\gamma)$ distribution [23] is given by

$$\mathbb{E}G^q(t) = \frac{K_{q-1/2}(\delta\gamma t)}{K_{-1/2}(\delta\gamma t)} \left(\frac{\delta t}{\gamma}\right)^q = \sqrt{\frac{2}{\pi}} \delta \left(\frac{\delta}{\gamma}\right)^{q-1/2} t^{q+1/2} e^{\delta\gamma t} K_{q-1/2}(\delta\gamma t). \tag{2.8}$$

Using the fact that $K_\nu(\omega) \sim \sqrt{\frac{\pi}{2}} e^{-\omega}\omega^{-1/2}$ for large ω [23], we obtain

$$\mathbb{E}G^q(t) \sim \left(\frac{\delta}{\gamma}\right)^q t^q \quad \text{as } t \to \infty. \tag{2.9}$$

From [24], the density functions $g(x,t)$ of $G(t)$ satisfy

$$\frac{\partial^2 g}{\partial t^2} - 2\delta\gamma \frac{\partial g}{\partial t} = 2\delta^2 \frac{\partial g}{\partial x}.$$

When $\gamma = 0$ the above equation is closely related to classical heat equation with time and space variables interchanged. Now let us consider the sequence $X_1, X_2, \ldots, X_n$ such that $\mathbb{E}X_i = \mu$ and $\mathrm{Var}(X_i) = 1$, $i = 1,2,\ldots,n$. Then we define the rescaled sum

$$T_n^m = \sum_{i=1}^n \left(\frac{1}{\sqrt{m}}(X_i - \mu) + \frac{\gamma}{m}\right), \quad n \geqslant 1,\ m > 0.$$

Using Donsker's theorem, it follows that $T_{[mt]}^m \Rightarrow B(t) + \gamma t$ in $(\mathbb{D}, J_1)$, as $m \to \infty$, where $\mathbb{D}$ is the Skorohod space and J_1 is the Skorohod metric. Consider the supremum process

$$M_n^m = \sup_{0\leqslant j\leqslant n} T_j^m. \tag{2.10}$$

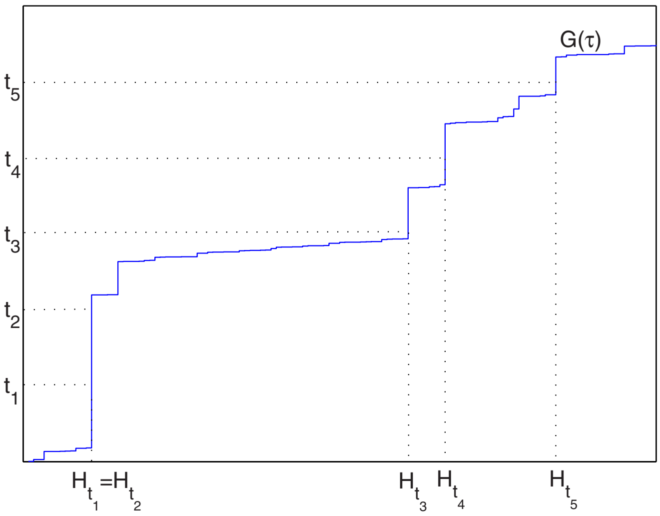

FIG. 1. Relation between the process $G(t)$ and its inverse $H(t)$. Observe that constant periods of $H(t)$ occur accordingly to the jumps of the process $G(t)$.

Then

$$G_t^m = \min\{n \geqslant 0 : M_n^m > t\} \tag{2.11}$$

is the first-exit time of M_n^m. As shown in [24], $m^{-1}G_t^m \Rightarrow G(t)$ in $(\mathbb{D}, J_1)$ as $m \to \infty$, where $G(t)$ is the IG process.

B. The IIG process

This section focuses on the IIG process. The results found are further used in deriving the properties of FBM time changed by IIG process.

The inverse to the inverse Gaussian process, denoted by $H(t)$, is the right continuous inverse of $G(t)$, defined by

$$H(t) = \inf\{s > 0 : G(s) > t\}, \quad t \geqslant 0. \tag{2.12}$$

Since the sample paths of $G(t)$ are strictly increasing with jumps, the sample paths of $H(t)$ are continuous and are constant over the intervals where $G(t)$ has jumps. In Fig. 1 we illustrate the relationship between the $H(t)$ and $G(t)$ processes. Note that $H(t)$ can also be considered as a first-passage time. For a general increasing non-negative process $G(t)$ with stationary independent increments corresponding to infinitely divisible distribution, the inverse process defined in (2.12) is always well defined [25]. We should mention that the general inverse subordinators have found various applications in probability theory. The connection between inverse subordinators and the theory of renewal processes can be found, for instance, in [26,27].

Note that

$$\begin{aligned}\{H(t) \leqslant x\} &= \{G(x) \geqslant t\}\\ &= \Big\{\sup_{s\leqslant t} B_\gamma(s) \leqslant \delta x\Big\}\\ &= \Big\{\delta^{-1}\sup_{s\leqslant t} B_\gamma(s) \leqslant x\Big\}\end{aligned} \tag{2.13}$$

and hence $H(t) \stackrel{\mathcal{L}}{=} \delta^{-1}\sup_{s\leqslant t} B_\gamma(s)$, where $\stackrel{\mathcal{L}}{=}$ means equivalence in law (or distribution). Thus the PDF of $H(t)$ can also be viewed as the density of supremum of BM with drift $B_\gamma(t)$. For a general strictly increasing process $G(t)$ with density function $g(x,t)$ and Laplace exponent Ψ_G, the density function $h(x,t)$ of the first-exit time process $H(t)$ has the LT [28]

$$\mathcal{L}_{t\to s}(h(x,t)) = \frac{1}{s}\Psi_G(s)e^{-x\Psi_G(s)}. \tag{2.14}$$

Since $G(t)$ is a strictly increasing process with the Laplace exponent $\Psi_G(u) = \delta(\sqrt{2u+\gamma^2}-\gamma)$, we obtain from (2.14) the LT of $h(x,t)$, the PDF of the $H(t)$ process, with respect to the time variable t, given by

$$\mathcal{L}_{t\to s}(h(x,t)) = \frac{1}{s}\delta(\sqrt{(2s+\gamma^2)}-\gamma)e^{-x\delta(\sqrt{(2s+\gamma^2)}-\gamma)}. \tag{2.15}$$

For $q > 0$, let $M_q(t) = \mathbb{E}H^q(t)$ be the qth-order moment of the process $H(t)$. Let $\tilde{M}_q(s)$ denote the LT of $M_q(t)$. Using [29], it follows that

$$\begin{aligned}\tilde{M}_q(s) &= \frac{1}{s[\delta(\sqrt{(2s+\gamma^2)}-\gamma)]^q}\\ &\sim \begin{cases}\frac{1}{(\delta\sqrt{2})^q}\frac{1}{s^{1+q/2}} & \text{for } \gamma = 0\\ \left(\frac{\gamma}{\delta}\right)^q\frac{1}{s^{q+1}} & \text{for } \gamma > 0\end{cases}\end{aligned} \tag{2.16}$$

as $s \to 0$. Using the Tauberian theorem [30], it follows, as $t \to \infty$, that

$$M_q(t) \sim \begin{cases}\frac{1}{(\delta\sqrt{2})^q}\frac{t^{q/2}}{\Gamma(1+q/2)} & \text{for } \gamma = 0\\ \left(\frac{\gamma}{\delta}\right)^q\frac{t^q}{\Gamma(1+q)} & \text{for } \gamma > 0.\end{cases} \tag{2.17}$$

Unlike the IG process, the one-dimensional distributions of the IIG process are not infinitely divisible and can be seen as follows. Note that for $u > 0$,

$$\begin{aligned}\mathbb{P}(H(t) > x) &= \mathbb{P}(G(x) \leqslant t) = \mathbb{P}(-uG(x) \geqslant -ut)\\ &= \mathbb{P}(e^{-uG(x)} \geqslant e^{-ut}) \leqslant \frac{\mathbb{E}e^{-uG(x)}}{e^{-ut}}\\ &= e^{ut-x\delta(\sqrt{2u+\gamma^2}-\gamma)}\\ &\leqslant e^{-\delta^2x^2/t+\delta\gamma x-\gamma^2t/2}.\end{aligned}$$

Thus $-\ln\mathbb{P}(H(t) > x) \geqslant \frac{\delta^2x^2}{t} - \delta\gamma x + \frac{\gamma^2t}{2}$. Hence,

$$\limsup_{x\to\infty}\frac{-\ln\mathbb{P}(H(t) > x)}{x\ln x} = \infty,$$

which implies, by Corollary 9.9 of [31], that $H(t)$ is Gaussian if $H(t)$ is infinitely divisible. This is a contradiction and hence $H(t)$ is not infinitely divisible. We state this result in the following proposition.

Proposition 1. The one-dimensional distributions of the IIG process are not infinitely divisible.

To find the exact tail behavior of $H(t)$, we employ Laplace's method [32], which is a general technique that allows us to generate an asymptotic expansion for Laplace integrals for large x. More precisely, suppose an integral has the form

$$I(x) = \int_a^b m(y)e^{x\phi(y)}dy,$$

where $m(y)$ and $\phi(y)$ are real-valued continuous functions. Suppose $\phi(\cdot)$ has a global maximum at $y = b$, $m(b) \neq 0$, and $\phi'(b) \neq 0$. Then, for large x (see [32]),

$$I(x) \sim \frac{m(b)e^{x\phi(b)}}{x\phi'(b)}. \tag{2.18}$$

Since $G(x)$ has density $g(y,x)$, defined in (2.2), we have

$$\begin{aligned}\mathbb{P}(H(t) > x) &= \mathbb{P}(G(x) \leqslant t)\\ &= \frac{\delta x}{\sqrt{2\pi}} e^{\delta\gamma x} \int_0^t y^{-3/2} e^{-(\gamma^2/2)y} e^{-(\delta^2/2y)x^2} dy\\ &\sim \frac{2}{\delta} t^{1/2} e^{-(\gamma^2/2)t} x^{-1} e^{\delta\gamma x - \delta^2 x^2/2t},\end{aligned} \tag{2.19}$$

which follows by using (2.18) and noting that $m(y) = y^{-3/2}e^{-(\gamma^2/2)y}$, $\phi(y) = -\frac{\delta^2}{2y}$, and ϕ attains global maximum at $y = t$.

Next we obtain the fractional partial differential equation (FPDE) satisfied by the PDF of the IIG process. The Riemann-Liouville (RL) fractional derivative of order α for the function $f(\cdot)$ is defined by [33]

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau\right], \tag{2.20}$$

where $\alpha \in (m-1,m)$. Consider the Laplace transform of the shifted fractional RL derivative given by [34]

$$\begin{aligned}&\mathcal{L}_{t\to s}\left\{\left(c + \frac{\partial}{\partial t}\right)^\nu f(t)\right\}\\ &= (c+s)^\nu \mathcal{L}_t\{f(t)\} - (c+s)^{\nu-1} f(0), \quad s > 0.\end{aligned} \tag{2.21}$$

We define $\mathcal{L}_{t\to s}(h(x,t)) = \hat{h}(x,s)$. Using (2.15) we obtain

$$\begin{aligned}\frac{\partial}{\partial x}\hat{h}(x,s) &= -\delta(\sqrt{(2s+\gamma^2)} - \gamma)\hat{h}(x,s)\\ &= -\delta(\sqrt{(2s+\gamma^2)}\hat{h}(x,s)\\ &\quad - (2s+\gamma^2)^{-1/2} h(x,0))\\ &\quad - \delta(2s+\gamma^2)^{-1/2} h(x,0) + \delta\gamma\hat{h}(x,s).\end{aligned}$$

Inverting the LT on both sides of this equation with the help of (2.21) and using $\mathcal{L}^{-1}[(s+a)^{-1/2}] = e^{-at}/\sqrt{\pi t}$ yields following result.

Proposition 2. The PDF $h(x,t)$ of the IIG process solves following FPDE involving the shifted power of the fractional derivative

$$\begin{aligned}\frac{\partial}{\partial x} h(x,t) &= -\delta\left(\gamma^2 + 2\frac{\partial}{\partial t}\right)^{1/2}\\ &\quad + \delta\gamma h(x,t) - \delta\sqrt{2}\frac{e^{-\gamma^2 t/2}}{\sqrt{\pi t}}\delta_0(x),\end{aligned} \tag{2.22}$$

with the initial condition $h(x,0) = \delta_0(x)$, the Dirac delta function concentrated at origin.

For $\gamma = 0$ and $\delta = 1/\sqrt{2}$, Eq. (2.22) reduces to

$$\frac{\partial}{\partial x} h(x,t) = -\frac{\partial^{1/2}}{\partial t^{1/2}} h(x,t) - \frac{\delta_0(x)}{\sqrt{\pi t}},$$

which is the FPDE corresponding to the density of $|B(t)|$.

It follows from (2.10) and Theorem 13.4.1 in [35] that

$$M^m_{[mt]} \Rightarrow \sup_{0\leqslant s\leqslant t} [B(s) + \gamma s] \quad \text{in space } \mathbb{D},$$

which is the same process as $H(t)$. Hence, for $\delta = 1$, $M^m_{[mt]} \Rightarrow H(t)$ in $\mathbb{D}$ as $m \to \infty$. Moreover, let

$$H^m_t = \min\left\{k \geqslant 0 : G^m_k > t\right\} \tag{2.23}$$

be the first-exit time of the process G^m_k defined in (2.11). We show that H^m_t converges weakly to the IIG process $H(t)$.

Proposition 3. As $m \to \infty$, it follows that

$$m^{-1} H^m_t \Rightarrow H(t) \quad \text{in } (\mathbb{D}, J_1). \tag{2.24}$$

Proof. First we establish the wconvergence of finite-dimensional distributions (FDDs). Note that as $m \to \infty$,

$$\begin{aligned}&\mathbb{P}\left\{m^{-1} H^m_{t_i} \leqslant x_i,\ i = 1,2,\dots,k\right\}\\ &= \mathbb{P}\left\{H^m_{t_i} \leqslant m x_i,\ i = 1,2,\dots,k\right\}\\ &= \mathbb{P}\left\{G^m_{[mx_i]} \geqslant t_i,\ i = 1,2,\dots,k\right\} \quad \text{(by 2.23)}\\ &= \mathbb{P}\left\{M^m_{[mt_i]} \leqslant x_i,\ i = 1,2,\dots,k\right\} \quad \text{(by 2.11)}\\ &\to \mathbb{P}\{H(t_i) \leqslant x_i,\ i = 1,2,\dots,k\}.\end{aligned}$$

Hence, $m^{-1} H^m_t \overset{\text{FDD}}{\Longrightarrow} H(t)$. Moreover, sample paths of H^m_t are monotonic and sample paths of $H(t)$ are continuous, since sample paths of $G(t)$ are strictly increasing. Hence, $H(t)$ is continuous in probability. Then the result follows by using Theorem 3 in [36], which states that in this situation convergence of the FDD is sufficient for weak convergence.■

In the next section we consider the IG and IIG processes as subordinators, i.e., processes that can replace the time of other processes. We analyze here the fractional Brownian motion as the external process. We will prove the main properties of the subordinated processes and introduce simulation and estimation procedures for the parameters corresponding to both systems.

III. FRACTIONAL BROWNIAN MOTION DRIVEN BY THE IG PROCESS

Fractional Brownian motion is a stationary-increment self-similar process that exhibits the so-called long-range dependence. It is also considered as a main model appropriate for the description of so-called anomalous diffusion phenomena. An anomalous diffusion property can be recognized, for example, by time-averaged mean square displacement (TAMSD). For a sample $\{X_i,\ i = 1,2,\dots,n\}$ with stationary increments, the TAMSD is defined as [37]

$$M_n(\tau) = \frac{1}{n-\tau} \sum_{k=1}^{n-\tau} (X_{k+\tau} - X_k)^2. \tag{3.1}$$

For FBM, it is known that the TAMSD $M_n(\tau) \overset{\mathcal{L}}{=} \tau^{2H}$, where $\overset{\mathcal{L}}{=}$ means equivalence in law and $H \in (0,1)$ is a parameter, called the Hurst exponent. We classify the FBM process with $H = 1/2$ as exhibiting linear dynamics. The process exhibits subdiffusive behavior if $H < 1/2$ and the superdiffusive behavior if $H > 1/2$. The FBM was introduced by Kolmogorov [38,39] and very often is considered as an extension of the classical BM. The FBM with index H is the mean-zero Gaussian process $B_H(t)$ defined as follows for $t > 0$ [38,40]:

$$B_H(t) = \int_{-\infty}^{\infty} [(t-u)_+^{H-1/2} - (-u)_+^{H-1/2}] dB(u), \quad (3.2)$$

where $(x)_+ = \max(x,0)$. For fixed t, the PDF of $B_H(t)$ is given by

$$f_{B_H(t)}(x) = \frac{1}{\sqrt{2\pi} t^H} e^{-x^2 t^{-2H}/2}, \quad x \in R. \quad (3.3)$$

Further, the covariance function of FBM with fixed s and as $t \to \infty$ behaves as

$$\mathbb{E}(B_H(s)B_H(t)) = \tfrac{1}{2}[s^{2H} + t^{2H} - (t-s)^{2H}] \sim Hst^{2H-1}. \quad (3.4)$$

The FBM time changed by the IG process $G(t)$ introduced in (2.1) is defined by

$$X(t) = B_H(G(t)), \quad (3.5)$$

under the assumption that $B_H(t)$ and $G(t)$ are independent.

In the following sections, the distributional properties of FBM time changed by the IG process are discussed. The CTRW connection was discussed in [24] and is provided here for completeness. More importantly, we establish the asymptotic behavior of the marginal density, the infinite divisibility, the long-range dependence of the process $X(t)$, and the ergodicity property.

A. Distributional properties of FBM time changed by the IG process

It is known that for standard normal random variable Z and for $q > 0$,

$$\mathbb{E}|Z|^q = \sqrt{\frac{2^q}{\pi}} \Gamma\left(\frac{1+q}{2}\right) \equiv c_q. \quad (3.6)$$

Using the self-similarity property of FBM and the independence of $B_H(t)$ and $G(t)$, it follows, as $t \to \infty$, that

$$\mathbb{E}|X(t)|^q = \mathbb{E}G^{qH}(t)\mathbb{E}|B_H(1)|^q = c_q \mathbb{E}G^{qH}(t) \sim c_q \left(\frac{\delta}{\gamma}\right)^{qH} t^{qH}. \quad (3.7)$$

Proposition 4. The density function $f_{X(t)}(x)$ of FBM time changed by the IG process has the asymptotic behavior

$$f_{X(t)}(x) = O(Mx^{-2(1+H)/(1+2H)} e^{-Nx^{2/(1+2H)}}), \quad (3.8)$$

while the tail probability $\mathbb{P}(X(t) > x)$ has the asymptotic behavior

$$\mathbb{P}(X(t) > x) = O\left(\frac{M(1+2H)}{2N} x^{-3/(1+2H)} e^{-Nx^2/(1+2H)}\right) \quad (3.9)$$

as $x \to \infty$, where

$$M = \frac{\delta t e^{\delta\gamma t}}{\sqrt{\pi H(1+2H)}} 2^{\frac{1}{4H}-\frac{1}{2}} \left(\frac{\gamma^2}{H2^{\frac{1}{2H}+2}}\right)^{\frac{1}{2(1+2H)}} \quad (3.10)$$

and

$$N = (1+2H)\left(\frac{\gamma^2}{H2^{1/2H+2}}\right)^{2H/(1+2H)}. \quad (3.11)$$

Proof. It follows that

$$\begin{aligned} f_{X(t)}(x) &= \frac{\delta t}{2\pi} e^{\delta\gamma t} \int_0^\infty \frac{1}{y^{H+3/2}} \exp\left[-\left(\frac{x^2}{y^{2H}} + \frac{\delta^2 t^2}{y} + \gamma^2 y\right)\Big/2\right] dy \\ &\leqslant \frac{\delta t}{2\pi} e^{\delta\gamma t} \int_0^\infty \frac{1}{y^{H+3/2}} \exp\left[-\left(\frac{x^2}{y^{2H}} + \gamma^2 y\right)\Big/2\right] dy \\ &= \frac{\delta t}{2\pi H} e^{\delta\gamma t} 2^{1/4H-1/2} \int_0^\infty z^{1/4H-1/2} \\ &\quad \times \exp\left(-x^2 z - \frac{\gamma^2}{2^{1/2H+1}} z^{-1/2H}\right) dz \\ &= C\int_0^\infty z^{c-1} e^{-x^2 z - az^{-b}} dz = Ch(x^2) \quad \text{(say)}, \end{aligned}$$

where $C = \frac{\delta t}{2\pi H} e^{\delta\gamma t} 2^{1/4H-1/2}$, $c = \frac{1}{4H} + \frac{1}{2}$, $a = \frac{\gamma^2}{2^{1/2H+1}}$, and $b = \frac{1}{2H}$. Using Example 2 of [41], we have, for large x,

$$h(x^2) \sim \left(\frac{ab}{x^2}\right)^{c/(b+1)} e^{-\lambda(1+1/b)} \sqrt{\frac{2\pi}{\lambda(b+1)}}, \quad (3.12)$$

where $\lambda = (abx^{2b})^{1/(b+1)}$. This implies, as $x \to \infty$, that

$$h(x^2) \sim \sqrt{\frac{2\pi}{b+1}} (ab)^{\frac{(2c-1)}{2(b+1)}} x^{-\frac{(b+2c)}{(b+1)}} e^{-\frac{(b+1)}{b}(ab)^{\frac{1}{(b+1)}} x^{\frac{2b}{(b+1)}}} \quad (3.13)$$

Thus, we have, as $x \to \infty$,

$$f_{X(t)}(x) = O(Mx^{-2(1+H)/(1+2H)}) e^{-Nx^{2/(1+2H)}}.$$

■

For $H < 1/2$ it follows that

$$\limsup_{x\to\infty} \frac{-\ln \mathbb{P}(X(t) > x)}{x \ln x} = \infty.$$

Thus, again by an application of Corollary 9.9 of [31], $X(t)$ is not infinitely divisible for $H < 1/2$. For $H \geqslant 1/2$, a different approach is required. The generalized gamma convolutions (GGCs) were introduced in [42] and have turned out to be a powerful tool in proving infinite divisibility of various distributions. A distribution on $\mathbb{R}^+$ is said to be a GGC if it is the weak limit of finite convolutions of gamma distributions [31]. Note that $G(t)$ is infinitely divisible and hence $G(t)^{2H}$ is also infinitely divisible for $2H \geqslant 1$ by using a result from [43]. Using the result in [44], $X(t) = B_H(G(t))$, which is a variance mixture of normal distribution, is infinitely divisible for $1/2 \leqslant H < 1$. Thus, we have following result for the infinite divisibility of $X(t)$.

Proposition 5. The one-dimensional distributions of the process $X(t)$ are infinitely divisible if and only if $1/2 \leqslant H < 1$.

Next we obtain the covariance function of process $X(t)$. Using standard conditioning argument and the relationship

$\mathbb{E}(X(s)X(t)) = \frac{1}{2}[\mathbb{E}(X^2(t)) + \mathbb{E}(X^2(s)) - \mathbb{E}(X(t) - X(s))^2]$ with (3.7), it follows that

$$\begin{aligned}\mathbb{E}X(t)X(s) = &\frac{\delta}{\sqrt{2\pi}}\left(\frac{\delta}{\gamma}\right)^{2H-1/2}\\ &\times [t^{2H+1/2}e^{\delta\gamma t}K_{1/2-2H}(\delta\gamma t)\\ &+ s^{2H+1/2}e^{\delta\gamma s}K_{1/2-2H}(\delta\gamma s)\\ &- |t-s|^{2H+1/2}e^{\delta\gamma(|t-s|)}K_{1/2-2H}(\delta\gamma(|t-s|))].\end{aligned} \tag{3.14}$$

For fixed s and large t, it follows from (2.7) that

$$\mathbb{E}X(t)X(s) \sim Hs\left(\frac{\delta}{\gamma}\right)^{2H} t^{2H-1}, \tag{3.15}$$

which is similar to the asymptotic behavior of covariance of FBM given in (3.4).

Next we discuss the LRD property of the process $X(t)$. A finite variance stationary process $W(t)$ is said to have the LRD property [22] if $\sum_{k=0}^{\infty}\gamma_k = \infty$, where

$$\gamma_k = \text{Cov}[W(t), W(t+k)].$$

Further, for a nonstationary process $W(t)$ an equivalent definition is given by the following definition.

Definition 1. Let $s > 0$ be fixed and $t > s$. Then the process $W(t)$ is said to have the LRD property if

$$\text{Corr}[W(s), W(t)] \sim c(s)t^{-d} \quad \text{as t} \to \infty, \tag{3.16}$$

where $c(s)$ is a constant depending on s and $d \in (0,1)$.

The process $X(t)$ is nonstationary. Using (3.7), we have $\text{Corr}[X(s), X(t)] \sim Hs^{1-H}t^{H-1}$ and hence by Definition 1, $X(t)$ possesses the LRD property for all H.

Since the FBM and IG processes both have stationary increments, the process $X(t)$ also has stationary increments. A stationary random process is also ergodic if the nth-order ensemble average is the same as the nth-order time average. We mention that the ergodicity properties of FBM and fractional Langevin motion are studied in [4]. The problems of ergodicity and nonergodicity of anomalous diffusion models is also discussed in [45]. Note that if a process is mean [or mean square displacement (MSD)] ergodic then one can estimate the mean (or MSD) using a sufficiently large single sample path of the process. The ensemble average MSD for the increments of $X(t)$ is given by

$$\begin{aligned}&\mathbb{E}(X(t+\tau) - X(t))^2\\ &= \mathbb{E}(B_H(G(t+\tau)) - B_H(G(t)))^2\\ &= \sqrt{2/\pi}\,\delta\left(\frac{\delta}{\gamma}\right)^{2H-1/2}|\tau|^{2H+1/2}e^{\delta\gamma|\tau|}K_{2H-1/2}(\delta\gamma|\tau|)\\ &\sim \left(\frac{\delta}{\gamma}\right)^{2H}|\tau|^{2H} \quad \text{for large } \tau.\end{aligned}$$

In this paper we show by simulation that the realized values of the TAMSD are concentrated around the ensemble average MSD for a different lag parameter τ. In Fig. 2 we present the theoretical ensemble average MSD (red solid line) and the realized TAMSD (blue star line) for the process $X(t)$ for two sets of parameters, namely, $H = 0.3$ and $\gamma = \delta = 1$ (top panel) and $H = 0.7$ and $\gamma = \delta = 1$ (bottom panel).

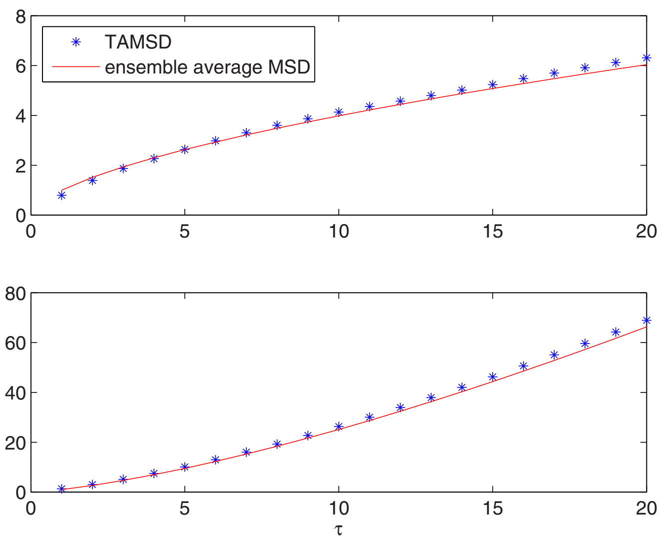

FIG. 2. Comparison of the TAMSD (blue star line) and ensemble average MSD (red solid line) for the process $X(t)$ for the case with $H = 0.3$ and $\gamma = \delta = 1$ (top) and $H = 0.7$ and $\gamma = \delta = 1$ (bottom).

B. The CTRW connection

Given independent identically distributed random variables Y_n representing the random jumps of a particle, the simple random walk $S_n = \sum_{i=1}^{n} Y_i$ locates the particle position after n steps. Let $X_n \geqslant 0$ be independent identically distributed waiting times between particle jumps; the random walk $T_n = \sum_{i=1}^{n} X_i$ gives the time of the nth jump. The counting process defined by $N(t) = \max\{n \geqslant 0 : T_n \leqslant t\}$ counts the number of jumps until time t. Further, $S_{N(t)} = \sum_{i=1}^{N(t)} Y_i$ gives the position of the particle at time t. As discussed in [24], the FBM time changed by the IG process can be obtained as the limit of a random walk with correlated jumps separated by independent identically distributed waiting times. More precisely, let Y_n be a stationary linear process defined by $Y_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$, where the Z_n are independent identically distributed and c_j are real constants such that $\sum_{j=0}^{\infty} c_j^2 < \infty$. Further, Y_n are, with mean zero and finite variance, independent of the independent identically distributed waiting times X_n. Suppose that the variance σ_n^2 of the sum $S_n = \sum_{j=1}^{n} Y_j$ varies regularly at ∞ with index $2H$ for some $0 < H < 1$, and for some constants $K > 0$ and $\rho > 1/H$ let $\mathbb{E}(S_n^{2\rho}) \leqslant K[E(S_n)^2]^{\rho}$. Then, as $m \to \infty$,

$$\sigma_{[m]}^{-1} S_{G_t^m} \Rightarrow B_H(G(t)) \quad \text{in space } (\mathbb{D}, J_1). \tag{3.17}$$

IV. FRACTIONAL BROWNIAN MOTION DRIVEN BY THE IIG PROCESS

We next discuss the FBM time changed by the IIG process $H(t)$, defined in (2.12). This system is defined by

$$Y(t) = B_H(H(t)), \tag{4.1}$$

where the processes $B_H(t)$ and $H(t)$ are assumed to be independent. In the following sections the asymptotic behavior of moments, the long-range dependence, and the CTRW connection of the process $Y(t)$ are discussed.

A. Distributional properties of FBM time changed by the IIG process

As in the previous case, we discuss the properties of the one-dimensional distribution of the process $Y(t)$ defined in (4.1). Using (2.17) and (3.6), it follows that

$$\mathbb{E}|Y(t)|^q = \mathbb{E}H^{qH}(t)\mathbb{E}|B_H(1)|^q \sim \begin{cases} \frac{c_q}{(\delta\sqrt{2})^{qH}}\frac{t^{qH/2}}{\Gamma(1+qH/2)} & \text{for } \gamma = 0 \\ c_q\left(\frac{\gamma}{\delta}\right)^{qH}\frac{t^{qH}}{\Gamma(1+qH)} & \text{for } \gamma > 0, \end{cases} \quad (4.2)$$

where c_q is defined in (3.6). Next we discuss the covariance and LRD behavior of the process $Y(t)$. The covariance structure for time-changed FBM is discussed in [46]. However, the authors have not explicitly discussed the covariance structure for time-changed FBM by the inverse of the inverse Gaussian process. Here we provide an explicit asymptotic behavior for the covariance structure of $Y(t)$. The covariance function for $Y(t)$ for $s < t$ is given by (see Theorem 3.1 in [46])

$$\mathbb{E}Y(s)Y(t) = M_{2H}(s) + 2H\int_0^s M_{2H-1}(t-y)dM_1(y),$$

where $M_q(t)$ is the qth-order moment of $H(t)$. For fixed s and large t, we have

$$\begin{aligned}\int_0^s M_{2H-1}(t-y)dM_1(y) &\sim \left(\frac{\gamma}{\delta}\right)^{2H-1}\frac{1}{\Gamma(2H)}\int_0^s (t-y)^{2H-1}M_1'(y)dy \\ &\sim \left(\frac{\gamma}{\delta}\right)^{2H-1}\frac{1}{\Gamma(2H)}(t-s)^{2H-1}M_1(s) \\ &\sim \left(\frac{\gamma}{\delta}\right)^{2H-1}\frac{1}{\Gamma(2H)}t^{2H-1}M_1(s).\end{aligned}$$

Thus $\mathbb{E}Y(s)Y(t) \sim M_{2H}(s) + (\frac{\gamma}{\delta})^{2H-1}\frac{2H}{\Gamma(2H)}t^{2H-1}M_1(s)$. For $\gamma > 0$, using (4.2), it gives

$$\text{Corr}[Y(s),Y(t)] \sim \left(\frac{\delta}{\gamma}\right)^{2H}\Gamma(1+2H)s^{-H}M_{2H}(s)t^{-H} + \left(\frac{\delta}{\gamma}\right)4H^2 s^{-H}M_1(s)t^{H-1}.$$

Hence, the process $Y(t)$ also has the LRD property, similar to the FBM and the time-changed FBM by the IG process.

Since the second moment of the $Y(t)$ process is a nonlinear function for large t, we conclude that the process is anomalous diffusive. Further, the process $Y(t)$ is not stationary and it does not have stationary increments, hence it is nonergodic.

B. The CTRW connection

In this section the CTRW connection of FBM time changed by the IIG process is established. Let Y_n be the linear process and S_n be the partial sum process discussed in Sec. III B. For H_t^m defined in (2.23), we have following result.

Proposition 6. As $m \to \infty$,

$$\sigma_{[m]}^{-1}S_{H_t^m} \Rightarrow B_H(H(t)) \quad \text{in } (\mathbb{D},J_1). \quad (4.3)$$

Proof. Theorem 4.6.1 in [35] yields $\sigma_{[m]}^{-1}S_{[mt]} \Rightarrow B_H(t)$. Further, Proposition 3 establishes that $m^{-1}H_t^m \Rightarrow H(t)$. Using the independence of underlying sequence X_n and Z_n, it follows that $(\sigma_{[m]}^{-1}S_{[mt]}, m^{-1}H_t^m) \Rightarrow (B_H(t),H(t))$ in $\mathbb{D}\times\mathbb{D}$. Since $B_H(t)$ has continuous sample paths, the result follows using Theorem 13.2.2 of [35] along with the continuous mapping theorem. ■

V. SIMULATION AND ESTIMATION METHODS

In this section simulation and estimation methods for the introduced processes are given. Similar methodology can be used for other time-changed processes with time change as subordinators and inverse subordinators.

A. Fractional Brownian motion driven by the IG process

In this section we present the simulation procedure and the method of parameter estimation for the FBM time changed by the IG process. The simulation of the $X(t)$ process defined in (3.5) is based on the assumption that this system is a superposition of two independent mechanisms, namely, fractional Brownian motion $B_H(t)$ and the inverse Gaussian process $G(t)$. Therefore, we need to simulate separately both processes and take their superposition at the end. The procedure of simulation of the FBM is described in detail, for example, in [40] and we refer the reader to this reference for further information. The algorithm of the simulation of the IG process $G(t)$ for time points $t_1 = 1/n, t_2 = 2/n, \dots, t_n = 1$ can be divided into the following steps.

(i) Since the inverse Gaussian process $G(t)$ has independent and stationary increments, $F_i \equiv G(t_i) - G(t_{i-1}) \stackrel{d}{=} G(dt) \sim \mathcal{G}(dt,1)$ for $i = 1,2,\dots,n$ and $dt = 1/n$. So we generate n independent identically distributed inverse Gaussian variables F_i as follows (see [22]), assuming $\delta = \gamma = 1$.

(a) Generate a standard normal random variable N.

(b) Assign $X = N^2$.

(c) Assign $Y = dt + \frac{X}{2} - \frac{1}{2}\sqrt{4dtX + X^2}$.

(d) Generate a uniform $[0,1]$ random variable U.

(e) If $U \leqslant \frac{dt}{dt+Y}$, return Y; otherwise return $\frac{(dt)^2}{Y}$.

(ii) Assign $G(t_0) = 0$ and $G(t_i) = \sum_{j=1}^{i} F_j$, $i = 1,2,\dots,n$.

(iii) $G(t_1), G(t_2), \dots, G(t_n)$ are n simulated values of the IG process at times $t_1, t_2, \dots, t_n$, respectively.

In order to simulate the trajectory of $G(t)$ process for $t_1, t_2, \dots, t_n$ not belonging to the $[0,1]$ interval we need to rescale the time and take an appropriate value of the δ parameter and in this case $dt = \delta/n$. At the end, we can obtain the $X(t) = B_H(G(t))$.

The exemplary trajectory of the FBM time changed by the IG process is given in Fig. 3. Here $n = 1000$, $\gamma = \delta = 1$, and $H = 0.3$ (top panel) and $H = 0.7$ (bottom panel).

The estimation algorithm of the parameters corresponding to the FBM time changed by the IG process is based on the main property of the $X(t)$ process, namely, the asymptotic behavior of TAMSD for the processes $X(t)$ and $B_H(t)$ is the

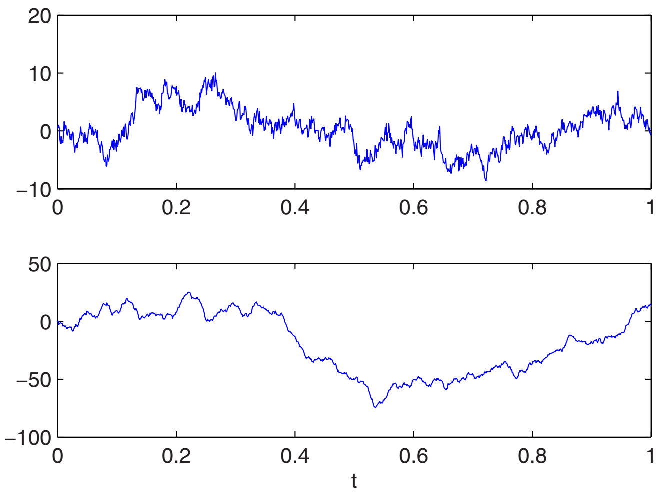

FIG. 3. Exemplary trajectories of the FBM time changed by the inverse Gaussian process for $\gamma = \delta = 1$ and $H = 0.3$ (top) and $H = 0.7$ (bottom).

same:

$$M_n(\tau) = \frac{1}{n-\tau}\sum_{k=1}^{n-\tau}[X(k+\tau)-X(k)]^2 \stackrel{\mathcal{L}}{=} \frac{1}{n-\tau}\sum_{k=1}^{n-\tau} G(\tau)^{2H} B_H(1)^2 \stackrel{\mathcal{L}}{=} \tau^{2H}.$$

The problem of asymptotic behavior of the TAMSD for subordinated processes is also discussed in [47,48]. However, those works discuss the models where the waiting times of corresponding CTRW models have a heavy-tailed distribution and hence the TAMSD changes the behavior in contrast to the TAMSD of the external process. This is in contrast to the case considered in this paper, where the waiting times of the corresponding CTRW model are finite.

Thus, we estimate the H parameter by fitting the power function τ^{2H} to the TAMSD calculated for real data by using the least-squares method. The TAMSD-based method of estimation of the Hurst exponent is well known. However, one can find other techniques that can be used in the problem of estimation of the H parameter [49–51].

The parameters γ and δ can be estimated on the basis of the moments' asymptotic behavior of the FBM time changed by the inverse Gaussian process (3.7). More precisely, if the number of trajectories of the process $X(t)$ is available for each t, we can calculate the empirical moment of the appropriate order and compare it to the theoretical one given in (3.7). This comparison allows for estimation of the parameters γ and δ. In order to check the effectiveness of the estimation procedure, we simulate 300 trajectories of the FBM time changed by the IG process. For each trajectory, we estimate the H parameter by using the method based on the TAMSD and the parameters corresponding to the inverse Gaussian process. To simplify the calculations, we assume that $\gamma = 1$ and estimate only the δ parameter on the basis of the second moment. In Table I we present the median of the estimated values for two sets of parameters: $H = 0.3$ and $\delta = 1$, and $H = 0.7$ and $\delta = 1$. Moreover, in Table I we present also the 95% confidence intervals for estimated values. As one can see, the estimated values coincide with the theoretical counterparts.

TABLE I. Median of estimated values of the parameters H and δ for FBM time changed by the inverse Gaussian process. The 95% confidence intervals for estimated parameters are given in square brackets.

Theory	Estimate	Theory	Estimate
$H = 0.3$	0.3047	$H = 0.7$	0.6953
	[0.2,0.35]		[0.63,0.75]
$\delta = 1$	1.0821	$\delta = 1$	0.9734
	[0.97,1.20]		[0.87,1.02]

B. Fractional Brownian motion driven by the IIG process

The procedure of simulation of the FBM time changed by the IIG process, similar to the previous case, is based on the assumption that the process $Y(t)$ defined in (4.1) is a superposition of two independent trajectories. The first one is a realization of FBM $B_H(t)$, while the second one is the IIG process $H(t)$. Then we need to have two independent realizations of both systems. Since the procedure of simulation of FBM is discussed above, we focus here only on the simulation of the process $H(t)$, which is the inverse of the $G(t)$ process. In order to simulate the approximate trajectory of the general inverse of the other process, we define $H_\Delta(t)$ with the step length Δ as follows:

$$H_\Delta(t) = [\min\{n \in \mathbb{N} : G(\Delta n) > t\} - 1]\Delta, \quad n = 1,2,\ldots,$$

where $G(\Delta n)$ is the value of the process $G(t)$ evaluated at Δn. In our case, the $G(t)$ process is the IG one and the procedure of simulation is described above. Finally, the trajectory of the process $Y(t)$ is obtained as the superposition of FBM and the process $H(t)$. In Fig. 4 we present the exemplary trajectories of the FBM time changed by the IIG process $H(t)$ for two different sets of parameters, namely, $H = 0.3$ and $\gamma = \delta = 1$, and $H = 0.7$ and $\gamma = \delta = 1$. From the trajectories we observe the constant time periods typical for processes time changed by inverse processes.

The estimation procedure of the parameters of FBM driven by the IIG subordinator is divided into two steps. We will make use of the fact that constant time periods in trajectories of the process $Y(t)$ correspond to the jumps of the process $G(t)$. In the first step, we divide the analyzed time series into two vectors. The first one (vector G) represents lengths of constant time periods observed in the data, i.e., the number of consecutive observations that are on the same level. According to the idea of constructing the inverse subordinators, the vector G constitutes a sample of independent identically distributed random variables from the same distribution as the subordinator. The second vector B_H arises after removing the constant time periods and it constitutes a trajectory of FBM. This scheme is a standard procedure in the analysis of the processes subordinated by inverse subordinators and was used in multiple applications; see, for instance, [52].

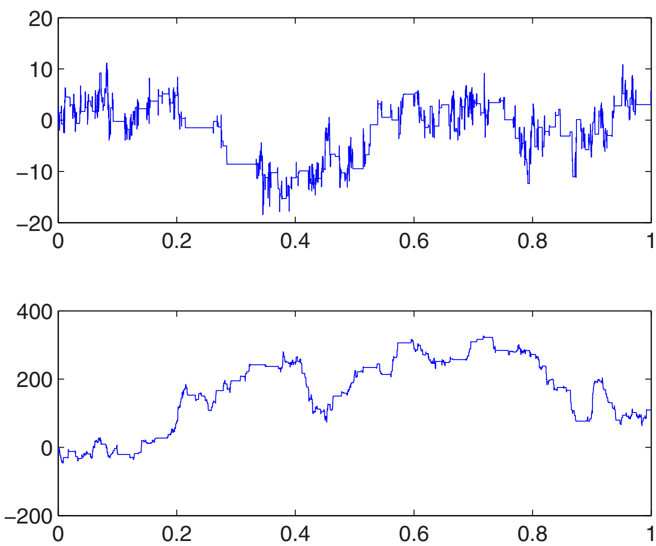

FIG. 4. Exemplary trajectories of the FBM time changed by the inverse to the inverse Gaussian process for $\gamma = \delta = 1$ and $H = 0.3$ (top) and $H = 0.7$ (bottom).

In the second step, we separately analyze the vectors G and B_H. On the basis of constant time periods, we estimate the parameter δ from the inverse Gaussian distribution given by the PDF (2.2) for $\gamma = 1$. The proposed methodology of fitting the distribution parameter δ corresponding to the IG process is based on the minimum distance estimation applied to the IG distribution. This procedure was proposed in [53] for different inverse subordinators and we here briefly sketch their idea. Let K and L denote two functions with a common support on $\mathbb{R}$. The distances considered are the Kolmogorov-Smirnov distance $\mathcal{D}_{\text{KS}}$,

$$\mathcal{D}_{\text{KS}}(K,L) = \sup_{x\in R} |K(x) - L(x)|,$$

the Cramér–von Mises distance $\mathcal{D}_{\text{CvM}}$,

$$\mathcal{D}_{\text{CvM}}(K,L) = \int_{-\infty}^{\infty} [K(x) - L(x)]^2 dL(x),$$

and the Anderson-Darling distance $\mathcal{D}_{\text{AD}}$,

$$\mathcal{D}_{\text{AD}}(K,L) = \int_{-\infty}^{\infty} \frac{[K(x) - L(x)]^2}{L(x)[1 - L(x)]} dL(x).$$

In our estimation procedure we consider the distance between the rescaled modified cumulative distribution function and empirical distribution function obtained for the vector G. The modified cumulative distribution function (CDF) of a given distribution with the CDF $F(\cdot)$ can be expressed as [53]

$$\tilde{F}(n) = \int_{n}^{n+1} F(x)dx. \tag{5.1}$$

The rescaled modified cumulative distribution function is defined in the following way:

$$G(0) = 0,$$
$$G(n) = \frac{\tilde{F}(n) - \tilde{F}(0)}{1 - \tilde{F}(0)} \quad \text{for } n \geqslant 1. \tag{5.2}$$

TABLE II. Median of estimated values of the parameters H and δ for FBM time changed by the inverse of the inverse Gaussian process. The 95% confidence intervals for estimated parameters are given in square brackets.

Theory	Estimate	Theory	Estimate
$H = 0.3$	0.3093	$H = 0.7$	0.6880
	[0.2777,0.3380]		[0.6375,0.7187]
$\delta = 1$	$\mathcal{D}_{\text{KS}} = 1.001$	$\delta = 1$	$\mathcal{D}_{\text{KS}} = 1.0401$
	[0.811,1.178]		[0.807,1.182]
	$\mathcal{D}_{\text{CvM}} = 0.997$		$\mathcal{D}_{\text{CvM}} = 1.0197$
	[0.841,1.143]		[0.852,1.155]
	$\mathcal{D}_{\text{AD}} = 0.998$		$\mathcal{D}_{\text{AD}} = 1.0230$
	[0.845,1.146]		[0.859,1.161]

Under the assumption that $\gamma = 1$, we find the parameter δ_0 that minimizes the distance between the empirical CDF and the rescaled modified CDF, i.e., satisfies the condition

$$D\big(G_{\delta_0}, \hat{F}\big) = \inf_{\delta\in\Theta} D(G_\delta, \hat{F}), \tag{5.3}$$

where D is one of the distances introduced ($\mathcal{D}_{\text{KS}}$, $\mathcal{D}_{\text{CvM}}$, and $\mathcal{D}_{\text{AD}}$), $\hat{F}$ is an empirical CDF, G is a rescaled modified CDF defined in (5.2), and Θ is the parameter space of δ. The estimation of the Hurst parameter H is done on the basis of the vector B_H, the procedure based on the TAMSD behavior has already been described in Sec. V A.

Similar to the previous case, we simulate 300 trajectories of the process $Y(t)$ and for each of them we estimate the unknown parameters. The δ parameter is estimated by using the three mentioned distances, namely, $\mathcal{D}_{\text{KS}}$, $\mathcal{D}_{\text{CvM}}$, and $\mathcal{D}_{\text{AD}}$. We consider here two sets of parameters: $H = 0.3$ and $\delta = 1$, and $H = 0.7$ and $\delta = 1$. The results obtained are presented in Table II. Similarly, like in Table I, we present also the 95% confidence intervals for the estimated values.

VI. CONCLUSION

One of the classical models used for the description of long-range-dependent processes is the fractional Brownian motion. One can find various applications of FBM in different disciplines, including physics, biology, and finance. However, for many real time series, which exhibit the long-range-dependent property, FBM cannot be used directly. Therefore, there is a need to introduce extensions of the classical model in order to keep the main properties of FBM and cover other characteristics adequate to analyzed phenomena. Generally, time-changed FBM models generate a long-range-dependent, semi-heavy-tail distribution, non-Gaussian behavior at small lags and Gaussian behavior at large lags, which is a common model requirement for modeling of hydraulic conductivity fields in geophysics [54]. In this paper we have studied two processes based on FBM. The first one is the FBM time changed by the inverse Gaussian process, known especially from the financial data description. The second process considered is the FBM delayed by the so-called inverse to the inverse Gaussian process. We have discussed the main properties of the IG and IIG processes. The IG process is known in the literature and here some results such as tail behavior and asymptotic behavior of moments are established. The features of the IIG process are not known in the literature. We have presented here, for example, the moments and their asymptotic behavior as well as the connections between the PDF of the IIG process and fractional partial differential equations.

We have also discussed the properties of FBM time changed by the IG and IIG processes. The first system considered is discussed in the literature. We have extended the theory related to this system and examined, for instance, the long-range-dependence property, infinite divisibility, and estimation of parameters. Our results generalize and complement the results available on the normal inverse Gaussian process whose density is known to solve the relativistic diffusion equation from statistical physics. We have presented a second process, namely, FBM delayed by the IIG process. We have compared the main properties of FBM time changed by the IIG process with similar features for FBM with the IG subordinator and indicated the differences between them. It is worth mentioning that the examined time-changed processes can be used for real-life applications. The first process considered can be used for data that exhibit the long-range-dependence property with non-Gaussian structure. This behavior is characteristic for many real time series. The second process considered, namely, FBM delayed by the IIG system, has very specific behavior, namely, constant time periods, which is typical for processes delayed by inverse subordinators. Moreover, this process exhibits also a long-range dependence that is typical for anomalous diffusion systems. We observe such characteristics, for example, in financial time series such as interest rates, where there are some time intervals with small volatility (so-called trapping behavior) and visible long-range dependence. The proposed subordinated processes can also be used on physical phenomena, for instance, to model data that exhibit some properties of FBM, but cannot be described by the classical Gaussian self-similar processes. An example of this kind can be found in [55], where the authors discovered self-similar properties of transport mediated by molecular motors on filament networks in *in vitro* and *in vivo* data. On the other hand, some characteristics indicate behavior related to a process that is between FBM and a CTRW. We also refer the reader to [56], where the diffusion in the plasma membrane of living cells is analyzed. The data are found to display anomalous dynamics; however, the mechanism underlying this diffusion pattern remains highly controversial. The authors show that an ergodic process and a nonergodic process coexist in the plasma membrane, which indicates that the existing models cannot be directly used to model such data. The proposed models in the paper can be useful in this context also.

To facilitate the possible applications of both processes considered to real phenomena, we have proposed the estimation procedures. The estimation techniques result from the properties of the processes, such as the TAMSD and the moments' behavior for the FBM time changed by the IG process and the TAMSD behavior and the existence of trapping events for FBM delayed by the IIG process. The presented theoretical results can be very useful in real data analysis. By comparing the empirical characteristics (such as empirical moments or the empirical cumulative distribution function) calculated for real time series with the theoretical counterparts, we can easily identify the possible probabilistic model that generates the data. The effectiveness of estimation methods and algorithms is compared for simulated trajectories. We have shown also the step-by-step procedures of simulations for both processes.

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